Math 42, Winter 2017

Homework set 4, due Wed Feb 1

Collaboration and discussion of the problems is permitted, and even recommended. But you should write up and hand in your own solutions independently.

1. Consider the quadratic surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = ax^2 + 2bxy + cy^2\}$, where a, b, c are three constants.

   (a) Find a general formula for the two principal curvatures κ_1, κ_2 (as defined by Euler) of S at the origin $(0, 0, 0)$.

   (b) Find a general formula for the Gaussian curvature $\kappa = \kappa_1\kappa_2$ of S at $(0, 0, 0)$.

   (c) Find a general formula for the so-called *mean curvature* $(\kappa_1 + \kappa_2)/2$ of S at $(0, 0, 0)$.

2. Consider the following atlas for the unit sphere $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ consisting of two charts. The first chart $\sigma_1 : U_1 \to S \cap W_1$ has domain $U_1 = \mathbb{R}^2$ and

$$\sigma_1(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right)$$

   The second chart $\sigma_2 : U_2 \to S \cap W_2$ has domain $U_2 = \mathbb{R}^2$ and

$$\sigma_2(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, -\frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right)$$

   (The formulas for σ_1 and σ_2 differ only in the sign of the z-coordinate.) Remark: W_1 and W_2 are appropriate open subsets of $\mathbb{R}^3$ that make the maps σ_1, σ_2 bijective.

   (a) What is the image $S \cap W_1$ of the map σ_1?

   (b) What is the image $S \cap W_2$ of the map σ_2?

   (c) What is the set $S \cap W_1 \cap W_2$?

   (d) What is the set $V_1 = \sigma_1^{-1}(S \cap W_1 \cap W_2)$?

   (e) What is the set $V_2 = \sigma_2^{-1}(S \cap W_1 \cap W_2)$?

   (f) Derive an explicit formula for the transition function $\Phi : V_2 \to V_1$.
   Hint. First find a formula for $u^2 + v^2$ as a function of z.

3. Consider the atlas for the unit sphere S defined in Exercise 4.1.2 on p.75 of Pressley's book (which we also discussed in Friday's lecture). In this problem we follow the notation of Pressley's Exercise 4.1.2.

   (a) Find the point in $U \subset \mathbb{R}^2$ that corresponds to the point $P = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ on the sphere S for the chart $\sigma_+^x : U \to \mathbb{R}^3$.

   (b) Calculate the tangent vectors σ_u, σ_v of S at the point P, when σ is the chart σ_+^x.

   (c) Now find the point in $U \subset \mathbb{R}^2$ that corresponds to P, but this time for the chart σ_+^z.

(d) Calculate the tangent vectors σ_u, σ_v of S at the point P for this chart σ_+^z.

According to Proposition 4.4.2, the tangent space $T_P S$ of the sphere S at point P is the 2 dimensional vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u, σ_v for *any* coordinate patch that contains P.

(a) Prove that the span of the vectors σ_u, σ_v for the chart σ_+^x of item (b) is the same subspace of $\mathbb{R}^3$ as the span of the vectors σ_u, σ_v for the chart σ_+^z of item (d).