Answers for the Math 20 Take-Home examination

1 (i) $P(X = 2) = p\bar{p}$, $P(X = 4) = q\bar{q}$, $P(X = 3) = p\bar{q} + q\bar{p}$.

(ii) If $P(X = 2) = P(X = 4)$ then we get $p\bar{p} = q\bar{q} = (1 - p)(1 - \bar{p}) = 1 - p - \bar{p} + p\bar{p}$ so it follows that $p + \bar{p} = 1$. We also get $q = (1 - p)$ and $\bar{q} = (1 - \bar{p}) = p$. Hence if $p = 1/3$ we may get $a = 2/9$.

(iii) Assuming $P(X = 2) = P(X = 3)$ and part (ii) we get $p\bar{p} = p(1 - p) = p - p^2 = p\bar{q} + q\bar{p} = p^2 + (1 - p)^2$ so $3p^2 - 3p + 1 = 0$. This polynomial has no real zeroes, so there is no solution.

2. If $k = 2$ then we get $E(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \ldots$, which is exactly the same as the expected value for $k = 1$, which we know is 2.

The cases $k = 3$ and $k = 4$ may be done similarly, but it may be easier to use conditional expectation. Let X_k denote the number of trials to achieve k. Then $E(X_k) = E(X_k|F_1)P(F_1) + E(X_k|F_2)P(F_2)$ where F_1 denotes getting a tail on the first toss and F_2 denotes getting a head on that toss. Obviously $P(F_1) = P(F_2) = \frac{1}{2}$. But now $E(X_3|F_1) = 1 + E(X_1) = 1 + 2 = 3$ and $E(X_3|F_2) = 1 + E(X_2) = 1 + 2 = 3$, so $E(X_3) = 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 3$. And $E(X_4|F_1) = 1 + E(X_2) = 1 + 2 = 3$ while $E(X_4|F_2) = 1 + E(X_3) = 1 + 3 = 4$, so $E(X_4) = \frac{3}{2} + 2 = \frac{7}{2}$.

3. (i) Let $\Omega = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{1, 3\}$ and $A = \{1\}$. Then $P(A|B) = P(A|C) = \frac{1}{2}$ and $PA|B \cup C) = \frac{1}{3}$. Note that $A \cap B = \{1\}$, so in this case the conclusion may fail.

(ii) If B, C are disjoint then $P(B \cup C) = P(B) + P(C)$. Also $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $\frac{P(A \cap B)}{P(B)} = p = \frac{P(A \cap C)}{P(C)}$. Thus

$$\frac{P(A \cap (B \cup C)}{P(B \cup C)} = \frac{P(A \cap B) + P(A \cap C)}{P(B \cup C)} = \frac{pP(B) + pP(C)}{P(B) + P(C)} = p$$

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4. 6/11. Nearly everyone got this.

5. (i) There are 36 cards, 20 of which are even, so the answer is

$$\frac{\binom{20}{4}}{\binom{36}{4}}.$$

(ii) The number of hands with at least two 2's is

$$\binom{4}{2}\binom{32}{2}+\binom{4}{3}\binom{32}{1}+\binom{4}{4}$$

so the probability is

$$\frac{\binom{4}{2}\binom{32}{2}+\binom{4}{3}\binom{32}{1}+\binom{4}{4}}{\binom{36}{4}}$$

(iii)

$$\frac{\binom{4}{2}\binom{16}{2}}{\binom{4}{2}\binom{32}{2}}=\frac{\binom{16}{2}}{\binom{32}{2}}$$