In these problems, you will need to use

$$\mathcal{F}(e^{-ax^2})(s) = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2}{a}s^2} = \mathcal{F}^{-1}(e^{-ax^2})(s).$$

11. Compute

$$e^{-ax^2} * e^{-bx^2}$$

where a and b are positive constants. Use the Fourier transform and its inverse.

12. In class we derived the solution $u(x,t)$ given in exercise 1.20, p.72, to the heat conduction problem for an infinite rod with initial temperature function $f = \mathcal{F}^{-1}A$. Now write the solution $u(x,t)$ as a convolution of f with a gaussian function. Briefly say how the gaussian changes with time.

In the following problems, you may want to express some of your work in terms of the heat kernel

$$K_t(x) = \frac{1}{\sqrt{4\pi\alpha^2 t}} e^{-x^2/4\alpha^2 t}.$$

13. In class we used the Fourier sine transform to solve the heat conduction problem

$$\alpha^2 u_{xx} = u_t, \quad x > 0, \quad t > 0 \tag{1}$$
$$u(0,t) = 0, \qquad t \geq 0 \tag{2}$$
$$u(x,0) = f(x), \quad x \geq 0 \tag{3}$$

for a semi-infinite rod, where f was the initial temperature function of the rod. The expression for the solution $u(x,t)$ was complicated and hard to interpret. Find a simpler form of the solution by following these steps:

(a) Solve the usual (boundaryless) heat conduction problem

$$\alpha^2 u_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0 \tag{4}$$
$$u(x,0) = g(x), \quad -\infty < x < \infty, \tag{5}$$

where g is the odd extension of f. Express the solution $u(x,t)$ as a convolution.

(b) Show that the solution $u(x,t)$ you found in part (a) satisfies the boundary condition $u(0,t) = 0$ for $t \geq 0$. This is condition (2). The function $u(x,t)$ automatically satisfies (1) and (3) since it satisfies (4) and (5).

14. (Heat-conduction with nonconstant coefficients) Find the solution to the partial differential equation

$$t u_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0$$

which satisfies the initial condition

$$u(x,0) = f(x), \quad -\infty < x < \infty.$$

Write the solution as a convolusion in as simple a form as possible.

15. We have seen that the solution to the usual heat conduction problem

$$\alpha^2 u_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x), \quad -\infty < x < \infty$$

is $u(x,t) = f * K_t(x)$, with K_t as just after problem 12. Show that if $t_0 > 0$ then the function $h(x) = u(x, t_0)$ has derivitives of all orders, even if f is not differentiable or even continuous. (Compare this with problem 4.)

16. Show that the function T defined on test functions by $T(\varphi) = 3\varphi''(2)$ or $\langle T, \varphi\rangle = 3\varphi''(2)$ is linear and so defines a distribution.

17. Does the formula $\langle T, \varphi\rangle = (\varphi(0))^2$ define a distribution. Why or why not?

18. Find, in the distribution sense, the first three derivatives of the function $f(x) = |x|$.

19. Which of these functions is rapidly decreasing? Which is a Schwartz function? Explain briefly.

(a) $\operatorname{sinc} x$ (b) $e^{-x^2} \sin x$ (c) $e^{-x^2} \sin(e^{x^2})$ (d) $e^{-2x^2} \sin(e^{x^2})$

20. (optional) In class we saw that the solution to the inhomogenous heat equation with homogeneous initial conditions

$$\alpha^2 u_{xx} + f(x,t) = u_t, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = 0, \quad -\infty < x < \infty$$

can be written as

$$u(x,t) = \int_0^t K_{t-w} * f(\cdot, w)(x)\, dw$$

where K_t is the usual heat kernel as in problem 12. The term $f(x,t)$ in the PDE may be interpreted as the rate (per unit time per unit length) at which heat is added to the wire at time t and position x. With this interpretation, what is the total amount of heat $\int_{-\infty}^{\infty} u(x, t_0)\, dx$ in the wire at time t_0 in terms of f? Check that this agrees with the value of $\int_{-\infty}^{\infty} u(x, t_0)\, dx$ if this integral is done using $u(x,t)$ as written above.

In the next two problems, $H(x)$ is the heaviside function.

21. Express these distributions in as simple a form as possible.

(a) $(H(x)\cos x)'$ (b) $(\delta \sin 2x)'$ (c) $\delta' \sin 2x$ (d) $x^2\delta'$ (e) $x^n\delta^{(n)}$

22.

(a) Find a distribution $F = f \cdot H$, where f is a twice continuously differentiable function which satisfies, in the distribution sense, the differential equation

$$F'' + 4F = \delta.$$

(b) Do the same for the equation

$$F'' - 4F = 2\delta - \delta'.$$