1 Introduction

In the oft intersecting worlds of physics and mathematics, one simple system allows for some extremely interesting analysis by hardened veterans of both fields. In basic definition and concept, the transformation of the popular game of billiards into a strictly mathematical system is elementary. However, the actual analysis of such a system proves to be a veritable obstacle course of some truly complex mathematics. The goal of this paper is to examine the formalism behind some billiards constructions as well as develop some tools to better understand their dynamics.

2 Basic Definition and Construction

In the conventional sense, a billiard game consists of a rectangular table with six pockets and several "hard" balls. What we are interested in, however, is somewhat more complicated.

Definition 1. Let $D \subset \mathbb{R}^2$ be the closure of a bounded, connected domain with smooth or piecewise-smooth boundary δD. We put a set of restrictions on δD as follows:

(i) δD is a union of a finite set of smooth, compact curves with degree of smoothness $l \geq 3$. So:

$$\delta D = \Gamma = \bigcup_{i=1}^{n} \Gamma_i$$

(ii) Γ_i can only intersect each other at their end-points.

(iii) An orientation is assigned to each Γ_i such that the billiard domain lies to the left at all times. Also, the second derivative of Γ_i (parametrized by arclength) is either zero everywhere on Γ_i or zero nowhere on Γ_i. So we break up the boundary according to where inflection points occur.

(iv) Let Γ_j and Γ_k be two boundary components where Γ_j is a 'focusing' curve and Γ_k is a 'dispersing' curve. They may NOT intersect at any point $q^* \in \Gamma_j \cap \Gamma_k$ such that the angle between their respective tangent vectors at q^* is 0. This angle is called the *interior angle.*

We call D a *billiard table* and Γ its *boundary.* The motion of the 'ball' is given by the free movement of a point in D with perfectly elastic collisions following the classical Law of Reflection.

Note: The class of billiards which have been excluded by our restrictions on the boundary as well as unbounded tables are not covered in this paper. Also, the terminology in part (iv) will be covered later.

Given restriction (iii), we are able to classify Γ_i into three separate categories. Each component curve can either be a straight line segment, a focusing (convex) curve or a dispersing (curve). This terminology merely refers to what the boundary curve would do if a set of parallel trajectories hit it. For example, a dispersing curve would increase the distance between the trajectories and a focusing one would decrease it (atleast in a small neighborhood around the set of collision points). As this classification is directly related to the second derivative restriction, we can assign a *curvature* $\mathcal{K}$ to each Γ_i.

$$\mathcal{K} = \begin{cases} 0, & \text{if line segment} \\ -||\Gamma_i''||, & \text{if focusing curve} \\ ||\Gamma_i''||, & \text{if dispersing curve} \end{cases}$$

3 Phase Space and Flow

As our billiard table is in two dimensions, we require two canonical coordinates q_1, q_2 to specify the point's location on the plane. The magnitude of the velocity is conserved at all times so we simply set it to 1. Now we can use an angle ω to fix the direction of velocity, and with q_1, q_2, these three coordinates completely define a particle's dynamics. The set of points corresponding to all allowed values of q_1, q_2, ω gives us the phase space of the billiard. Note that the phase space is 3-dimensional. We denote it using Ω.

$$\Omega = (q_1, q_2, \omega) = D \times S^1$$

We tweak Ω a little bit more to satisfy continuity. Let $\widetilde{\Gamma} \in \Gamma$ be the subset of the boundary given by removing all the intersection (corner) points. Then all boundary points (q_1, q_2, ω^-) are identified with (q_1, q_2, ω^+) where $(q_1, q_2) \in \widetilde{\Gamma}$ and ω^- and ω^+ represent precollision and postcollision velocity direction for a trajectory which collides at (q_1, q_2). This is no longer visualizable, but fortunately, we only really need to realize that we have made the flow continuous over regular collision points.

We now realize that at some points of our billiard table, the trajectory is no longer clearly defined. After all, what happens when the particle hits some type of corner or grazes the boundary? Since the 'ball' has no dimensions of its own, these problems cause much more of a headache than a true physical system. For now, let us denote by $\widetilde{\Omega} \subset \Omega$ the piece of phase space where trajectories exist for all time $t \in (-\infty, \infty)$.

We are finally in a position to define a billiard flow Φ^t.

$$\Phi^t : \widetilde{\Omega} \to \widetilde{\Omega} \text{ with } t \text{ continuous and } \in \mathbb{R}$$

Φ^t takes a point in phase space and maps it to where it will be once time 't' has passed. Every curve in phase space, therefore, corresponds to a trajectory of the flow Φ^t. The projection of the curve onto the billiard domain D simply gives us a set of lines and collision points that the moving particle would draw out.

Note: There is a family of billiards where the domain is not bounded. We can project such a billiard domain onto a 2D Torus so that $D \subset Tor^2$ by identifying the sides. But we exclude such billiard domains for our purposes.

Returning to the Flow map on the phase space, let us try to construct a more definitive description. Take a point $(q_1^-, q_2^-, \omega^-) \in \Omega$ and look at where the trajectory will end up after a time $t \in \mathbb{R}$. This new point $(q_1^+, q_2^+, \omega^+) \in \Omega$ is given by $\Phi^t(q_1^-, q_2^-, \omega^-)$. There are 2 distinct possibilities here. Either the trajectory experienced a collision during time t, or it did not. The map for no collision is simple:

$$(q_1^+, q_2^+, \omega^+) = \Phi^t(q_1^-, q_2^-, \omega^-) = (q_1^- + t\cos\omega^-, q_2^- + t\sin\omega^-, \omega^-)$$

The case where there *is* a collision, however, proves a bit more complicated. Let there be *one* regular collision at a boundary Γ_i during time t. For our convenience, and to be able to properly define the map, we introduce some new variables.

$(\overline{q_1}, \overline{q_2})$:	The collision point on the boundary.
$\mathbf{T}$:	The tangent vector to the boundary at $(\overline{q_1}, \overline{q_2})$.
s^- :	The time it takes to arrive at the collision point from (q_1^-, q_2^-)
s^+ :	The time it takes to get from $(\overline{q_1}, \overline{q_2})$ to (q_1^+, q_2^+).
γ :	The angle between $\mathbf{T}$ and the axis spanned by vector q_1
ψ :	The angle between post-collision trajectory and $\mathbf{T}$.

Now, we can explicitly state the flow map in terms of these new variables.

$$\begin{aligned} q_1^- &= \overline{q_1} - s^-\cos\omega^- & q_1^+ &= \overline{q_1} + s^+\cos\omega^+ \\ q_2^- &= \overline{q_2} - s^-\sin\omega^- & q_2^+ &= \overline{q_2} + s^+\sin\omega^+ \\ \omega^- &= \gamma - \psi & \omega^+ &= \gamma + \psi \end{aligned}$$

3.1 Measure Preserving Flow

Our goal here is to find a Lebesgue measure and check whether it is invariant under the flow map. Let r be the arc-length of Γ_i. Then we can use the following relations to eventually compute the volume form of the phase space. First, we note the following:

$$\begin{aligned} d\overline{q_1} &= \cos\gamma dr \\ d\overline{q_2} &= \sin\gamma dr \\ d\gamma &= -\mathcal{K}dr \end{aligned}$$

Note that Γ_i is always oriented such that the billiard domain D is to its left. This was a choice we made when assigning curvature. Then we differentiate the flow equations and plug in the arclength parametrizations. This will allow us to find an infinitesimal volume in phase space around the initial and final trajectory points.

$$\begin{aligned}
dq_1^- &= d\overline{q_1} - \cos\omega^- ds^- + s^- \sin\omega^- d\omega^- \\
&= \cos\gamma dr - \cos\omega^- ds^- + s^- \sin\omega^- d\omega^- \\
dq_2^- &= d\overline{q_2} - \sin\omega^- ds^- - s^- \cos\omega^- d\omega^- \\
&= \sin\gamma dr - \sin\omega^- ds^- - s^- \cos\omega^- d\omega^- \\
d\omega^- &= -d\gamma - d\psi \\
&= -\mathcal{K} dr - d\psi
\end{aligned}$$

And

$$\begin{aligned}
dq_1^+ &= d\overline{q_1} + \cos\omega^+ ds^+ - s^+ \sin\omega^+ d\omega^+ \\
&= \cos\gamma dr + \cos\omega^+ ds^+ - s^+ \sin\omega^+ d\omega^+ \\
dq_2^+ &= d\overline{q_2} + \sin\omega^+ ds^+ + s^+ \cos\omega^+ d\omega^+ \\
&= \sin\gamma dr + \sin\omega^+ ds^+ + s^+ \cos\omega^+ d\omega^+ \\
d\omega^+ &= -d\gamma + d\psi \\
&= -\mathcal{K} dr + d\psi
\end{aligned}$$

Subbing in for $d\omega$ in each of the equations, we get a set of maps from (r, s, ψ) to $(q_1^\pm, q_2^\pm, \omega^\pm)$.

$$\begin{aligned}
dq_1^- &= \cos\gamma dr - \cos\omega^- ds^- + s^- \sin\omega^- (-\mathcal{K} dr - d\psi) \\
&= (\cos\gamma - s^- \mathcal{K} \sin\omega^-) dr - \cos\omega^- ds^- - s^- \sin\omega^- d\psi \\
dq_2^- &= \sin\gamma dr - \sin\omega^- ds^- - s^- \cos\omega^- (-\mathcal{K} dr - d\psi) \\
&= (\sin\gamma + s^- \mathcal{K} \cos\omega^-) dr - \sin\omega^- ds^- + s^- \cos\omega^- d\psi \\
d\omega^- &= -\mathcal{K} dr - d\psi
\end{aligned}$$

And

$$\begin{aligned}
dq_1^+ &= \cos\gamma dr + \cos\omega^+ ds^+ - s^+ \sin\omega^+ (-\mathcal{K} dr + d\psi) \\
&= (\cos\gamma + s^+ \mathcal{K} \sin\omega^+) dr + \cos\omega^+ ds^+ - s^+ \sin\omega^+ d\psi \\
dq_2^+ &= \sin\gamma dr + \sin\omega^+ ds^+ + s^+ \cos\omega^+ (-\mathcal{K} dr + d\psi) \\
&= (\sin\gamma - s^+ \mathcal{K} \cos\omega^+) dr + \sin\omega^+ ds^+ + s^+ \cos\omega^+ d\psi \\
d\omega^+ &= -\mathcal{K} dr + d\psi
\end{aligned}$$

From here it is a simple process to create the Jacobian matrix and find its determinant. The coefficients of the system of equations gives us the matrix quite easily. We can use the result to write an expression for the differential volume forms at the two points.

$$
\mathcal{J}^{-}= \begin{pmatrix} \cos\gamma - s^{-}\mathcal{K}\sin\omega^{-} & -\cos\omega^{-} & -s^{-}\sin\omega^{-} \\ \sin\gamma + s^{-}\mathcal{K}\cos\omega^{-} & -\sin\omega^{-} & s^{-}\cos\omega^{-} \\ -\mathcal{K} & 0 & -1 \end{pmatrix}
$$

$$
\mathcal{J}^{+}= \begin{pmatrix} \cos\gamma + s^{+}\mathcal{K}\sin\omega^{+} & \cos\omega^{+} & -s^{-}\sin\omega^{-} \\ \sin\gamma - s^{+}\mathcal{K}\cos\omega^{+} & \sin\omega^{+} & s^{-}\cos\omega^{-} \\ -\mathcal{K} & 0 & 1 \end{pmatrix}
$$

Finally, we can calculate the Jacobian of the maps by taking the determinants.

$$
\begin{aligned}
det(\mathcal{J}^{-}) &= -\sin(\gamma - \omega^{-}) = -\sin(\psi) \\
det(\mathcal{J}^{+}) &= -\sin(\gamma - \omega^{+}) = -\sin(-\psi) = \sin(\psi)
\end{aligned}
$$

The infinitesimal phase space volume elements at $(q_1^{\pm}, q_2^{\pm}, \omega^{\pm})$ can now be written in terms of variables that are parameters of the boundary curve and the time dependence. We also know that since the flow acts over a total time t, a constant, $s^{-} + s^{+} = t$. Taking the derivative, we get the result $ds^{-} + ds^{+} = 0$ or $ds^{-} = ds^{+}$. This result comes in handy when evaluating the effect of the flow map on a volume element in phase space.

From the Jacobians, we have:

$$
\begin{gathered}
dq_1^{-}dq_2^{-}d\omega^{-} = \mathcal{J}^{-}dr ds^{-}d\psi = -\sin\psi dr ds^{-}d\psi \\
dq_1^{+}dq_2^{+}d\omega^{+} = \mathcal{J}^{-}dr ds^{+}d\psi = \sin\psi dr ds^{+}d\psi \\
ds^{-} = ds^{+}
\end{gathered}
$$

And therefore,

$$
dq_1^{-}dq_2^{-}d\omega^{-} = dq_1^{+}dq_2^{+}d\omega^{+}
$$

Lemma 1. *If for some time $t \in \mathbb{R}$ a trajectory in the phase space experiences a regular collision, then the Flow map $\Phi^t : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}$ preserves the measure $dq_1dq_2d\omega$ on $\widetilde{\Omega}$.*

If there are no collisions during time t, then we have a different representation for Φ^t. Note that t is a constant and as there are no collisions, $\omega^{-} = \omega^{+} = \omega$ for ω some constant. So taking the map, we differentiate to find a somewhat obvious result.

$$
\begin{gathered}
(q_1^{+}, q_2^{+}, \omega^{+}) \xrightarrow{\Phi^t} (q_1^{-} + t\cos\omega^{-}, q_2^{-} + t\sin\omega^{-}, \omega^{-}) \\
dq_1^{+} = dq_1^{-}\ ,\ dq_2^{+} = dq_2^{-}\ ,\ d\omega^{+} = d\omega^{-}
\end{gathered}
$$

Which gives us, yet again:

$$
dq_1^{-}dq_2^{-}d\omega^{-} = dq_1^{+}dq_2^{+}d\omega^{+}
$$

Lemma 2. *If for some time $t \in \mathbb{R}$ a trajectory in the phase space experiences no collisions, then the Flow map $\Phi^t : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}$ preserves the measure $dq_1dq_2d\omega$ on $\widetilde{\Omega}$.*

This measure is a Lebesgue measure on Ω.

$$d\mu_{\widetilde{\Omega}} = dq_1 dq_2 d\omega$$

The particle moves in two dimensions. With the lack of any complicating potential function and a conserved velocity magnitude, the change in momentum can be represented with the single parameter of velocity direction. Clearly, $d\mu_{\widetilde{\Omega}}$ is the Liouville Measure $dq_1 dq_2 d\omega$ of our system .

3.2 Irregularities and other Considerations in Flow

It becomes readily apparent that there are points in phase space where the dynamics of the system are not clearly defined. Up to now, we have had the luxury of considering only an easy-to-work-with subset of Ω. Let us examine a couple of the issues that lead to our aggravating inability to include all points of phase space, and hopefully, find some resolutions.

Proposition 1. If trajectory is not defined for all $t \in \mathbb{R}$ then there is some finite time t_f such that for $t > t_f$ the trajectory started at (q_1, q_2, ω) at $t = 0$ no longer exists. This may happen in one of several ways:

(i) The particle hits some corner point $(q_1, q_2, \omega) \in \Gamma^*$ where $\Gamma^* = \Gamma \backslash \widetilde{\Gamma}$.

(ii) The trajectory has an *accumulation point* t_a finite in $\mathbb{R}$.

3.3 Accumulation Point of Collision Times

Let us examine this phenomenon first. Let $\{t_n\}$ represents the series of collision times of the given trajectory. If $\lim_{n\to\inf} \{t_n\} \longrightarrow t_a$ for some finite $t_a \in \mathbb{R}$, then t_a is the accumulation point of time. Note that if this were the case, then for some finite time $t > t_a$, the particle experiences infinite bounces. Indeed it obviously follows that a trajectory satisfying Proposition 1(iii) is not defined for all t. If there is an accumulation point for time, then there must be an accumulation point $(q_{1a}, q_{2a}) \in \Gamma$ where $\lim_{t\to\infty}(q_1, q_2) \longrightarrow (q_{1a}, q_{2a})$. So either

(1) $q^* = (q_{1a}, q_{2a}) \in \Gamma^*$, a corner point, or

(2) $\widetilde{q} = (q_{1a}, q_{2a}) \in \widetilde{\Gamma}$, a regular point.

3.3.1 Accumulation Points at a Positive Angle Corner Point

Chernov presents an elementary inductive argument to eliminate possibility 1, but here we choose a more rigorous path. We will set up several pieces of the proof before consolidating it into a theorem.

Let there be a neighborhood around the corner point q^* given by:

$$\mathcal{N}_\epsilon(q^*) = \{q \in D : ||q - q^*|| < \epsilon, \epsilon > 0\}$$

We also define the following items:

q_i	The i-th collision point *within* the neighborhood $\mathcal{N}_\epsilon(q^*)$. The very first collision point in $\mathcal{N}_\epsilon(q^*)$ is therefore $q_1 \in \Gamma_1$.
Γ_1, Γ_2	The two boundaries that form q^*.
θ_i	An angle defined only at each collision point. We define it as the inside angle between the velocity vector at q_i and the tangent vector $\mathbf{T}$ of the boundary of collision. Note that if a trajectory link is not *incident* upon Γ_1 or Γ_1 then θ_i is no longer defined.
γ	The interior angle of the corner q^*.

Assumption 1. For the sake of convenience, we assume that the first collision point in $\mathcal{N}_\epsilon(q^*)$ occurs on Γ_1. Also, we let $\gamma \in (0, 2\pi)$ and $\theta_1 \in (0, \pi)$ and conduct our analysis over their allowable values. Finally, we set the curvature of the boundaries to zero (flat) for now.

Observation 1. Trivially, if $\gamma \in [\pi, 2\pi)$, then a trajectory entering $\mathcal{N}_\epsilon(q^*)$ bounces once at q_1 and then experiences no other collisions regardless of initial angle $\theta_1 \in (0, \pi)$. This is equivalent to an accumulation point on a regular boundary point and will be treated later.

Observation 2. If $\gamma \in (\pi/2, \pi)$ then we may have one of two situations. If initial angle θ_1 is $\in (\pi - \gamma, \gamma]$ then again we have only one collision at q_1. But for $\theta_1 \in (\gamma, \pi)$ we have exactly two collision points and then the trajectory leaves $\mathcal{N}_\epsilon(q^*)$. Also note that if $\theta_1 < \pi - \gamma$, then there was a local collision on Γ_2 prior to q_1 which violates our assumptions if the collision point $q^- \in \mathcal{N}_\epsilon(q^*)$. Otherwise, such a trajectory also leaves with one collision at q_1.

Finally, we come to $\gamma \in (0, \pi/2)$, where the main results lie. To begin, we define two maps $\mathcal{L} : \mathcal{S}^1 \to \mathcal{S}^1$ and $\mathcal{R} : \mathcal{S}^1 \to \mathcal{S}^1$. These two maps effectively act as our collision maps on only one parameter, namely θ_i. $\mathcal{L}$ maps θ_i from a collision point i in the 'lower' boundary Γ_1 to θ_{i+1} at Γ_2. $\mathcal{R}$ does a similar task from the 'upper' boundary to the 'lower' one. Together, these two maps deterministically define every future collision angle given an initial θ_1.

$$\begin{aligned}
\mathcal{L}(\theta) &= \gamma + \pi - \theta \\
\mathcal{R}(\theta) &= -\gamma + \pi - \theta
\end{aligned}$$

The first thing to realize is that the set G of alternating mappings (in other words a 'trajectory' on the set spanned by θ_i) given by $\{\mathcal{I}, \mathcal{L}, \mathcal{R}, \mathcal{LR}, \mathcal{LRL}, \mathcal{RLR} \ldots\}$ is a group (G, $\circ$) under function composition. The checking of this fact is elementary and has been omitted. Note that $\mathcal{I}$ is the identity element and more importantly that $\mathcal{L}$ and $\mathcal{R}$ are their own inverses (from which one can construct the other inverses and ultimately be able to analytically trace backwards the trajectory through θ-space). We now can use the maps in an iterative capacity. Given $n \in \mathbb{N}$ collisions, we can define the angle

$\theta_n = g\theta_1$ where $g \in G$ is and element of size $n-1$.

We assumed that the very first collision happened on the 'lower' boundary, so the $\mathcal{L}$ map is applied first. Now we also realize that n being odd or even decides which boundary the last collision point is on and then we begin construction.

First for the composite maps:

$$\begin{aligned} \mathcal{LR} &= 2\gamma + \theta \\ \mathcal{RL} &= -2\gamma + \theta \end{aligned}$$

And by induction, this yields:

$$\begin{aligned} (\mathcal{LR})^N &= 2N\gamma + \theta \\ (\mathcal{RL})^M &= -2M\gamma + \theta \end{aligned}$$

If n is odd, q_n is on the 'lower' boundary:

$$\begin{aligned} \theta_n &= \mathcal{RL}\ldots\mathcal{RLRL}\theta_1 \\ &= (\mathcal{RL})^{\frac{n-1}{2}}\theta_1 = -(n-1)\gamma + \theta_1 \end{aligned}$$

If n is even, q_n is on the 'upper' boundary and we also change to initial parameter θ_2:

$$\begin{aligned} \theta_n &= \mathcal{LRL}\ldots\mathcal{RLRL}\theta_1 \\ &= (\mathcal{LR})^{\frac{n-2}{2}}\mathcal{L}\theta_1 = (\mathcal{LR})^{\frac{n-2}{2}}\theta_2 \\ &= (n-2)\gamma + \theta_2 \end{aligned}$$

Now let there be some θ_n such that, after the the n-th collision, the particle no longer collides with Γ_1 or Γ_2. It is simple to see that for the two cases of odd and even n, this sets a limit on n. For odd n:

$$0 < \theta_n \leq \gamma$$

Which, we solve:

$$\begin{aligned} \theta_n &> 0 \\ -(n-1)\gamma + \theta_1 &> 0 \\ n &< \frac{\theta_1}{\gamma} + 1 \end{aligned} \qquad \begin{aligned} \theta_n &\leq \gamma \\ -(n-1)\gamma + \theta_1 &\leq \gamma \\ n &\geq \frac{\theta_1}{\gamma} \end{aligned}$$

And for even n:

$$\pi - \gamma \leq \theta_n < \pi$$

Which, we solve and substitute in $\theta_2 = \gamma + \pi - \theta_1$:

$$
\begin{array}{rclcrcl}
\theta_n & \geq & \pi - \gamma & \qquad & \theta_n & < & \pi \\
(n-2)\gamma + \theta_2 & \geq & \pi - \gamma & & (n-2)\gamma + \theta_2 & < & \pi \\
n\gamma - \gamma + \pi - \theta_1 & \geq & \pi - \gamma & & n\gamma - \gamma - \theta_1 + \pi & < & \pi \\
n & \geq & \frac{\theta_1}{\gamma} & & n & < & \frac{\theta_1}{\gamma} + 1
\end{array}
$$

Now, we substitute in $\pi - \theta_1 = \alpha$ where α is the angle the incoming trajectory makes with Γ_1 at the first collision point. This is for ease of visualization as our θ seems a bit counterintuitive. Combined with the restrictions above, we note that for both odd and even n, we get the same result.

$$
\frac{\pi}{\gamma} - \frac{\alpha}{\gamma} \leq n < \frac{\pi}{\gamma} - \frac{\alpha}{\gamma} + 1
$$

And it follows that since $n \in \mathbb{N}$, that there must exist only one n that fits the above condition. Also, as $\alpha \neq 0$, we finally have the following theorem. Note that for any corner point we can use the idea of approximation to see that for really small ϵ any type of boundary is a line segment.

Theorem 1. *If a trajectory enters a neighborhood $\mathcal{N}_\epsilon(q^*)$ around the corner point q^* with interior angle $\gamma > 0$, then given that the first collision does not occur at q^*, the trajectory MUST leave within $\frac{\pi}{\gamma} + 1$ bounces. There may be no accumulation points at a corner of positive interior angle.*

3.3.2 Accumulation Points in Cusps

Definition 2. A *cusp* is defined as a corner with interior angle zero

Observation 3. A cusp can only be made by a combination of a dispersing side and a flat side or two dispersing sides. By Def 1(iv), we cannot have a cusp created by a dispersing side and a focusing side.

Lemma 3. *If a billiard particle enters a neighborhood of a cusp corner, then within a finite number of collisions, the trajectory will exit the neighborhood.*

Proof. Let the cusp occur at a point q^* and take a trajectory that enters the neighborhood given by:

$$
\mathcal{N}_\epsilon(q^*) = \{q \in D : ||q - q^*|| < \epsilon, \epsilon > 0\}
$$

For collision points q_i in $\mathcal{N}_\epsilon(q^*)$, let d_i denote the perpendicular distance from q^* to a line drawn along the incoming trajectory segment of the i-th collision point. Every time the partile collides with a dispersing wall, the positive curvature ensures that the perpendicular distance d_i increased. For flat walls, a curvature is zero, we note that d_i is left unaltered. So, for any cusp point, $d_{i+1} \geq d_i$. As a cusp must have a dispersing side and the particle can not bounce twice on the same wall (for flat and dispersing at least), every two iterations/collisions, we have $d_{i+2} > d_i$. It follows that no matter where the first collision in $\mathcal{N}_\epsilon(q^*)$ is, that eventually, for some $i \geq 2$, $d_i > \epsilon$. By definition of d_i, the (extended) trajectory of the i-th collision point is tangent to a circle of radius d_i around q^*, and as a tangent can not lie within a circle, $||q_i - q^*|| > \epsilon$ meaning that the trajectory has left $\mathcal{N}_\epsilon(q^*)$. □

And finally, we state a theorem written by Halpern in his study of 'Strange Billiard Tables' along with an observation.

Observation 4. Given a regular collision point, the curvature of a dispersing wall and a flat wall necessitates that any non-grazing trjectory will leave a neighborhood around it.

Theorem 2. *There can be no accumulation points of collision times on a focusing wall with a bounded third derivative and nowhere vnishing curvature.*

We note that from Def 1 restrictions that Halpern's theorem applies to all of our billiard tables. In conjunction with the rest of our Lemmas, we have ruled out the possibility of accumulation of collision points.

3.4 Corner Point Collision

Take some non boundary point $q \in D$ where $q = (q_1, q_2)$ and a corner point q^*. Quite trivially, An angle ω is *uniquely* defined for each point q given that q^* is immediately accessible from q. Therefore, the set of all points $x \in \Omega$ (the phase space) given by $x = (q_1, q_2, \omega)$ that collide with q^* *first* forms a two dimensional hypersurface embedded in the 3D Phase Space Ω.

Via the fact that the flow is $l - 1$ smooth at regular collision points (though important, we do not prove it here), we arrive at the following theorem:

Theorem 3. *The set given by $\Omega\backslash\Omega$, namely the set of all phase space points where the dynamics are not clearly defined, is a countable union of two-dimensional hypersurfaces embedded in Ω.*

This, fortunately, means that the measure of $\Omega\backslash\widetilde{\Omega}$ is zero and the flow can be said to be defined μ-almost everwhere. To wrap that up, we can now normalized the Lebesgue measure by integrating over the entire phase space.

$$d\mu_{\widetilde{\Omega}} = d\mu_\Omega = \frac{1}{2\pi(\text{Area})} dq_1 dq_2 d\omega$$

4 Collision Map and Space

We turn to examining a subset of Ω known as the collision space. This is a hypersurface embedded in phase space and serves as a useful tool in billiards analysis. Like how Ω consists of points in $D \times S^1$, the collision space that we construct is $\mathcal{M} = \Gamma \times S^1$, the subset of Ω whose real space points lie on the boundary of the billiard table. However, the identification of the precollisional and post collisional velocity at a regular point on the boundary implies that we must make S^1 a half circle.

Definition 3. The *collision space* $\mathcal{M}$ is given by a finite, countable union of $\mathcal{M}_i$ where $\mathcal{M}_i = \{(q,v) \in \Omega : q \in \Gamma_i \text{ and } <v,n> \geq 0\}$. Here n is the normal vector of Γ_i at q and $<v,n>$ denotes a scalar product between v and n.

For a non-corner point $q \in \Gamma$, the flow is defined for a small time interval $t \in (0,\epsilon)$, $\epsilon > 0$ only if it is a regular collision or a grazing collision at a dispersing boundary. As this covers all the possibilities, the restriction is not too troublesome. For a collision point $x = (q,v) \in \mathcal{M}$ with a defined flow at q, Φ^t will eventually map x to some $x_1 \in \mathcal{M}$, namely the next collision point. Since we are looking at bounded billiard tables, all trajectories in $\widetilde{\Omega}$ give an infinite number of collisions (points in $\mathcal{M}$).

Definition 4. We define a map $\mathcal{F} = \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$ by $\mathcal{F}(x) = \Phi^{\tau(x)}$, $x \in \widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}} = \mathcal{M} \cap \widetilde{\Omega}$ is the set of all collision points from trajectories with clearly defined dynamics (see previous sections). $\tau(x)$ is the distance from x to the next collision point.

Without going into too much detail or rigor, we define a coordinate system on $\mathcal{M}$ using the variables r and ϕ. r_i is the arclength of each boundary Γ_i and $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the angle between velocity v and the normal vector n at the point of collision.

Let us take two points (r,ϕ) and (r_1,ϕ_1) in $\mathcal{M}$ such that $\mathcal{F}(r,\phi) = (r_1,\phi_1)$ and let (x,y), (x_1,y_1) be the corresponding x-y coordinate points. We also define τ as the distance from (x,y) to (x_1,y_1) and then refer to the flow map analysis for the definition of ω, γ, and ψ. First we note:

$$x_1 - x = \tau\cos\omega, \quad y_1 - y = \tau\sin\omega, \quad \psi = \tfrac{\pi}{2} - \phi \tag{1}$$

and, following a similar line of calculation as with the flow map,

$$\begin{aligned} dx &= \cos\gamma dr, \quad dx_1 = \cos\gamma_1 dr_1 \\ dy &= \sin\gamma dr, \quad dy_1 = \sin\gamma_1 dr_1 \\ d\gamma &= -\mathcal{K}dr, \quad d\gamma_1 = -\mathcal{K}_1 dr_1 \end{aligned} \tag{2}$$

Taking ω and differentiating:

$$\omega = \gamma + \psi = \gamma_1 - \psi_1 \tag{3}$$

$$d\omega = -\mathcal{K}dr + d\psi = -\mathcal{K}_1 dr_1 - d\psi_1 \tag{4}$$

Differentiating (1) and substituting from (2), we get:

$$\begin{aligned} dx_1 - dx &= d(\tau\cos\omega) = \tau d(\cos\omega) + \cos\omega d\tau \\ &= \cos\omega d\tau - \tau\sin\omega d\tau \\ \cos\gamma_1 dr_1 - \cos\gamma dr &= \cos\omega d\tau - \tau\sin\omega d\tau \end{aligned} \tag{5}$$

$$\begin{aligned} dy_1 - dy &= d(\tau\sin\omega) = \tau d(\sin\omega) + \sin\omega d\tau \\ &= \sin\omega d\tau + \tau\cos\omega d\tau \\ \sin\gamma_1 dr_1 - \sin\gamma dr &= \sin\omega d\tau + \tau\cos\omega d\tau \end{aligned} \tag{6}$$

Using (5), we solve for $d\tau$ and then plug it into (6) (Mathematica helps here...) and get:

$$\sin\psi_1 dr_1 + \sin\psi dr = \tau d\omega \tag{7}$$

Plugging (4) into (7):

$$\begin{aligned} \sin\psi_1 dr_1 + \sin\psi dr &= \tau(-\mathcal{K}dr + d\psi) \\ \sin\psi_1 dr_1 &= (-\tau\mathcal{K} - \sin\psi)dr + \tau d\psi \end{aligned} \tag{8}$$

$$\begin{aligned} \sin\psi_1 dr_1 + \sin\psi dr &= \tau(-\mathcal{K}_1 dr_1 - d\psi_1) \\ \sin\psi_1 dr_1 + \tau\mathcal{K}_1 dr_1 &= -\sin\psi dr - \tau d\psi_1 \\ \sin\psi_1 dr_1(1 + \frac{\tau\mathcal{K}_1}{\sin\psi_1}) &= -\sin\psi dr - \tau d\psi_1 \\ ((-\tau\mathcal{K} - \sin\psi)dr &+ \tau d\psi)(\sin\psi_1 + \tau\mathcal{K}_1) \\ &= -\sin\psi_1\sin\psi dr - \tau\sin\psi_1 d\psi_1 \\ (-\tau\mathcal{K}\sin\psi_1 - \sin\psi\sin\psi_1 - \tau^2\mathcal{K}_1\mathcal{K} &- \tau\mathcal{K}_1\sin\psi)dr + (\tau\sin\psi_1 + \tau^2\mathcal{K}_1)d\psi \\ &= -\sin\psi_1\sin\psi dr - \tau\sin\psi_1 d\psi_1 \\ -\tau(\mathcal{K}\sin\psi_1 + \tau\mathcal{K}\mathcal{K}_1 + \mathcal{K}_1\sin\psi)dr &+ \tau(\sin\psi_1 + \tau\mathcal{K}_1)d\psi \\ &= -\tau\sin\psi_1 d\psi_1 \end{aligned}$$

$$\sin\psi_1 d\psi_1 = (\mathcal{K}\sin\psi_1 + \tau\mathcal{K}\mathcal{K}_1 + \mathcal{K}_1\sin\psi)dr + (\sin\psi_1 + \tau\mathcal{K}_1)d\psi \tag{9}$$

Using $d\psi = -d\phi$, $\sin\phi = \cos\psi$, $\sin\psi = \cos\phi$ we rewrite (8) and (9) in terms of ϕ's and r's.

$$\begin{aligned} -\cos\phi_1 dr_1 &= (\tau\mathcal{K} + \cos\phi)dr + \tau d\phi \\ -\cos\phi_1 d\phi_1 &= (\mathcal{K}\cos\phi_1 + \tau\mathcal{K}\mathcal{K}_1 + \mathcal{K}_1\cos\phi)dr + (\cos\phi_1 + \tau\mathcal{K}_1)d\phi \end{aligned} \tag{10}$$

Finally, (10) allows us to build the Jacobian matrix for the map $\mathcal{F} : (r, \phi) \to (r_1, \phi_1)$ with the coefficient matrix.

$$D_x\mathcal{F} = \mathcal{J}_{\mathcal{F}} = -\frac{1}{\cos\phi_1}\begin{pmatrix} \tau\mathcal{K} + \cos\phi & \tau \\ \mathcal{K}\cos\phi_1 + \tau\mathcal{K}\mathcal{K}_1 + \mathcal{K}_1\cos\phi & \cos\phi_1 + \tau\mathcal{K}_1 \end{pmatrix} \tag{11}$$

And so the Jacobian is:

$$det(\mathcal{J}_{\mathcal{F}}) = \frac{\cos\phi}{\cos\phi_1}$$

Which is useful for the next theorem, the last result we present here. We do not prove it, but note that $\mathcal{F}$ is a diffeomorphism of smoothness $l-1$.

Theorem 4. *The measure* $\cos\phi dr d\phi$ *is preserved by the collision map* $\mathcal{F}$ *on* $\mathcal{M}$*.*

Proof. We apply the change of variables theorem on some set $A \subset \mathcal{M}$ for a continuous bounded function $f(\mathcal{F}(r,\phi)) = \cos\phi$ and a diffeomorphism $A \mapsto \mathcal{F}(A)$. This gives us

$$\int\int_{\mathcal{F}(A)} f(r_1,\phi_1)dr_1 d\phi_1 = \int\int_A \mathcal{J}_{\mathcal{F}} f(\mathcal{F}(r,\phi))dr d\phi$$

$$\int\int_{\mathcal{F}(A)} \cos\phi_1 dr_1 d\phi_1 = \int\int_A \frac{\cos\phi}{\cos\phi_1}\cos\phi_1 dr d\phi = \int\int_A \cos\phi dr d\phi$$

□

And so we end our foray into the world of billiards with that final result. Though the benefits of proving that the collision map is invariant under the measure $\cos\phi dr d\phi$ are not included here, it has many applications. The possibilities for illuminating analysis of billiards are endless. As such, there is still a lot of ongoing research in the field. In truth, we have barely scratched the surface.