Math 38: Graph Theory

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Dartmouth College

7.1.26. Let G be a regular graph with cut-vertex. We will show that $\chi'(G) > \Delta(G)$.

Suppose that G is k-regular and that vertex $v \in V(G)$ is a cut-vertex. Furthermore, assume that we have a proper k-edge-coloring of $E(G)$ and thus $\chi'(G) = k = \Delta(G)$.

Vertex v has k edges incident with it. These edges are colored 1 through k. Suppose that the edge colored i has endpoints v and v_i. We will show that for all $1 \leq i < j \leq k$, vertices v_i and v_j are connected by a path in $G - v$.

Starting with vertex v_i, find a maximum walk whose edges are alternately colored i and j. This path must begin with color j, since v_i does not have an edge incident to it colored i in $G - v$. The walk can never return to a previously encountered vertex, otherwise we would have two edges colored the same that are incident to the same vertex, which would contradict the assumption that our edge-coloring is proper. And finally, each edge colored j leads us to a vertex that is incident with an edge colored i. This is due to the fact that we started at v_i, the only vertex that is not incident with an edge colored i, and we know we can never return to vertex v_i. Therefore, our walk ends when we reach a vertex that does not have an edge incident with it that is colored j. But v_j is the only such vertex. Thus our maximum walk must be a v_i, v_j-walk.

Therefore, any vertex in the component of $G - v$ containing v_i is connected to any vertex in the component of $G - v$ containing v_j. Since this is true for all i and j, we conclude that $G - v$ has the same number of components as G. But this contradicts the assumption that v was a cut-vertex.

Therefore, if G is a k-regular graph with a cut-vertex then $\chi'(G) > k = \Delta(G)$, as promised.