Math 42, Winter 2017

Homework set 2, due Wed Jan 18

Collaboration and discussion of the problems is permitted, and even recommended. But you should write up and hand in your own solutions independently.

1. Prove that the curvature of curve in $\mathbb{R}^n$ is zero at every point if and only if it is a straight line.

2. If an ellipse in $\mathbb{R}^2$ has circumference C, prove that there is at least one point on the ellipse where the curvature κ equals $2\pi/C$.

3. (a) Prove that the signed curvature κ_s of a regular curve γ in $\mathbb{R}^2$ is a smooth function.

   (b) Prove that there exists a regular curve γ in $\mathbb{R}^2$ for which the curvature κ (without sign) is *not* a smooth function. (*Hint. Use the fact that* $\kappa = |\kappa_s|$.)

4. A circle in $\mathbb{R}^2$ is defined as the set of points (x, y) satisfying

$$(x-a)^2 + (y-b)^2 = r^2$$

   for given $a, b, r \in \mathbb{R}$ with $r > 0$. Prove formally that an isometry of $\mathbb{R}^2$ sends circles to circles.

5. Recall from linear algebra: If A is an $n \times n$ matrix with real coefficients, then λ is an eigenvalue of A if there exists a non-zero vector $v \in \mathbb{R}^n$ with $Av = \lambda v$.

   Proposition 1 *The real eigenvalues of an* $n \times n$ *real matrix* A *are the real roots* λ *of the characteristic equation* $\det(A - \lambda I) = 0$.

   Proof. If $Av = \lambda v$ then $(A - \lambda I)v = 0$, while $v \neq 0$. So the matrix $A - \lambda I$ is singular (i.e. not invertible), and it follows that $\det(A - \lambda I) = 0$. Conversely, if $\det(A - \lambda I) = 0$ then $A - \lambda I$ is singular, and there exists a non-trivial solution of the vector equation $(A - \lambda I)v = 0$. Then $Av = \lambda v$ with $v \neq 0$, and so λ is an eigenvalue of A.

   (a) If A is an $n \times n$ matrix, what is the degree of the characteristic equation? Explain.

   (b) Prove that a 3×3 matrix with real coefficients has at least one real eigenvalue.

   (c) Prove that at least one of the eigenvalues of a 3×3 orthogonal matrix must be ± 1. (*Hint. Consider the equation* $\|Av\| = \|\lambda v\|$.)

   (d) Consider the isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = Av$ for an orthogonal matrix A. Assume that one of the eigenvalues of A is 1. Prove that there exists a line l in $\mathbb{R}^3$ all of whose points are fixed by T. (In other words, T is a rotation of $\mathbb{R}^3$ around the axis l).

   (e) If $+1$ is not an eigenvalue of the orthogonal matrix A, give a geometric description of the type of isometry represented by $T : \mathbb{R}^3 \to \mathbb{R}^3$.