## The Invertible Matrix Theorem

Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent.

| | |
|---|---|
| a. $A$ is an invertible matrix. | |
| b. $A$ is row equivalent to the $n \times n$ identity matrix. | equiv to (a) by §2.2 |
| c. $A$ has $n$ pivot positions. | easily shown equiv to (b) |
| d. The equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. | equiv to (c): no free variables |
| e. The columns of $A$ form a linearly independent set. | equiv to (d): linearly dep cols $\Leftrightarrow$ nontrivial zeros |
| f. The linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is one-to-one. | equiv to (d) by §1.9 |
| g. The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for each $\boldsymbol{b}$ in $\mathbb{R}^n$. | equiv to (c) by §1.2 |
| h. The columns of $A$ span $\mathbb{R}^n$. | equiv to (g) by def of matrix mult |
| i. The linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is onto $\mathbb{R}^n$. | equiv to (h) by §1.9 |
| j. There is an $n \times n$ matrix $C$ such that $CA = I_n$. | both (j), (k) equiv to (a) by showing $C$, $D$ two-sided inverse: eg $C = I_nC = CAC = C(AC)$, so $AC = I_n$ |
| k. There is an $n \times n$ matrix $D$ such that $AD = I_n$. | |
| l. $A^T$ is an invertible matrix. | equiv to (a) by §2.2 |

**Table of equivalences as in margin:**

$$
\begin{array}{ccccccccccc}
 & & (i) & \Longleftrightarrow & (h) & \Longleftrightarrow & (g) & & (f) & & \\
 & & & & & & \Updownarrow & & \Updownarrow & & \\
(l) & \Longleftrightarrow & (a) & \Longleftrightarrow & (b) & \Longleftrightarrow & (c) & \Longleftrightarrow & (d) & \Longleftrightarrow & (e) \\
 & & \Updownarrow & & & & & & & & \\
 & & (j,k) & & & & & & & &
\end{array}
$$