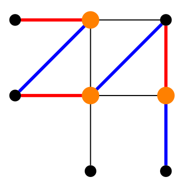
Quiz 3, Math 38, Spring 2012

Instructions:
Always (briefly) explain or demonstrate why, unless told otherwise.

(1) For the graph below,
   (a) give an optimal matching M;
   (b) give a maximum M-alternating path;
   (c) give an optimal vertex covering;
   (d) and justify the optimality of your answers to (a) and (c) by comparing those answers.

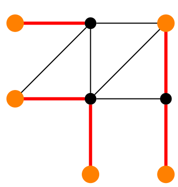
Red edges are in the matching.
Blue edges are the others in the alternating path.
Orange vertices are the cover.



Any matching and vertex cover of size three will do for (a) and (c). For (d), since those are the same size, and you have equality, you know you've optimized. For (b), I found a path which has all edges in the matching, and with one endpoint unsaturated; you can't have it any longer, since that would be an augmenting path.

(2) For the same graph again,
   (a) exhibit a maximum independent set, and list which answer from (1) guarantees this is optimal and why;
   (b) exhibit a minimum edge cover, and list which answer from (1) guarantees this is optimal and why;

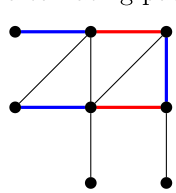
Red edges are in the cover.
Orange vertices are the indep set.



Both have size 5. You can check this against your answers to (a) and (c) from number 1: since there are 8 vertices, a minimum vertex cover of size 3 implies a maximum independent set of size $8 - 1 = 5$; similarly a maximum matching of size 3 implies a minimum edge cover of size $8 - 3 = 5$.

(3) For the same graph one more time, give an example of a maximal matching M which is not a maximum matching, and provide an M-augmenting path.

Red edges are in the matching.
Blue edges are the others in the alternating path.



Just check that the saturated vertices and their neighborhoods cover the entire vertex set, and so it is maximal. This matching has an augmenting path and so is not maximum.

(4) Give an example of a graph with the following properties (and explain why it does), or explain why it's not possible.

(a) A graph for which a minimum vertex cover has size different from that of a maximum matching. (I do mean min/maximum, not just min/maximal)

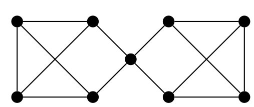
Answer. Any odd cycle will have min'm vertex covering of size one larger than the max'm matching. □

(b) A graph (with at least one edge) for which the minimum degree, the edge-connectivity, and the connectivity all have the same values.

Answer. Any complete graph K_n will have all equal to $n-1$. □

(c) A graph for which the minimum degree, the edge-connectivity, and the connectivity all have different values. (explain)

Answer. The following graph has $\delta = 3$, $\kappa' = 2$ (there are no cut-edges, but deleting two edges to one side of the middle disconnects), $\kappa = 1$ (the middle vertex is cut).



□

(5) State four ways to characterize 2-connected graphs.
(No need to prove, just state them.)

For all, assume $n(G) \geq 3$.
1. Connected with no cut-vertex.
2. Every pair of vertices have two internally disjoint paths.
3. Every pair of vertices share a common cycle.
4. Every pair of edges share a common cycle, and $\delta(G) \geq 1$.