Math 22

Fall 2018

Midterm 2

October 23, 2018

NAME: ____

SECTION (check one box): Section 1 (S. Allen 12:50) ☐
Section 2 (A. Babei 2:10) ☐

Instructions:

1. Write your name *legibly* on this page, and indicate your section by checking the appropriate box.
2. Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer. Make it clear and legible so we can follow it.
3. It is fine to leave your answer in a form such as $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $\cos(\pi)$ or $(3-2)$), you should simplify it.
4. You may use the last 2 pages and the back of each page for scrap paper. Unless you run out of space and clearly indicate that your solutions are on any designated space for scrap paper, *scrap paper will not be graded.*
5. This exam is closed book. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource. It is a violation of the honor code to give or receive help on this exam.

Problem	Points	Score
1	7	
2	12	
3	3	
4	12	
5	9	
6	10	
7	5	
Total	58	

1. *[7 points]* Please indicate whether the following statements are TRUE (T) or FALSE/NOT NECESSARILY TRUE (F) (no working needed, just circle the answer):

T An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

T The map $T : \mathbb{P}_2 \to \mathbb{R}^2$ such that $T(\mathbf{p}(t)) = \begin{bmatrix} 0 \\ \mathbf{p}(3) \end{bmatrix}$ is a linear transformation. (Here, $\mathbb{P}_2$ is the vector space consisting of polynomials of degree at most 2.)

T The space $\mathbb{P}$ of all polynomials is infinite dimensional.

F The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

T Let A be any matrix with n rows. If there is such a matrix C such that $CA = I_n$, then A is invertible.

T Any set of n vectors that span $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

F Let A be a 3×7 matrix. It is possible that $\text{dimNul}A = 3$.

2. *[12 points]* Consider the matrix $A = \begin{bmatrix} 1 & 0 & 6 \\ 1 & -2 & 8 \\ 0 & 3 & 0 \end{bmatrix}$.

(a) Find $\det A$.

$$\det A = 0 - 3 \begin{vmatrix} 1 & 6 \\ 1 & 8 \end{vmatrix} + 0 = -3(8-6) = -6$$

(b) Find $\det(A^5)$. You do not need to simplify your answer.

$$\det(A^5) = (\det A)^5 = (-6)^5$$

(c) Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & 0 & 0 \\ 1 & -2 & 8 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & -2 & 2 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 2 & -1 & 1 & 2/3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 & -2 \\ 0 & 0 & 1/3 \\ -1/2 & 1/2 & 1/3 \end{bmatrix}$$

(d) Suppose B is a 3×3 matrix such that $\det B = 5$. Based on your answer for part (a), find each of the following.

i. the number of pivot positions of A.

3

ii. $\text{rank}(BA^{-1})$.

3

iii. Span Col (A^4).

$\mathbb{R}^3$

iv. dim Nul B.

0

3. *[3 points]* Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$ be the set of vectors lying in the first quadrant of the xy-plane. Prove that V is not a vector space. State clearly which axiom does not hold and explain why.

V does not satisfy the axiom "$c\mathbf{u}$ is in V" because $-1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$ so if $\begin{bmatrix} x \\ y \end{bmatrix}$ is in V, $x \geq 0$ and $y \geq 0$, i.e., $-x \leq 0$ and $-y \leq 0$, thus $\begin{bmatrix} -x \\ -y \end{bmatrix}$ is not in V.

4. *[12 points]*

(a) Using the definition of a subspace (and no other theorems), show that the set W consisting of all vectors of the form $\begin{bmatrix} s+3t \\ 4t \\ 0 \end{bmatrix}$ is a subspace of $\mathbb{R}^3$.

- **0** is in W:

  Let $s = 0$ and $t = 0$, then $\begin{bmatrix} 0+3(0) \\ 4(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W.

- W is closed under addition:

  Let $\mathbf{u} = \begin{bmatrix} s_1+3t_1 \\ 4t_1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} s_2+3t_2 \\ 4t_2 \\ 0 \end{bmatrix}$ be in W. Then

  $$\mathbf{u}+\mathbf{v} = \begin{bmatrix} s_1+3t_1 \\ 4t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} s_2+3t_2 \\ 4t_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (s_1+s_2)+3(t_1+t_2) \\ 4(t_1+t_2) \\ 0 \end{bmatrix}$$

  which is also in W.

- W is closed under scalar multiplication:

  Let $\mathbf{u} = \begin{bmatrix} s+3t \\ 4t \\ 0 \end{bmatrix}$ be in W and c be a scalar. Then $c\mathbf{u} = \begin{bmatrix} (cs)+3(ct) \\ 4(ct) \\ 0 \end{bmatrix}$ which is also in W.

(b) Find a basis B for the subspace W.

$$\begin{bmatrix} s+3t \\ 4t \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$ is a basis for W. (They clearly span W, and are linearly independent because neither is a multiple of the other.)

(c) What is dim W?

$$\dim W = 2$$

(d) Geometrically, what does W represent?
W is a plane through the origin

(e) Add an appropriate number of vectors to the set B from part (b) so that the new set is a basis for $\mathbb{R}^3$.
$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

5. *[9 points]* Let V be a vector space with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Prove that the coordinate map is one-to-one and onto.

Solution. We have the coordinate map

$$V \to \mathbb{R}^3$$

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$$

One-to-one: Let $\mathbf{u}, \mathbf{v} \in V$ such that $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$. Then

$$\mathbf{u} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + r_3\mathbf{b}_3$$

$$\mathbf{v} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + r_3\mathbf{b}_3$$

so

$$\mathbf{u} = \mathbf{v},$$

and the transformation is one-to-one.

(Alternatively, we may want to show that $[\mathbf{x}]_{\mathcal{B}} = \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$. Then

$$\mathbf{x} = 0\mathbf{b}_1 + 0\mathbf{b}_2 + 0\mathbf{b}_3 = \mathbf{0},$$

and the map is again one-to-one.)

Onto: Given any $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \in \mathbb{R}^3$, we want to find its preimage under the coordinate map. Let $\mathbf{x} = t_1\mathbf{b}_1 + t_2\mathbf{b}_2 + t_3\mathbf{b}_3$, then $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$, and the map is onto.

6. *[10 points]* Consider the matrix $A = \begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ -3 & 4 & -3 & -2 & -4 \\ -2 & 4 & -2 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for Nul$A$. What is dim Nul$A$?

Solution.

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ -3 & 4 & -3 & -2 & -4 \\ -2 & 4 & -2 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ -2 & 4 & -2 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 2R_1} \begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We get

$$\begin{aligned} x_1 &= -x_3 - 2x_4 - 4x_5 \\ x_2 &= -x_4 - 2x_5 \\ &x_3, x_4, x_5 \text{ free.} \end{aligned}$$

Since there are 3 free variables, dim Nul$A = 3$. Furthermore

$$\text{Nul}A = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and a basis is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for Col$A$. What is $\dim \mathrm{Col}A$?

Solution. There are 2 basic variables, so $\dim \mathrm{Col}A = 2$. Furthermore, a basis corresponds to the pivot columns in A and we have a basis

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix} \right\}.$$

(c) Find a basis for Row$A$.

Solution. We have $\dim \mathrm{Col}A = \dim \mathrm{Row}A = 2$. Furthermore, a basis corresponds to the pivot rows in the echelon form of A and we have a basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

7. *[5 points]* Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$.

Find the change of basis matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

Solution. We have $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [\ [\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}}\ ]$, so we row-reduce concomitantly

$$[\ \mathbf{c}_1 \quad \mathbf{c}_2 \mid \mathbf{b}_1 \quad \mathbf{b}_2\ ] \Longrightarrow [\ I_2 \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\ ]$$

$$\begin{bmatrix} 2 & 1 & 1 & 4 \\ -1 & 0 & -1 & -3 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{bmatrix} 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow -R_2} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

so

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}.$$

Since

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = (\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1}$$

we get

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \frac{1}{-2-(-3)} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

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