Selected answers to suggested problems: 5.1, 5.2

5.1

2. (b) $T(3+4x) = -2(3+4x)$, $T(2+3x) = -3(2+3x)$, so both are eigenvectors (and a basis) and

$$[T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

(d) $T(x-x^2) = -4(-1-x+x^2)$, $T(-1+x^2) = -2(-1+x^2)$, $T(-1-x+x^2) = 3(x-x^2)$. This is not a basis of eigenvectors;

$$[T]_\beta = \begin{pmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{pmatrix}.$$

(f) $T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$, $T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, and $T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, so this is a basis of eigenvectors, and

$$[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. (b) (i) eigenvalues 1, 2, 3

(ii) $E_1 = \left\{ \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$. $E_2 = \left\{ \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$.

$$E_3 = \left\{ \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} : x \in \mathbb{R} \right\}.$$

(iii) One possible basis is $\{(1,1,-1), (1,-1,0), (1,0,-1)\}$.

(iv) $Q = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(d) (i) eigenvalues 0, 1, 1

(ii) $E_0 = \left\{ \begin{pmatrix} \frac{1}{2}z \\ -6z \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$. $E_1 = \left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} : y \in \mathbb{R} \right\}$.

(iii) Not possible; E_1 is not of large enough dimension.

4. (c) Matrix representation of T in terms of the standard ordered basis is

$$\begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{pmatrix}.$$

Determinant of $[T] - tI$ is $2 + 3t - t^3$; eigenvalues $2, -1, -1$.

$$E_2 = \left\{ \begin{pmatrix} \frac{1}{2}z \\ z \\ z \end{pmatrix} : z \in \mathbb{R} \right\}. \quad E_{-1} = \left\{ \begin{pmatrix} y + 2z \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\}.$$

One possible basis: $\{-\frac{1}{2}, 1, 1), (1, 1, 0), (2, 0, 1)\}$.

(d) Matrix representation of T in terms of standard ordered basis ($\{1, x\}$) is

$$\begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}.$$

Determinant of $[T] - tI$ is $6 + 5t + t^2$; eigenvalues $-2, -3$.

$$E_{-2} = \left\{ \begin{pmatrix} 2y \\ y \end{pmatrix} : y \in \mathbb{R} \right\}. \quad E_{-3} = \left\{ \begin{pmatrix} 3a \\ 2a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

One possible basis: $\{2 + x, 3 + 2x\}$.

(e) Matrix representation of T in terms of standard ordered basis ($\{1, x, x^2\}$) is

$$\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

Determinant of $[T] - tI$ is $-8t + 6t^2 - t^3$; eigenvalues 0, 2, 4.

$$E_0 = \left\{ \begin{pmatrix} -3y \\ y \\ 0 \end{pmatrix} : y \in \mathbb{R} \right\}. \quad E_2 = \left\{ \begin{pmatrix} -3a \\ -13a \\ 4a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

$$E_4 = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

One possible basis: $\{-3+x, -3-13x+4x^2, 1+x\}$.

(g) Matrix representation of T in terms of standard ordered basis ($\{1, x, x^2, x^3\}$) is

$$\begin{pmatrix} -1 & -2 & -2 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Eigenvalues can be read off: $-1, 1, 2, 3$.

$$E_{-1} = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}. \quad E_1 = \left\{ \begin{pmatrix} x \\ -x \\ 0 \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

$$E_2 = \left\{ \begin{pmatrix} -2a \\ 0 \\ 3a \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}. \quad E_3 = \left\{ \begin{pmatrix} -7a \\ 6a \\ 0 \\ 2a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

One possible basis: $\{1, 1-x, -2+3x^2, -7+6x+2x^2\}$.

10. (a) λI_V is the linear transformation which takes every vector to the λ-multiple of itself, in particular the vectors of any basis β.

(b) If V is of dimension n, the characteristic polynomial of λI_V is $(\lambda - t)^n$.

(c) Since the characteristic polynomial of λI_V has only one distinct root, λI_V has only one eigenvalue. It is diagonalizable because any basis of V is a basis of eigenvectors for λI_V.

11. (a) To be similar to λI is to be $B^{-1}\lambda I B$ for some invertible B. However, $B^{-1}\lambda I B = B^{-1}(\lambda B) = \lambda(B^{-1}B) = \lambda I$, so every matrix similar to λI is λI.

(b) If a matrix is diagonalizable, it is similar to a diagonal matrix which has the matrix's eigenvalues as its diagonal entries. If the matrix has only one eigenvalue, the diagonal matrix is a scalar matrix, so by part (a) the original matrix must already be a scalar matrix.

(c) The given matrix has characteristic polynomial $(1-t)^2$, so it has only one eigenvalue. However, it is not a scalar matrix, so by part (b) it is not diagonalizable.

14. Since subtracting tI affects only the diagonal elements, which are the same between A and A^t, we have $\det(A - tI) = \det((A - tI)^t) = \det(A^t - tI)$, so they have the same characteristic polynomial.

15. (a) For any m, $T^m(x) = T^{m-1}T(x) = T^{m-1}(\lambda x) = \lambda T^{m-1}(x)$, so by induction we get $T^m(x) = \lambda^m x$.

(b) The eigenvalues of the powers of a matrix A are the corresponding powers of the eigenvalues of A.

17. (a) T leaves the diagonal of a matrix unchanged, so on matrices with nonzero diagonal elements, it can clearly have no other eigenvalue than 1. This is indeed an eigenvalue, with eigenvectors the symmetric matrices.

If the diagonal of a matrix is all zero and the matrix is not symmetric, for λ to be an eigenvalue of T we need $a_{ij} = \lambda a_{ji}$ for all $i \neq j$. To have $a_{ij} = \lambda a_{ji}$ as well as $a_{ji} = \lambda a_{ij}$ requires $\lambda = \pm 1$. We have seen 1 is an eigenvalue. -1 is as well, with eigenvectors the matrices with zero along the diagonal and all other symmetric pairs of entries different by a factor of -1.

(b) done

(c) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

(first three go with 1, last with -1.)

(d) n matrices with a single 1 somewhere along the diagonal and other entries 0, $\frac{1}{2}(n^2 - n)$ matrices with a pair of symmetric 1s and 0s elsewhere, $\frac{1}{2}(n^2 - n)$ matrices with a pair 1, -1 in symmetric positions and 0s elsewhere.

19. This is essentially working backwards through section 2.5. If A and B are similar, there is some invertible Q such that $Q^{-1}AQ = B$. Viewing A as $[T]_\beta$ for β the standard basis we can consider Q to be a change-of-coordinate matrix from the columns of Q, which we call γ, to β. With that interpretation $B = [T]_\gamma$.

22. $g(T)$ means plug T into g; for example, if $g(u) = 2u^2 + u + 5$ the resulting linear transformation is $2T^2 + T + 5I_V$, where T^2 means T composed with itself and I_V is T^0. We will show that the result holds for the powers of u and then that it is preserved under addition and scalar multiplication.

If $g(u) = 1$, then $g(T) = I_V$ and $g(\lambda) = 1$, so $g(T)(x) = I_V(x) = x = 1x = g(\lambda)x$. If $g(u) = u^n$ for some $n > 0$, then $g(\lambda) = \lambda^n$ and $g(T) = T^n$. By exercise 15, $T^n(x) = \lambda^n x$, so $g(T)(x) = g(\lambda)x$.

Now suppose the result holds for $g(u), h(u)$ and consider $g + h$. $(g + h)(T)(x) = (g(T) + h(T))(x) = g(T)(x) + h(T)(x) = g(\lambda)x + h(\lambda)x = (g(\lambda) + h(\lambda))x = (g + h)(\lambda)x$.

Finally consider $h = cg$, where the result holds for g. $h(T)(x) = cg(T)(x) = cg(\lambda)x = h(\lambda)x$.

5.2

2. (b) $Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$.

(d) $Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & \frac{3}{2} \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(f) Not diagonalizable - eigenvalues 1, 1, 3; E_1 has dimension 1.

3. (b) $\{1 - x^2, 1 + x^2, x\}$ (first with eigenvalue -1, last two with 1)

(e) $\{(1, 1), (1, -1)\}$ (first with eigenvalue $1 + i$, second with $1 - i$)

(f) $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$
(first three with eigenvalue 1, last with -1)

8. The dimension of any eigenspace is at least 1, by definition of eigenvalue. Therefore since $\dim(E_{\lambda_1}) = n - 1$ and $\dim(E_{\lambda_2}) \geq 1$ (in fact it must be equal to 1), A is diagonalizable.

9. (a) The characteristic polynomial of an $n \times n$ upper triangular matrix with diagonal elements $a_1, a_2, \ldots, a_n$ is $(a_1 - t)(a_2 - t) \ldots (a_n - t)$, and T has the same characteristic polynomial no matter what basis its matrix representation is with respect to. Therefore the characteristic polynomial for T splits.

(b) If A is similar to an upper triangular matrix, then the characteristic polynomial of A splits.

10. Again, the key is that the characteristic polynomial is the same no matter what the basis is, and that an upper triangular matrix gives a characteristic polynomial of $(a_1 - t)(a_2 - t) \ldots (a_n - t)$ where a_i are the diagonal entries. Therefore, since we know the characteristic polynomial is $(\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2} \ldots (\lambda_k - t)^{m_k}$, we know the entries of the diagonal are m_i copies each of λ_i.

13. (a) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, a matrix over $\mathbb{R}$. Its eigenvalues are 1 and 2. A's eigenspaces are $E_1 = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R} \right\}$ and $E_2 = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{R} \right\}$.

A^t's eigenspaces, on the other hand, are $E_1' = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : x \in \mathbb{R} \right\}$ and $E_2' = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R} \right\}$.

(b) The dimension of E_λ is $n - \text{rank}(A - \lambda I)$, and that of E'_λ is $n - \text{rank}(A^t - \lambda I)$. As in 5.1#14, $(A - \lambda I)^t = (A^t - \lambda I)$, so they have the same rank.

(c) Follows immediately from (b), since A diagonalizable means A has eigenvalues all of which have eigenspace dimension equal to their multiplicities. Since A^t has the same eigenvalues with the same multiplicities as A, its eigenspaces also have maximal dimension.