Math 22

Fall 2018

Midterm 1

October 2, 2018

NAME: ______________________________

SECTION (check one box): Section 1 (S. Allen 12:50) ☐
Section 2 (A. Babei 2:10) ☐

Instructions:

1. Write your name *legibly* on this page, and indicate your section by checking the appropriate box.

2. Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer. Make it clear and legible so we can follow it.

3. It is fine to leave your answer in a form such as $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $\cos(\pi)$ or $(3-2)$), you should simplify it.

4. You may use the last 2 pages and the back of each page for scrap paper. Unless you run out of space and clearly indicate that your solutions are on any designated space for scrap paper, *scrap paper will not be graded.*

5. This exam is closed book. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource. It is a violation of the honor code to give or receive help on this exam.

Problem	Points	Score
1	6	
2	12	
3	10	
4	10	
5	10	
6	3	
7	6	
Total	57	

1. *[6 points]* Please indicate whether the following statements are TRUE (T) or FALSE/NOT NECESSARILY TRUE (F) (no working needed, just circle the answer):

   T The linear system

$$\begin{aligned} 3x + 7y + z &= -1 \\ -x - y + z &= -3 \\ -2y - 2z &= 5 \end{aligned}$$

   is consistent.

   T A vector $\mathbf{b}$ is in the span of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

   F If there is a pivot in each row of a coefficient matrix, there are no free variables.

   F If there is a pivot in each row of an augmented matrix, the system is consistent.

   T A set of 4 vectors in $\mathbb{R}^3$ is always linearly dependent.

   F The vectors

$$\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 9 \\ -6 \\ -12 \end{bmatrix}$$

   are linearly independent.

2. *[12 points]* Let

$$A = \begin{bmatrix} 3 & 0 & 3 & 0 \\ 1 & 2 & 3 & -4 \\ 2 & -1 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$$

a) Solve the matrix equation $A\mathbf{x} = \mathbf{b}$, and write the solution in parametric vector form.

Solution.

$$\begin{bmatrix} 3 & 0 & 3 & 0 & 6 \\ 1 & 2 & 3 & -4 & 0 \\ 2 & -1 & 1 & 2 & 5 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{bmatrix} 0 & -6 & -6 & 12 & 6 \\ 1 & 2 & 3 & -4 & 0 \\ 0 & -5 & -5 & 10 & 5 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} 0 & -6 & -6 & 12 & 6 \\ 1 & 2 & 3 & -4 & 0 \\ 2 & -1 & 1 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 \leftarrow -1/5 R_3]{R_1 \leftarrow -1/6 R_1} \begin{bmatrix} 0 & 1 & 1 & -2 & -1 \\ 1 & 2 & 3 & -4 & 0 \\ 0 & 1 & 1 & -2 & -1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_3 - R_1} \begin{bmatrix} 0 & 1 & 1 & -2 & -1 \\ 1 & 2 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 & -4 & 0 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ The solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ -1 - x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

b) Without doing any row operations, write the solution set to the matrix equation $A\mathbf{x} = 5\mathbf{b}$.

Solution.

$$\begin{bmatrix} 10 \\ -5 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

c) Write the solution to the homogenous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

Solution.

$$x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

d) Is the solution set of $A\mathbf{x} = \mathbf{0}$ a point, a line, a plane, a 3-dimensional space, or all of $\mathbb{R}^4$? Explain your answer.

Solution. The solution is the span of the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, which are linearly independent. The span of two linearly independent vectors is a plane.

3. *[10 points]* Let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ h \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

a) Find all the real values h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans all of $\mathbb{R}^3$. Explain your answer.

Solution. Let $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$, then the columns of A span $\mathbb{R}^3$ if and only if the matrix equation $A\mathbf{x} = \mathbf{b}$ is always consistent, if and only if there is a pivot in every row of A. To find out when this is the case, we do row operations as follows:

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & h \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 3R_1} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 3+h \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1+h \end{bmatrix}$$

There is a pivot in every row if and only if $h \neq -1$. Therefore, the vectors span $\mathbb{R}^3$ for $h \neq -1$.

b) Let $h = 1$, and write $\mathbf{v}_4$ as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution. We try to solve the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{v}_4$, which is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}_4$. To achieve this, we create the augmented matrix $[A \quad \mathbf{v}_4]$, and do the following row operations:

$$\begin{bmatrix} -1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 3R_1} \begin{bmatrix} -1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 4 & -2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} -1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & -6 \end{bmatrix}$$

$$\xrightarrow[R_3 \leftarrow 1/2 R_3]{R_1 \leftarrow -R_1} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + 3R_3} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Therefore, we get $-7\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{v}_4$

4. *[10 points]* Suppose that T is a one-to-one linear transformation and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent vectors in $\mathbb{R}^n$. Prove that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\}$ is also linearly independent.

Proof. We assume T is one to one, and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ are linearly independent. We want to show $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\}$ is also linearly independent, which is equivalent to showing that the vector equation $x_1 T(\mathbf{v}_1) + x_2 T(\mathbf{v}_2) + \cdots + x_p T(\mathbf{v}_p) = \mathbf{0}$ only has the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

Consider the equation

$$x_1 T(\mathbf{v}_1) + x_2 T(\mathbf{v}_2) + \cdots + x_p T(\mathbf{v}_p) = \mathbf{0}.$$

Since T is a linear transformation, and $cT(\mathbf{u}) + dT(\mathbf{v}) = T(c\mathbf{u} + d\mathbf{v})$, this is equivalent to

$$T(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p) = \mathbf{0}.$$

Since T is one-to-one, the equation $T(\mathbf{y}) = \mathbf{0}$ only has the trivial solution, so

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = 0.$$

Since the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent, by definition this vector equation only has the trivial solution $x_1 = x_2 = \cdots = x_p = 0$, and we are done.

Alternative Proof. We want to show $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\}$ is linearly independent. This is equivalent to showing the system corresponding to the augmented matrix

$$[T(\mathbf{v}_1) \quad T(\mathbf{v}_2) \quad \ldots \quad T(\mathbf{v}_p) \quad \mathbf{0}]$$

only has the trivial solution $\mathbf{x} = \mathbf{0}$.

But the coefficient matrix above is the product between the standard matrix $[T]$ and the matrix $B = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_p]$:

$$[T(\mathbf{v}_1) \quad T(\mathbf{v}_2) \quad \ldots \quad T(\mathbf{v}_p)] = [T]B$$

Therefore, we want to show that the system

$$[T](B\mathbf{x}) = \mathbf{0}$$

only has the trivial solution. Since T is one to one, the system $[T]\mathbf{y} = \mathbf{0}$ only has the trivial solution, so

$$B\mathbf{x} = \mathbf{0}.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent , the system

$$B\mathbf{x} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_p]\mathbf{x} = \mathbf{0}$$

only has the trivial solution $\mathbf{x} = \mathbf{0}$, and we are done.

5. *[10 points]* Let T be the linear transformation given by $T(x_1, x_2) = (\frac{3}{2}x_1 + \frac{1}{2}x_2, x_1 - x_2, x_2)$.

a) Find the standard matrix for T.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2}x_1 + \frac{1}{2}x_2 \\ x_1 - x_2 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Standard matrix: $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

b) Find a vector $\mathbf{x}$ whose image under T is $(5, 2, 1)$.

$$\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 5 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ \frac{3}{2} & \frac{1}{2} & 5 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

c) Is T one-to-one? Justify your answer.

Yes, because the columns of the standard matrix are linearly independent.

Yes, because the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

6. *[3 points]* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which vertically contracts by a factor of 1/3, with standard matrix $[T] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects through the line $x = y$, with standard matrix $[S] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   Find the standard matrix for the linear transformation which first vertically contracts by a factor of 1/3, and then reflects through the line $x = y$.

$$S \circ T(\mathbf{x}) = S(T(\mathbf{x})) = S([T]\mathbf{x}) = [S][T]\mathbf{x}$$

$$[S][T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{bmatrix}$$

7. *[6 points]* Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

For each of the following matrix operations, indicate whether the operation is defined. If an expression is undefined, explain why. If an expression is defined, evaluate it.

a) $BA + C$

Undefined. The product BA will have dimensions 2×3. Since C is a 2×1 matrix, and matrix addition is only defined for matrices of the same size, this expression is undefined.

a)BC

$$BC = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 - 2(8) \\ 9 - 2(10) \end{bmatrix} = \begin{bmatrix} -9 \\ -11 \end{bmatrix}$$

a) $B + 3I_2$

$$B + 3I_2 = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 9 & 13 \end{bmatrix}$$