Problem 1

(a) $\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ -8 \\ -19 \\ -9 \end{bmatrix}, \begin{bmatrix} 10 \\ 16 \\ 31 \\ 13 \end{bmatrix} \right\}$

(c) 3

(d) $\left\{ \begin{bmatrix} 8 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -5/2 \\ 0 \\ 0 \\ -1/2 \\ 1 \end{bmatrix} \right\}$

(e) $\text{rank}(A^T) = \text{rank}(A) = 3$ so

$\dim \text{Nul}(A^T) = 4 - 3 = 1.$

$$\begin{bmatrix} 1 & -3 & 4 & 2 & 5 & 1 \\ 0 & 1 & -4 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 1 & -3 & 4 & 2 & 5 & 1 \\ 0 & 1 & -4 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 1 & -3 & 4 & 2 & 0 & -3/2 \\ 0 & 1 & -4 & 3 & 0 & 5/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -8 & 11 & 0 & 6 \\ 0 & 1 & -4 & 3 & 0 & 5/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 8x_3 - 11x_4 - 6x_6$

$x_2 = 4x_3 - 3x_4 - 5/2\, x_6$

$x_5 = -1/2\, x_6$

x_3, x_4, x_6 free

$$\bar{x} = x_3 \begin{bmatrix} 8 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -11 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -6 \\ -5/2 \\ 0 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}$$

Problem 2

(a) $A\bar{v} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\bar{v}$

$A\bar{v} = 2\bar{v}$ eigen-value 2

$A\bar{w} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\bar{w}.$

$A\bar{w} = 3\bar{w}$ eigen-value 3.

(b) $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$

(c) $A^k = PD^kP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2^k & 3^k \\ 2^k & 2\times 3^k \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} - 3^k & 3^k - 2^k \\ 2^{k+1} - 2\times 3^k & -2^k + 2\times 3^k \end{pmatrix}$

Problem 3

(a) $P = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 2 & 2 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

$A = PDP^{-1}$

(b) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 1 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 1 & 0 \\ 0 & 0 & 2 & -6 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1/2 \end{array}\right]$

$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -1 & 1/2 \end{bmatrix}.$

$A = PDP^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -1 & 1/2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ -4 & 2 & 0 \\ -2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -1 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 20 & 6 & -1 \end{pmatrix}.$

Problem 4

(a) $A - \lambda I_3 = \begin{pmatrix} 7-\lambda & 0 & 0 \\ 3 & 2-\lambda & 0 \\ 3 & 3 & 2-\lambda \end{pmatrix}$.

$\det(A - \lambda I_3) = (7-\lambda)(2-\lambda)^2$.

Eigen-values are 7 and 2.

(b) $\lambda = 7$ multiplicity 1
$\lambda = 2$ multiplicity 2.

(c) $\lambda = 7$ geometric multiplicity 1.

$\lambda = 2$: $A - 2I_3 = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 3 & 0 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

x_3 is free. Geometric multiplicity 1.

(d) No! For $\lambda = 2$, geometric multiplicity is smaller than the algebraic multiplicity.

Problem 5

$$\bar{x} = P_B [\bar{x}]_B.$$

$$P_B = \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}; \quad \bar{x} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}.$$

$$P_B^{-1} = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}.$$ So,

$$[\bar{x}]_B = P_B^{-1} \bar{x} = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 9 \\ -28 \end{pmatrix}.$$

Problem 6

$$1 = 1 \Rightarrow [1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$t = 2 \cdot 1 + 1 \cdot (t-2) \Rightarrow [t]_B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$t^2 = (t-2+2)^2 = 4 + 4(t-2) + (t-2)^2$$

$$[t^2]_B = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

$$[x]_B = P_B [4 + 7t - 3t^2]_B =$$

$$= 4[1]_B + 7[t]_B - 3[t^2]_B$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}.$$

Ad hoc:

$$\bar{x} = 4 + 7(t-2+2) - 3((t-2)+2)^2 =$$

$$= 4 + 7(t-2) + 14 - 3((t-2)^2 + 4(t-2) + 4)$$

$$= (4 + 14 - 3 \times 4) + (7 - 4 \times 3)(t-2) - 3(t-2)^2$$

$$= 6 - 5(t-2) - 3(t-2)^2. \Rightarrow$$

$$[\bar{x}]_B = \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}$$

Problem 7

(a) $A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \end{pmatrix}$

$$\det(A - \lambda I_2) = \det \begin{vmatrix} 3-\lambda & 3 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)(5-\lambda) - 6 = 15 - 8\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 8\lambda + 9.$$

(b) $B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$

$$\det(B - \lambda I_3) = \det \begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} =$$

$$= (4-\lambda) \det \begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} =$$

$$= (4-\lambda)\left((1-\lambda)(2-\lambda) - 2\right) = (4-\lambda)(\lambda^2 - 3\lambda + 2 - 2)$$

$$= (4-\lambda)(3-\lambda)\lambda = -\lambda^3 + 7\lambda^2 - 12\lambda.$$

(c) $\lambda^2 - 8\lambda + 9 = 0, \quad \lambda_{1,2} = 4 \pm \sqrt{16-9} = 4 \pm \sqrt{7}.$

(d) $\lambda(4-\lambda)(3-\lambda) = 0, \quad \lambda_{1,2,3} = 0, 3, 4.$

Problem 8

(a) i). 2

ii). 2

iii). 2

iv). 5

(b) because the eigen-values of the shown matrix are its diagonal elements (matrix is upper triangular), so, 2, 7, 12, 17 but not -1.

(c) No! Matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.

(d) $H = \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 ; \; p(2) = a_0 + 2a_1 + 4a_2 + 8a_3 = 0 \}$.

$H = Nul(A)$; $A : \mathbb{R}^4 \to \mathbb{R}$

$$A \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = [1, 2, 4, 8] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_0 + 2a_1 + 4a_2 + 8a_3.$$

So matrix $A = [1, 2, 4, 8]$ (is 1×4)

(e) Same as (c)

(f) YES by a theorem in class.

Problem 9

(i) $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $C = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

$$BC = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

$$CB = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} wa + cx & wb + xd \\ ay + zc & by + zd \end{bmatrix}$$

$tr(BC) = aw + by + cx + dz$: they are
$tr(CB) = wa + cx + by + zd$ equal

(ii) $tr(PAP^{-1}) = tr((PA)P^{-1})$
$= tr(P^{-1}(PA)) = tr(((P^{-1}P)A) = tr\,A$.

other. $tr(A)$ is coefficient of $(-\lambda)$ in the characteristic polynomial for A.
Since PAP^{-1} and A have same characteristic polynomial (theorem in class), then they have same coefficient for $(-\lambda)$. So, $tr(A) = tr(PAP^{-1})$.

· Problem 10

i) $$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 2 & 2 & -1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & -1 & -2 & 3 \end{bmatrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1 & 2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & -5 & 10 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -5/2 & 5 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} -5/2 & 5 \\ 2 & -3 \end{bmatrix}$$

ii) $$\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P}\right)^{-1} = \frac{1}{-\frac{5}{2}} \begin{bmatrix} -3 & -5 \\ -2 & -5/2 \end{bmatrix}$$

$$= \begin{bmatrix} 6/5 & 2 \\ 4/5 & 1 \end{bmatrix}.$$