Math 22, Summer 2013, Midterm II

Name (Print): Solutions
Last First

Instructions: You are not allowed to use calculators, books, or notes of any kind. You may not look at a classmate's exam for "inspiration." You must explain your reasoning behind each solution to receive full credit. Credit will not be awarded for correct answers with no explanation (with the exception of problem #1).

You may use pages 10-13 of the exam as scratch paper, but any work you intend to be graded should be on the exam itself in the space provided. If you run out of room, clearly indicate which page of scratch paper your solution is on and circle the solution that should be graded.

Before beginning the exam, skim through the problems to verify that you have one true/false question and five free-response questions.

Warning: Due to having different syllabi, the following topics are not exam material

- the row space of a matrix
- the vector space of polynomials
- any proof-based question (e.g question 3)

Be careful, sometimes parts of a question would still be exam material even if others aren't

Non-exam questions are denoted by NEX

Problem #	Points	Score
1	7	
2	7	
3	6	
4	9	
5	15	
6	6	
Total	50	

1. Determine whether each statement below is true or false and indicate your answer by circling the appropriate choice (1pt each):

   (a) (True / False) Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix such that $AB = O$ (where O represents the $m \times p$ zero matrix). Then, the columns of B are in Nul$A$.

   NEW (b) (True / False) Let $\mathbb{P}_n$ denote the vector space of polynomials $p(x)$ of degree at most n. The set of all polynomials in $\mathbb{P}_n$ with $p(0) = 1$ is not a subspace of $\mathbb{P}_n$.

   (c) (True / False) Suppose A is a 5×5 matrix with exactly 3 distinct eigenvalues. Suppose further that two eigenspaces of A are 2-dimensional. ~~It is possible that~~ A is not diagonalizable.

   (d) (True / False) Suppose $B_3 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & c \end{bmatrix}$ is an echelon form of A obtained through the following series of elementary row operations: B_1 is obtained by interchanging two rows of A; B_2 is obtained from B_1 by performing a row replacement; and lastly, a scaling of each row is performed so that $B_3 = \frac{1}{5}B_2$. Then $det(A) = -5^3abc$.

   (e) (True / False) If $\mathbf{v}_1$ is an eigenvector of A corresponding to λ_1 and $\mathbf{v}_2$ is an eigenvector of A corresponding to λ_2, where λ_1 and λ_2 are distinct eigenvalues of A, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

   (f) (True / False) Any linearly independent set in a subspace H is a basis for H.

   (g) (True / False) If A is a 4×3 matrix whose null space has dimension 2, then A can have rank 2.

(a) $AB = O \Rightarrow Ab_1 = 0 \ \dots \ Ab_n = 0$ where b_i are the columns of B.

(b) If $p(x) = 0$ (the zero polynomial), then $p(0) = 0 \neq 1$, so the zero polynomial is not in the subspace $\Rightarrow$ it's not a subspace

(c) Note: It was not stated as a T/F question, so I modified it.
Because A has 3 distinct eigenvalues and two 2-dim eigenspaces we have five LI eigenvectors $\Rightarrow A$ is diagonalizable.

(d) $B_2 = 5B_3$, $B_2 = B_1$ w/ row repl, $B_1 = A$ w/ row interchange. By the determinant properties, $\det(A) = -\det(B_1) = -\det(B_2) = -5^3\det(B_3)$ and $\det(B_3) = abc$.

(e) Theorem 2 p270

(f) We need the set of vectors to span H as well

(g) $\text{rank}(A) + \dim(\text{Nul}(A)) = \#\text{ columns} = 3 \Rightarrow \text{rank}(A) = 3 - 2 = 1$.

2. Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 3 & 7 & 10 & 1 \end{bmatrix}$. Determine a basis for the following subspaces:

(a) Col$A$ (3pts)

$$A \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & 1 & -2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{Col}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} \right\}$ (remember to pick the columns of A, not of the echelon form)

(b) Row$A$ (2pts)

NEX

$\text{Row}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

(c) Nul$A$ (2pts)

$$X = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{v_1} + x_4 \underbrace{\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{v_2}$$

$\text{Nul}(A) = \text{Span}\{v_1, v_2\}$

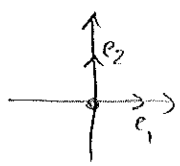
NEX

3. Suppose $H = Span\{\mathrm{e}_1\}$, $K = Span\{\mathrm{e}_2\}$, where $\{\mathrm{e}_1, \mathrm{e}_2\}$ is the standard basis for $\mathbb{R}^2$.

(a) Explain why H and K are subspaces of $\mathbb{R}^2$. (2pts)

They're subspaces b/c they're the span of some vectors

(b) Is the intersection of H and K ($H \cap K$) a subspace of $\mathbb{R}^2$? Explain. (2pts)



$Span\{e_1\} =$ x-axis

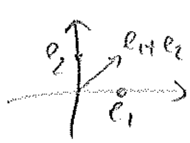
$Span\{e_2\} =$ y-axis

$H \cap K = \{0\} \Rightarrow$ It's a subspace (the trivial one)

(c) Is the union of H and K ($H \cup K$) a subspace of $\mathbb{R}^2$? Explain. (2pts)

Note: $Span\{e_1\} \cup Span\{e_2\} \neq Span\{e_1, e_2\}$

$H \cup K$ is the union of the two axes. It's not a subspace because $e_1 + e_2$ is not in it



4. Determine whether the set $\mathrm{B} = \{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_2$ (the set of all polynomials of degree at most 2), where $p_1(x) = 3x^2 + x + 1$, $p_2(x) = 2x + 1$, and $p_3(x) = 2$. Fully justify your answer. (9pts)

NEX

The best way is to use the standard basis $\mathcal{E} = \{1, t, t^2\}$ for $\mathbb{P}^2$ to write $[p_1(x)]_{\mathcal{E}}$, $[p_2(x)]_{\mathcal{E}}$ and $[p_3(x)]_{\mathcal{E}}$ and see if those vectors are LI (see Section 4.4 p 221)

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{pivot in every column} \Downarrow \text{LI}$$

Because we have three LI vectors in a space of dimension 3 they form a basis non-invertible

5. (a) Suppose that an $n \times n$ matrix A has a zero eigenvalue. Explain why A must be a singular matrix. (1pt)

0 eigenvalue $\Rightarrow$ $Ax = 0x = 0$ for some non zero x $\Rightarrow$ x is a nontrivial solution to the homogeneous system $Ax=0$ $\Rightarrow$ A is not invertible

(b) For the remaining parts of this problem, let $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. Determine the eigenvalue(s) of A. (2pts)

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{pmatrix} \qquad \det(A - \lambda I) = (1-\lambda)(-2-\lambda) + 2 =$$
$$= \lambda^2 + \lambda = \lambda(\lambda+1)$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = -1$$

(c) Determine a basis for each eigenspace of A. (4pts)

$$A - 0I = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A + I = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \left\{ \text{or } \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

(d) Explain why A is diagonalizable. (1pt)

A is a 2x2 matrix with two distinct eigenvalues

or : The dimension of each eigenspace corresponds to the algebraic multiplicity of the eigenvalue

(e) Use the fact that A is diagonalizable to calculate A^{1000}. (7pts)

$$A = PDP^{-1} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\underbrace{\begin{matrix} 2 & -1 \\ -1 & 1 \end{matrix}}_{P^{-1}}$$

$$A^{1000} = PD^{1000}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0^{1000} & 0 \\ 0 & 1^{1000} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = PDP^{-1} = A$$

6. Find a basis for $H = \left\{ \begin{bmatrix} a+2b-4c \\ -5b+15c \\ a+b-c \\ a+b+3c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$. (6pts)

Note: I'd probably just give you the expression of H as a span of vectors

$H = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 15 \\ -1 \\ 3 \end{pmatrix} \right\}$, but the three vectors might not be LI

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & -5 & 15 \\ 1 & 1 & -1 \\ 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 \\ 0 & -5 & 15 \\ 0 & -1 & 3 \\ 0 & -1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \\ 0 & -1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ pivot in every column $\Rightarrow$ vectors are LI and thus form a basis