Traces on graph algebras

Danny W. Crytser

Dartmouth College

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C^*-algebras

Definition

A *C^*-algebra* is a complex $*$-algebra A with norm $||\cdot||$ such that

1. $||ab|| \leq ||a||||b||$ for any $a, b \in A$
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Example: If X is a locally compact Hausdorff space, then $C_0(X) := \{f : X \to \mathbb{C} | f \text{ is continuous and vanishes at } \infty\}$ is a C^*-algebra under the $||\cdot||_\infty$-norm and pointwise operations.

C^*-algebras generated by partial isometries

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Theorem (Cuntz, '77)

If A is generated by elements s, t satisfying

$$s^*s = t^*t = ss^* + tt^* = 1$$

then $A \cong \mathcal{O}_2$, the Cuntz algebra.

Directed graphs

Definition

A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$, where E^0 and E^1 are (countable) sets and $r, s : E^1 \to E^0$ are functions called the *range* and *source* map, respectively.

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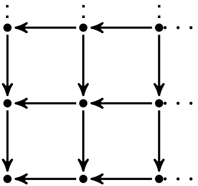
(All the graphs in this talk will be directed, so we might start just referring to them as graphs.) You can visualize a directed graph by drawing a point in the plane for each $v \in E^0$ and drawing for each edge $e \in E^1$ an arrow from $s(e)$ to $r(e)$.



Graph C^*-algebras

Operator algebraists like graphs because they give us a standard way to study a wide class of C^*-algebras generated by partial isometries. The basic idea is that you keep track of the relations between the generators using the edge matrix of a directed graph.

Graph C^*-algebras

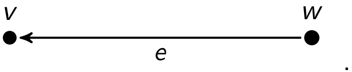
Definition

Given a directed graph $E = (E^0, E^1, r, s)$ the *graph algebra* $C^*(E)$ is the universal C^*-algebra generated by a family $\{s_e, p_v : e \in E^1, v \in E^0\}$, where the p_v are mutually orthogonal projections and the s_e are partial isometries with mutually orthogonal range projections satisfying

1. $s_e^* s_e = p_{s(e)}$
2. $s_e s_e^* \leq p_{r(e)}$
3. $p_v = \sum_{r(e)=v} s_e s_e^*$ if $0 < |r^{-1}(v)| < \infty$.

Graph algebras

If E is the graph



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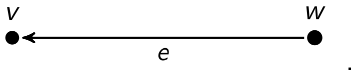

then you can show that $C^*(E) \cong M_2(\mathbb{C})$.

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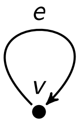

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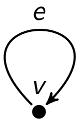

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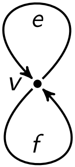

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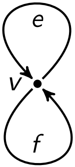

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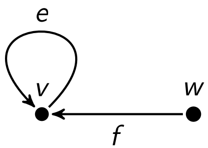

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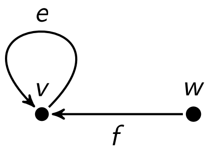

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We aim to characterize two C^*-algebraic properties for graph algebras. First, we determine which graphs yield *continuous-trace* graph algebras. Then we examine existing theorems determining which graphs yield *stable* graph algebras.

Part I: Continuous-trace graph algebras

Hausdorff spectrum

The set of unitary equivalence classes of irreducible representations of a C^*-algebra A forms a topological space called the *spectrum* of A, denoted by $\hat{A}$. This can be a poorly-behaved topological space.

Example

The spectrum of $B(H)$ is uncountable and non-Hausdorff.

Many people have studied various topological aspects of the spectrum. Goehle determined when a suitably nice graph E yields a graph algebra with Hausdorff spectrum.

Continous-trace C^*-algebras

If A has Hausdorff spectrum then for any point $t = [\pi]$ in the spectrum and any element $a \in A$, you can consider $a(t) = a + \ker \pi \in A/\ker \pi$. Since $\hat{A}$ is Hausdorff, this has a well-defined rank.

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Definition

Let A be a C^*-algebra with Hausdorff spectrum. Then A has *continuous trace* if for every point $t \in \hat{A}$, there is a neighborhood U of t and an element $a \in A$ such that $a(s)$ is a rank-one projection for all $s \in U$.

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The upshot of this is that continuous-trace C^*-algebras act like "locally trivial non-commutative fiber bundles." These algebras are well-studied and have nice representation theory.

Continuous-trace C^*-algebras

Example: Let X be a locally compact Hausdorff space and let $A = C_0(X, \mathcal{K})$ denote the set of all continuous functions from X to $\mathcal{K}$ which vanish at infinity. Then A has continuous trace.

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$$A = \{f : [0,1] \to M_2(\mathbb{C}) : f \text{ is continuous, } f(0) = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}\}.$$

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We characterize those graphs which yield continuous-trace graph algebras.

Groupoids

In order to determine when a graph E yields a continuous-trace graph algebra, we use groupoids. A *groupoid* G is a set along with

1. a subset $G^{(2)} \subset G \times G$ of *composable pairs*;
2. an associative operation $G^{(2)} \to G$ written $(\alpha, \beta) \to \alpha\beta$ called *composition*;
3. a map $G \to G$ written $\gamma \to \gamma^{-1}$ called *inversion* which allows cancellation on the left and right

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There is no longer any identity element in a groupoid but there are "partial identities" called units. A *unit* of G is an element u such that $u = u^2 = u^{-1}$. In general there are many units; they form the *unit space* of G, denoted by $G^{(0)}$.

Groupoids

Let $r : G \to G^{(0)}$ be given by $r(\gamma) = \gamma\gamma^{-1}$ and $s : G \to G^{(0)}$ be given by $s(g) = \gamma^{-1}\gamma$. Then r and s are referred to as the *range* and *source* maps of G.

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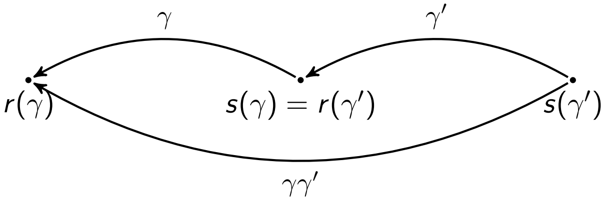

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A *topological groupoid* is a groupoid with a topology that makes the operations continuous. *Examples*: Any topological group (such as $\mathbb{R}, \mathbb{T}, \mathbb{C}, \mathbb{Z}$) is an example of a topological groupoid. Any discrete groupoid is a topological groupoid.

Definition

A topological groupoid is *étale* if the range and source maps are local homeomorphisms.

If G is étale then $r^{-1}(u)$ and $s^{-1}(u)$ are discrete for any $u \in G^{(0)}$.

Groupoids

Groupoids are interesting for many reasons, but we mostly use them to construct C^*-algebras. If G is a second countable locally compact Hausdorff étale groupoid, then we can define operations on $C_c(G)$ by

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These operations make $C_c(G)$ into a $*$-algebra. You can give $C_c(G)$ a norm by taking a supremum over certain representations into C^*-algebras. Completing yields the *groupoid C^*-algebra* $C^*(G)$.

Groupoids

If E is a directed graph then there is an affiliated *path groupoid* G_E. The elements of G_E are built out of *infinite paths*: sequences of edges $e_1e_2\dots$ with $s(e_i) = r(e_{i+1})$. The collection of such paths is denoted E^∞. There is for any integer $k \geq 0$ a *shift map* on E^∞: $\sigma^k(e_1e_2\dots) = e_{k+1}e_{k+2}\dots$.

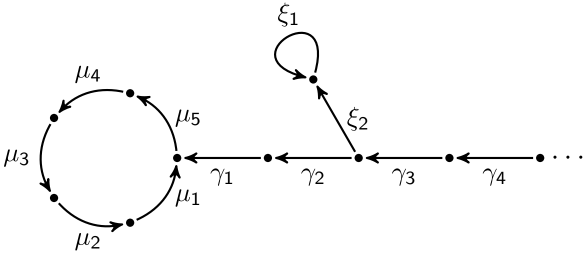
Definition

The path groupoid $G_E \subset E^\infty \times \mathbb{Z} \times E^\infty$ consists of all triples (x, n, y) such that there exist p, q with $\sigma^p x = \sigma^q y$ and $p - q = n$.

The unit space of G_E is identified with E^∞.

Groupoids

Let E be the graph



If $x = \mu_1\mu_2\mu_3\mu_4\mu_5\gamma_1\gamma_2\gamma_3\ldots$ and $y = \xi_1\xi_2\gamma_3\ldots$, then the triple $(x, 5, y)$ belongs to G_E because $\sigma^7 x = \sigma^2 y$.

Groupoids

The path groupoid carries a natural topology with basis consisting of all sets of the form

$$Z(\alpha, \beta) = \{(\alpha z, |\alpha| - |\beta|, \beta z) : \alpha, \beta \in E^*, r(z) = s(\alpha) = s(\beta)\},$$

where E^* denotes the finite path space. This topology makes G_E into a locally compact Hausdorff second countable étale groupoid, so we can construct its groupoid C^*-algebra.

Theorem (KPRR, '98)

If E is a row-finite graph with no sources, then there is an isomorphism $C^*(E) \to C^*(G_E)$ which carries the edge partial isometry s_e onto the characteristic function $\chi_{Z(e,s(e))} \in C_c(G_E) \subset C^*(G_E)$.

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Now we can study $C^*(E)$ by studying G_E: we look for conditions on a groupoid that yield a continuous-trace algebra, and then

Groupoids

Definition

Let G be a groupoid. If $u \in G^{(0)}$, the *stabilizer subgroup of u* is the set $G(u) = \{g \in G : r(g) = u = s(g)\}$. A groupoid is *principal* if $G(u) = \{u\}$ for each $u \in G^{(0)}$.

If G is a groupoid then there is a principal groupoid $R = \{(u,v) \in G^{(0)} \times G^{(0)} : (u,v) = (r(g), s(g)) \text{ for some } g \in G\}$ and a groupoid homomorphism $\pi : G \to R$ given by $\pi(g) = (r(g), s(g))$. We call this the *orbit groupoid* of G. If G is a nice topological groupoid then R is a topological groupoid carrying the quotient topology.

Groupoids

Any groupoid acts on its unit space via the formula

$$g \cdot s(g) = r(g).$$

We say that a topological groupoid acts *properly* on its unit space if the map

$$\Phi : G \to G^{(0)} \times G^{(0)}$$

given by $g \to (r(g), s(g))$ is proper.

Groupoids

A toplogical groupoid G has *continuously varying stabilizers* if the map $u \to G(u)$ which assigns to each unit its stabilizer subgroup is continuous. (Here the set of stabilizer subgroups is topologized with the *Fell topology*.)

Continuous-trace groupoid algebras

Now we can say when a groupoid yields a C^*-algebra with continuous trace.

Theorem (MRW, '96)

Suppose that G is a second countable locally compact Hausdorff groupoid with unit space $G^{(0)}$, abelian stabilizers, and Haar system. Then $C^*(G)$ has continuous trace if and only if

1. *the stabilizer map $u \mapsto G(u)$ is continuous, and*
2. *the orbit groupoid R acts properly on its unit space $R^{(0)} = G^{(0)}$.*

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As $C^*(G_E) \cong C^*(E)$ (when E is nice), determining which graphs yield continuous-trace graph algebras is reduced to the question of determining which graphs yield path groupoids satisfying the above conditions.

Continuous-trace graph algebras

Definition

An *entrance to a cycle* $\lambda = e_1 \dots e_n$ is an edge f with $r(f) = r(e_k)$ for some k such that $f \neq e_k$

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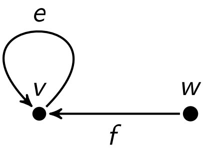


Here's a simple example of an entrance to a cycle.

Continuous-trace graph algebras

Proposition (Goehle, '13)

Let E be a row-finite graph with no sources. Then G_E has continuously varying stabilizers if and only if no cycle of E has an entrance.

Thus the only thing that remains is to find conditions on E that ensure the orbit groupoid R_E acts properly on E^∞.

Continuous-trace graph algebras

Let v, w be vertices. An *ancestry pair* is a pair of edges $(\lambda, \mu) \in E^* \times E^*$ such that

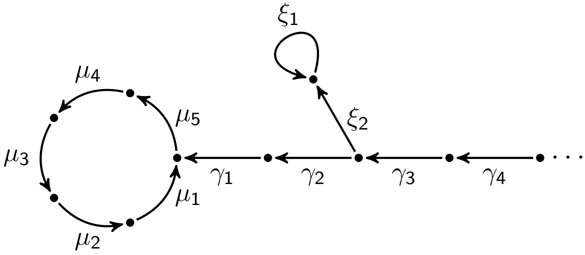
1. $r(\lambda) = v, r(\mu) = w$
2. $s(\mu) = s(\lambda)$,
3. neither path contains a cycle.

An ancestry pair is *minimal* if there is no factorization $(\lambda, \mu) = (\lambda'\nu, \mu'\nu)$ for another ancestry pair (λ', μ').

Definition

A graph has *finite ancestry* if given any two vertices v and w there are only finitely many minimal ancestry pairs for v and w.

Continuous-trace graph algebras



Here $(\gamma_1\gamma_2\gamma_3, \xi_2\gamma_3)$ is an ancestry pair which is not minimal. The ancestry pair (γ_2, ξ_2) is minimal.

Continuous-trace graph algebras

Theorem (C., '13)

Let E be a row-finite graph with no sources. Then $C^*(E)$ has continuous trace if and only if

1. *no cycle of E has an entrance, and*
2. *E has finite ancestry.*

The restriction on E allows us to use groupoid methods. Using a Drinen-Tomforde desingularization we can extend this to arbitrary graphs.

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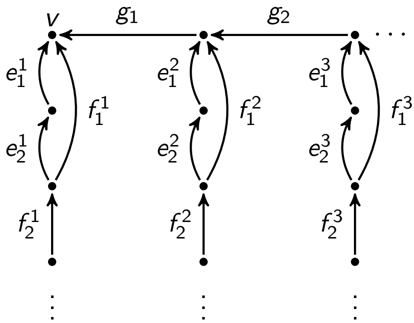
Example

Let E be the graph



It can be shown that $C^*(E)$ has Hausdorff spectrum. While E has no cycles, and hence no entrance to a cycle, it does not have finite ancestry. Thus $C^*(E)$ does not have continuous trace.

Part II: Stable graph algebras

Stability

Tensor products are common in C^*-algebras. Often you form from a C^*-algebra A its *stabilization* $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on an infinite dimensional Hilbert space.

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Definition

A C^*-algebra A is *stable* if it is isomorphic to $A \otimes \mathcal{K}$.

Stability

Example

The algebra $\mathcal{K}$ is stable because $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$.

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Example

Any stable C^*-algebra is non-commutative and non-unital, so we get a wealth of non-stable C^*-algebras: $C_0(X), B(H), \mathcal{T}$, $\mathcal{O}_2$, and others.

Stability

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Lemma

Let A be a stable C^*-algebra. Then A has no tracial states.

If I is a two-sided closed ideal in a C^*-algebra then there is a quotient C^*-algebra A/I and a canonical homomorphism $q : A \to A/I$.

Lemma

Let A be a stable C^*-algebra. Then A has no nonzero unital quotients.

Question

What conditions must a graph E satisfy in order for $C^*(E)$ to be stable?

Stability

Discussing stability of graph algebras requires some new graph theory terminology.

Definition

A *graph trace* on a directed graph E is a function $g : E^0 \to [0, \infty)$ satisfying

1. $g(v) \geq \sum_{r(e)=v} g(s(e))$ for all v
2. $g(v) = \sum_{r(e)=v} g(s(e))$ if $0 < |r^{-1}(v)| < \infty$

A graph trace is *bounded* if its norm $\sum_{v \in E^0} g(v)$ is finite. The (possibly empty) set of graph traces on E with norm 1 is denoted by $T(E)$.

Tracial states on graph algebras

Graph traces lift to tracial states.

Theorem (Tomforde '03)

If $g \in T(E)$ then there is a tracial state τ_g on $C^*(E)$ such that $\tau_g(p_v) = g(v)$.

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Stable C^*-algebras possess no tracial states. This shows that a graph with bounded graph traces cannot yield a stable C^*-algebra.

Left finite vertices

Definition

If $v, w \in E^0$, then we say that $w \leq v$ if there is a directed path from v to w. We say that v is *left finite* if

$$L(v) = \{w \in E^0 : w \leq v\}$$

is finite.

The following lemma tells us why we care about left finite vertices. Recall that a *singular vertex* receives either zero edges or infinitely many edges.

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The following lemma tells us why we care about left finite vertices. Recall that a *singular vertex* receives either zero edges or infinitely many edges.

Lemma

If E has a left-finite vertex which lies on a cycle or is singular, then $C^*(E)$ has a nonzero unital quotient.

Projection comparison

Definition

Let p, q be projections. We say that p is *subequivalent* to q if there exists an element x such that $x^*x = p$ and $xx^* \leq q$.

Usually we will be comparing different projections of the form $p = \sum_{v \in V} p_v$ for some finite subset $V \subset E^0$.

Statement of theorem

The following abridged theorem generalizes previous work of Hjelmborg [3].

Theorem (Tomforde '04)

Let E be a directed graph. Then the following are equivalent:

1. $C^*(E)$ *is stable.*
2. $C^*(E)$ *has no tracial states and no nonzero unital quotients.*
3. E *has no left finite cycles and no nonzero bounded graph traces.*
4. *For any* $v \in E^0$ *and any subset* $F \subset E^0$*, there exists* $W \subset E^0 \setminus F$ *such that* $p_v \lesssim \sum_{w \in W} p_w$.

Proof

One part of the theorem needs reproving: the implication from (4) to (5).

4. E has no left finite cycles, no left finite singular vertices, and no nonzero bounded graph traces.
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Idea of proof: Show that if we cannot construct the comparison by using the "obvious" strategy, then the graph must carry a bounded graph trace. First, let's take a look at what this "obvious" strategy might be.

Comparison of range and source

For any directed path $\lambda = e_1 e_2 \ldots e_n$ in a directed graph E, we have a partial isometry $s_\lambda = s_{e_1} s_{e_2} \ldots s_{e_n}$. The partial isometry s_λ gives a subequivalence between $p_{s(\lambda)}$ and $p_{r(\lambda)}$, as $s_\lambda^* s_\lambda = p_{s(\lambda)}$ and $s_\lambda s_\lambda^* \leq p_{r(\lambda)}$.

Lemma

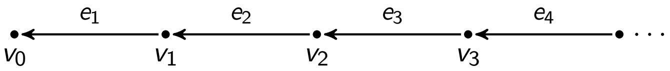
Suppose that v is a left infinite vertex and $F \subset E^0$ is a finite set. Then there exists finite $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$.

This allows us to restrict our attention to left finite vertices when we are constructing graph traces later on.

If p_v is a vertex projection and $F \subset E^0$ then a *cover for v that avoids F* is a set of vertices W with $p_v \precsim \sum_{w \in W} p_w$ and $W \cap F = \emptyset$.

Example

Let E be the graph



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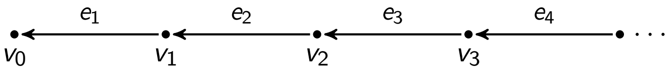

We can cover vertex v_0 and avoid any finite $F = \{v_0, v_1, \ldots, v_n\}$. For

$$p_v = s_{e_1} s_{e_1}^* \sim s_{e_1}^* s_{e_1} = p_{v_1} = s_{e_2} s_{e_2}^* \sim s_{e_2}^* s_{e_2} = p_{v_2} \sim \ldots \sim p_{v_{n+1}}.$$

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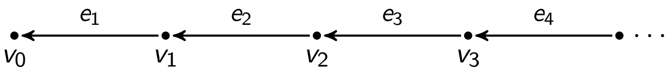

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Thus $p_{v_{n+1}}$ is a cover for p_v and we can take $W = \{v_{n+1}\}$. Notice that this graph does not carry a nonzero bounded graph trace.

Example

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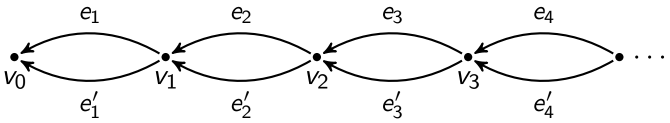

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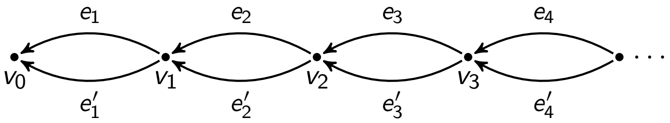

Claim: we can't cover v_0 and avoid $F = \{v_0\}$. We have $p_{v_0} = s_{e_1}s_{e_1}^* + s_{e_1'}s_{e_1'}^*$. Then $s_{e_1}s_{e_1}^* \sim s_{e_1}^*s_{e_1} = p_{v_1}$ and likewise for $s_{e_1'}s_{e_1'}^*$. However we can't write $p_v \lesssim p_{v_1} + p_{v_1}$ because the sum is not a projection. So we cover one range projection and split the other. But this lands us exactly where we started. This process goes on forever.

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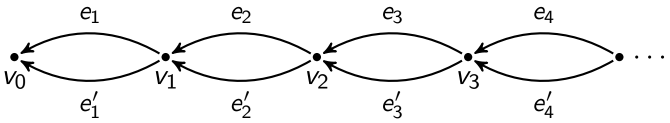

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Constructing the graph trace

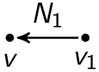
Now let's sketch the proof of

4. E has no left finite cycles, no left finite singular vertices, and no nonzero bounded graph traces.
5. For any $v \in E^0$ and any subset $F \subset E^0$, there exists $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$.

Suppose that v is a regular vertex of E and F is a finite subset of E^0 such that for all $W \subset E^0$, we have $p_v \not\lesssim \sum_{w \in W} p_w$.

Constructing the graph trace

Assume that all N_1 edges entering v have common source

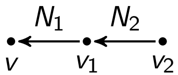


Then $p_v = \sum_{r(e)=v} s_e s_e^*$. Let d_1 be the number of paths $\lambda_1, \ldots, \lambda_{d_1}$ which start at v_1 and terminate at a vertex not in F. If $d_1 \geq N_1$, then we can write $p_v \lesssim \sum_{w \in r(\{\lambda_i\})} p_w$.

Thus we must have $d_1 < N_1$, or equivalently $\frac{d_1}{N_1} < 1$.

Constructing the graph trace, part II

Now assume we couldn't find a comparison using edges going into v.



Let N_2 be the number of edges from v_2 to v_1, and let d_2 be the number of paths which start at v_2, don't include the N_1 edges from v_1 to v, and don't terminate in F. If $d_2 \geq N_2(N_1 - d_1)$, then we can construct the comparison. So we must have that $d_2 < N_2(N_1 - d_1)$, or equivalently that $\frac{d_1}{N_1} + \frac{d_2}{N_1 N_2} < 1$.

Definition of the graph trace

Inductively we find a chain of vertices $v, v_1, \dots$ with N_i vertices from v_i to v_{i-1}, and d_i paths out of v_i which do not terminate at a vertex in F. The nice thing about this chain is

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If $w \in E^0$, define

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You can check that this is a graph trace.

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You can check that this is a graph trace. Bounded? Need to worry about the paths which terminate in the finite set F, but they just multiply the trace norm by a finite constant.

Stability of graph algebras

Thus we have seen that the failure of comparison within a C^*-algebra associated to a graph with left infinite cycles and singular vertices yields a nonzero graph trace on the graph, and hence a tracial state on the C^*-algebra. This seals the gap in the theorem on stability for graph algebras.

Stable k-graph algebras

Directed graphs can be generalized to more combinatorially rich objects called *k-graphs*.

Definition

A *k-graph* Λ is a category equipped with a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the *factorization property*: if $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there is a unique factorization of λ as $\lambda = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$. The objects of Λ are precisely $d^{-1}(0) = \Lambda^0$. In general if $n \in \mathbb{N}^k$, then Λ^n denotes $d^{-1}(n)$.

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We can assign a C^*-algebra to a well-behaved k-graph in a manner very similar to the definition of graph algebras. It then becomes interesting to ask which k-graphs yield stable C^*-algebras.

Stable k-graph algebras

I wanted to look at a class of k-graphs which is amenable to the construction of k-graph traces developed by Evans, Rennie and Sims.

Definition

A k-graph is *balanced* if for any basis elements $e_i, e_k \in \mathbb{N}^k$ we have $|v\Lambda^{e_i}w| = |v\Lambda^{e_k}w|$.

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Theorem (work in progress)

Let Λ be a row-finite balanced k-graph with no sources. Then the following are equivalent.

1. *$C^*(\Lambda)$ is stable;*
2. *$C^*(\Lambda)$ has no tracial states and no nonzero unital quotients;*
3. *no left finite $v \in \Lambda^0$ lies on a cycle and Λ has no nonzero bounded k-graph traces.*

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Thank you!