Math 22 Fall 2013 Midterm Exam I
Tuesday, October 9, 2013

PRINT NAME: ───────────────────────────

INSTRUCTIONS: READ CAREFULLY!

This is a closed book, closed notes exam. Use of calculators is not permitted. You have two hours, do all problems.

On all **free response** questions below you must show your step-by-step work and make sure it is clear *how* you arrived at your solution. Whenever you answer a question, don't just say "Yes" or "No", but always justify your answers. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** Guessing is allowed, and no points are subtracted for wrong answers.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

FREE RESPONSE SECTION

Show your step-by-step work. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

1. (a) [5 pt] Solve the following system of equations:

$$\begin{array}{ccccccccc}
2x_1 & + & x_2 & & & - & x_4 & = & 5 \\
4x_1 & + & 2x_2 & + & 3x_3 & & & = & 11 \\
2x_1 & + & x_2 & - & 15x_3 & - & 11x_4 & = & 0
\end{array}$$

(b) [3 pt] Describe the solution set in *parametric vector form*.

2. In items (a), (b), (c) below A is the 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 2 & 3 \\ 2 & 6 & 0 \end{pmatrix}$$

(a) [5 pt] Describe the solution set of the matrix equation $A\mathbf{x} = \mathbf{0}$ as a *Span* of vectors.

(b) [3 pt] Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for every possible vector $\mathbf{b}$ in $\mathbb{R}^3$?

(c) [3 pt] If the equation $A\mathbf{x} = \mathbf{b}$ is consistent (for a specific choice of $\mathbf{b}$), is the solution set a point, a line, a plane, or all of $\mathbb{R}^3$? Explain your answer.

3. Consider the following vectors in $\mathbb{R}^3$,

$$\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \mathbf{v_1} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(a) [5 pt] Is the vector $\mathbf{b}$ a linear combination of the three vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$?

(b) [3 pt] Do the three vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ span $\mathbb{R}^3$? Explain your answer.

4. [5 pt] Determine if the three vectors **a**, **b**, **c** are linearly independent. Explain your answer.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

5. (a) [5 pt] Write the plane $5x + 3y + z = 3$ in $\mathbb{R}^3$ in parametric vector form $\mathbf{x} = \mathbf{p} + t\mathbf{u} + s\mathbf{v}$.
    Hint. Treat the equation as if it is a linear system with only one row.

    (b) [3 pt] The set $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is a plane in $\mathbb{R}^3$ as well. What is the *geometric* relation between the plane $5x + 3y + z = 3$ and $\text{Span}\{\mathbf{u}, \mathbf{v}\}$?

6. [5 pt] Consider the 2×2 matrix A,

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

Describe the set of all vectors $\mathbf{b}$ in $\mathbb{R}^2$ for which the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

MULTIPLE CHOICE SECTION

Write your answer choice in the box that is provided. For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** *Guessing is allowed, and no points are subtracted for wrong answers.*

7. [5 pt] What is the standard matrix of the vertical shear depicted below?

$(A) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $(B) \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $(C) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$(D) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $(E) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $(F) \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

ANSWER

8. [5 pt] What is the *reduced echelon form* of the matrix A below?

$$A = \begin{pmatrix} 0 & 2 & 0 & -2 \\ 1 & 2 & 3 & 0 \\ -1 & 0 & -3 & -1 \\ 2 & 7 & 6 & 0 \end{pmatrix}$$

(A) $\begin{pmatrix} 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(B) $\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(C) $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
(E) $\begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(F) $\begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

ANSWER

9. [5 pt] Consider the 4×4 matrix
$$\begin{pmatrix} 0 & 6 & 8 & 0 \\ 2 & 3 & -4 & 5 \\ 2 & 3 & -4 & 0 \\ 4 & 6 & -8 & 0 \end{pmatrix}$$

Which of the following pictures has dots that correctly indicate the pivot positions of this matrix?

Hint. The calculations simplify if you use the "interchange" row operation wisely.

(A)
$$\begin{pmatrix} \bullet & * & * & * \\ * & * & * & \bullet \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

(B)
$$\begin{pmatrix} \bullet & * & * & * \\ * & * & \bullet & * \\ * & * & * & \bullet \\ * & * & * & * \end{pmatrix}$$

(C)
$$\begin{pmatrix} \bullet & * & * & * \\ * & \bullet & * & * \\ * & * & * & \bullet \\ * & * & * & * \end{pmatrix}$$

(D)
$$\begin{pmatrix} \bullet & * & * & * \\ * & \bullet & * & * \\ * & * & \bullet & * \\ * & * & * & \bullet \end{pmatrix}$$

(E)
$$\begin{pmatrix} \bullet & * & * & * \\ * & \bullet & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

(F)
$$\begin{pmatrix} \bullet & * & * & * \\ * & \bullet & \bullet & * \\ * & * & * & \bullet \\ * & * & * & * \end{pmatrix}$$

ANSWER

10. [5 pt] Which of the following sets of vectors is linearly independent?

(A) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (B) $\begin{pmatrix} 4 \\ -2 \\ -6 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ 9 \end{pmatrix}$ (C) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

(D) $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (E) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ (E) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

ANSWER

TRUE or FALSE?

For each of the statements below indicate whether it is true or false. You do not need to justify your answers. You get one point for each correct answer. Guessing will not hurt you: No points are subtracted for wrong answers.

11. [10 pt]

(a) **True / False**   If the $m \times n$ matrix A has a pivot in every column then the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for every possible choice of $\mathbf{b}$ in $\mathbb{R}^m$.

(b) **True / False**   If the matrix A has a pivot in every row then the columns of A are linearly independent.

(c) **True / False**   If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$ then A has a pivot in every row.

(d) **True / False**   A system of 5 linear equations in 7 unknowns $x_1, x_2, \ldots, x_7$ never has a unique solution.

(e) **True / False**   A system of 5 linear equations in 7 unknowns $x_1, x_2, \ldots, x_7$ is always consistent.

(f) **True / False**   The vector $\mathbf{v} - \mathbf{u}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.

(g) **True / False**   If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$ then $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is always a plane in $\mathbb{R}^3$.

(h) **True / False**   A set of 5 vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ in $\mathbb{R}^4$ is always linearly dependent.

(i) **True / False**   The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ is *onto* if the columns of the matrix A span $\mathbb{R}^m$.

(j) **True / False**   The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ is *one-to-one* if the matrix equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

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