Mathematics 33

Homework Assignment #1

Due Wednesday, April 5

1. (5 points). Show that the (short) distance from a point to a straight line is the length of the perpendicular. Use vector notation.

Solution. Let $\mathbf{x} \in R^2$ and the straight line be defined as $\mathbf{a}+\lambda\mathbf{b}$, where $\mathbf{a} \in R^2$ is the translation vector and $\mathbf{b} \in R^2$ is the direction vector. We need to find λ_* such that $\|\mathbf{x}-(\mathbf{a}+\lambda_*\mathbf{b})\|^2 = \min$. The squared norm can be rewritten as

$$\begin{aligned}\|\mathbf{x}-(\mathbf{a}+\lambda\mathbf{b})\|^2 &= \|(\mathbf{x}-\mathbf{a})-\lambda\mathbf{b})\|^2 = \|\mathbf{x}-\mathbf{a}\|^2 - 2\lambda(\mathbf{x}-\mathbf{a},\mathbf{b}) + \|\lambda\mathbf{b}\|^2 \\ &= \|\mathbf{x}-\mathbf{a}\|^2 - 2\lambda(\mathbf{x}-\mathbf{a},\mathbf{b}) + \lambda^2\|\mathbf{b}\|^2\end{aligned}$$

where $(\mathbf{x}-\mathbf{a},\mathbf{b})$ is the scalar (dot) product. The squared norm is a quadratic function of λ. It takes minimum if and only if its gradient vanishes, $-2(\mathbf{x}-\mathbf{a},\mathbf{b})+2\lambda\|\mathbf{b}\|^2 = 0$. Let λ_* be the solution to this equation, then

$$0 = (\mathbf{x}-\mathbf{a},\mathbf{b}) - \lambda_*(\mathbf{b},\mathbf{b}) = (\mathbf{x}-\mathbf{a}-\lambda_*\mathbf{b},\mathbf{b}) = (\mathbf{x}-(\mathbf{a}+\lambda_*\mathbf{b}),\mathbf{b}).$$

This means that the vector from $\mathbf{x}$ to the line, $\mathbf{x}-(\mathbf{a}+\lambda_*\mathbf{b})$ is orthogonal to the direction vector $\mathbf{b}$. Therefore, the shortest distance is the length of the perpendicular.

2. (3 points). Let physical system be defined by n points $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$ with masses $m_1, m_2, ..., m_n$ as vectors on the plane. Where is the center of the system (centroid)?

Solution. The center, $\mathbf{x}$ is where the moment mass is zero, i.e.

$$m_1(\mathbf{x}-\mathbf{a}_1) + m_2(\mathbf{x}-\mathbf{a}_2) + ... + m_n(\mathbf{x}-\mathbf{a}_n) = \mathbf{0}.$$

This implies

$$\mathbf{x} = \frac{\sum_{i=1}^n m_i\mathbf{a}_i}{\sum_{i=1}^n m_i}.$$

3. (4 points). Derive the equation for the straight line $y = a + bx$ in polar coordinates.

Solution. In polar coordinates $x = r\cos\theta, y = r\sin\theta$. Substituting this into equation for the straight line we obtain

$$r = \frac{a}{\sin\theta - b\cos\theta}.$$

4. (6 points). After the Big Bang the universe diverges following Archimedian spiral $r = \theta$. What distance it covers after 3 light years?

Solution. In polar coordinates the Archimedian spiral takes the form $r = \theta$ where $0 \le \theta < \infty$. In Cartesian system of coordinates the Archimedian spiral takes the form $x(\theta) = \theta\cos\theta, y(\theta) = \theta\sin\theta$. The length along the curve from 0 to 3 is calculated as

$$\int_0^3 \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

But

$$\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta, \frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$$

so that we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2 \\ &= \cos^2\theta - 2\theta\sin\theta\cos\theta + \theta^2\sin^2\theta + \sin^2\theta + 2\theta\sin\theta\cos\theta + \theta^2\cos^2\theta \\ &= (\cos^2\theta + \sin^2\theta) + \theta^2(\cos^2\theta + \sin^2\theta) = 1 + \theta^2.\end{aligned}$$

Thus, the distance is

$$\int_0^3 \sqrt{1+\theta^2}d\theta = \frac{3}{2}\sqrt{10} + \frac{1}{2}\operatorname{arcsinh} 3 = 5.6526.$$

Thus, over 3 light years after explosion the universe covers the distance of 5.65 light years.

5. (7 points). A body is thrown from certain height with certain initial speed. Is it true that the minimum velocity attains at the maximum height? Is it true for any other trajectory, e.g. $10 - 5t(t+1)(t-5)$?

Solution. The trajectory of a free fall body is specified by a quadratic equation $S(t) = a + bt - ct^2$ where $c = g/2$ and g is the gravity acceleration. The squared velocity is $1 + (dS/dt)^2$ which takes minimum when $dS/dt = 0$. But this corresponds to the maximum height. Therefore, for a free fall trajectory minimum velocity attains at maximum height. It is not true for *any* trajectory because a force may be involved: Imagine a plane which accelerates the speed but flies horizontally.