# Chaotic Billiards

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# Part I
# Introduction to Dynamical Billiards

## 1 Review of the Different Billiard Systems

In investigating dynamical billiard theory, we focus on two important examples that demonstrate a variety of behaviors and represent clear gradation in complexity. This paper mixes analytic and computational billiard theory to give a detailed picture of a circular billiards and billiards in a stadium—a two semi-circular regions separated by a rectangular region. We investigate sensitive dependence on initial coordinates and the structure of periodic orbits before considering whether each exhibits chaos.

### 1.1 Case Examples of the Billiards

The circular billiard is one of the simplest finite billiards to analyze because the collision map can be described by a very simple system of equations. If $\theta$ is the angular representation of a collision point relative to the positive $x$ axis and $\psi$ is the angle of the trajectory against the tangent line to the circle, we can easily relate an element in the phase space by the system

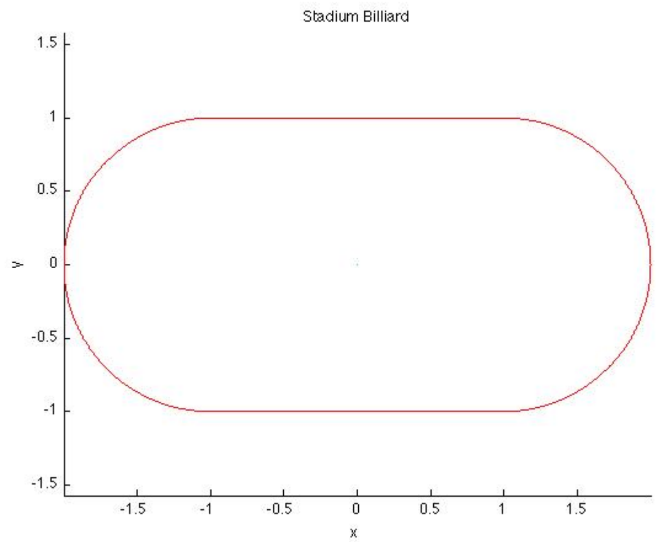
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \Theta_n \\ \Psi_n \end{pmatrix} = \begin{pmatrix} \Theta_{n+1} \\ \Psi_{n+1} \end{pmatrix}$$

Since $\theta \in [0, 2\pi)$ and $\psi \in [0, \pi]$ we can define the phase space as $C^1 \times [0, \pi]$. We see that the map simply "twists" the $\theta$ coordinate by how high it is on the cylinder. By also calculating the determinant of this map, we see that the area of a shape in the phase space is preserved under the mapping.

Computationally however, we represent the boundary as a single curve parameterized as $(cos(t), sin(t))$ for $t \in [0, 2\pi)$.

The stadium billiard however, represents an even more difficult problem to model mathematically but proves still tractable to computational analysis. The boundary of the stadium billiard can be thought of as a union of four curves.

There exists an upper horizontal portion, a right semi-circular portion, a lower horizontal portion, and a left semi-circular portion. The left and right curves are parameterized by the vectors $(-cos(t) - a, -sin(t))$ and $(cos(t) + a,\, sin(t))$, respectively, for $-\pi/2 \leq t < \pi/2$. The other two horizontal curves which complete the billiard vary from $-a$ to $a$, $a \geq 0$, at heights of $y = 1$ and $y = -1$. For depiction below, $a$ is equal to 1.



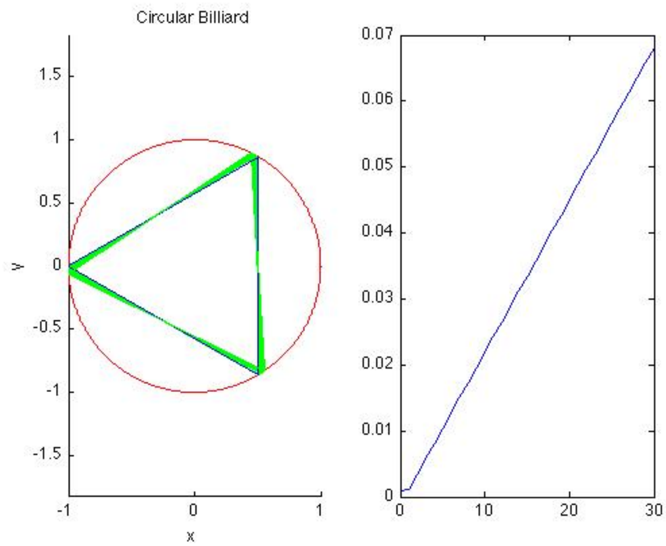

# Part II
# Numerical Calculations with Circular and Stadium Billiards

## 2 Numerical Calculations

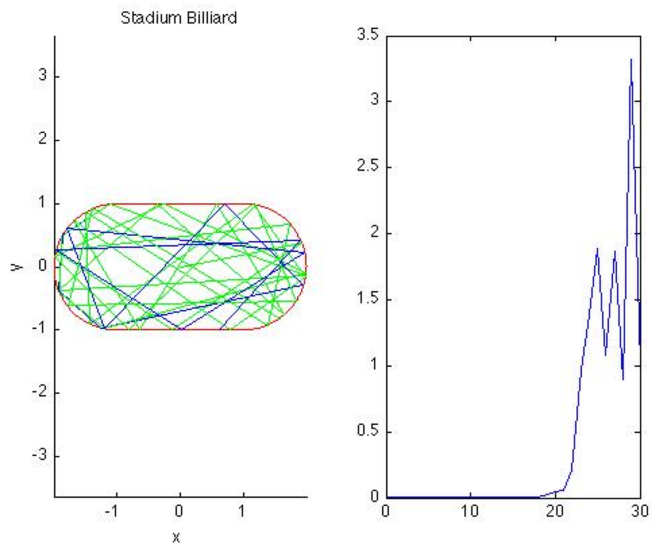
### 2.1 Sensitive Dependence on Initial Conditions

For both of the circular and the stadium billiards, we can demonstrate sensitive dependence on initial conditions by simply varying any on of the parameters of one orbit and let the others remain constant. As is the case for both the stadium and circular billiards, we start with a known initial condition for two given orbits of each billiards. We then create another orbit in each billiard, which starts at a small distance from the first initial condition in each billiard.

We start by observing the case of the circular billiard. In this particular example, we start with an initial point $(0.5, 0)$, and set the initial $x$ velocity to 0 and $y$ velocity to $-1$. For the other initial condition in the billiard, each initial condition is the same except the starting $x$-coordinate of the second initial condition. In this case, we retain the same initial $x$ and $y$ velocities, but instead start at the point $(0.5 + 1 \times 10^{-3}, 0)$. Thus, we seek to find the first collision at which two collision points are at least 0.05 units apart. The graphs on the next page show that this separation is reached by the twenty-fourth bounce along the circular billiard. The right-hand plot measures the Euclidean distance between each collision point on the vertical axis the nth collision along the boundary on the horizontal axis.



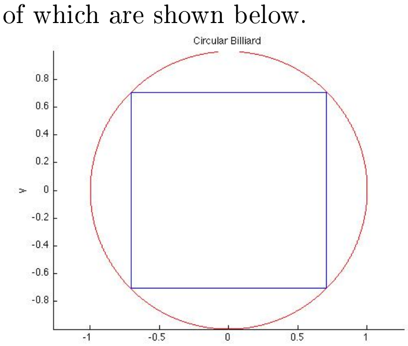

Sensitive dependence on initial conditions for the stadium billiard follows the same premise as the circular billiard example, but with slight modifications. That is, we start with an initial point $(0, 0)$ in the stadium billiard, and let the initial x and y velocities equal 1 and 0.2, respectively:



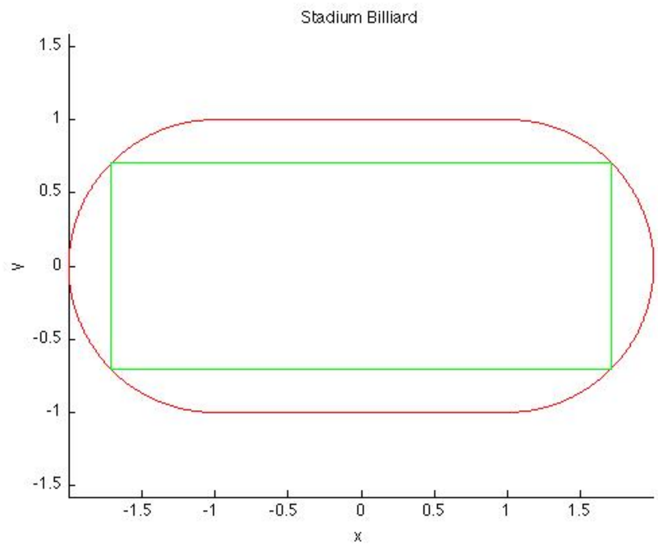

In this particular plot, the blue trajectory is the is the initial condition starting at $(0, 0)$, whereas the green trajectory starts at $(1 \times 10^{-10}, 0)$. To show sensitive dependence, we specify that the trajectories must separate by a distance $d$ of at least 0.25 after a certain number of collisions. The graph on the right indicates that this separation occurs at the twentieth collision along the billiard. As with the circular billiard, the horizontal axis specifies the $nth$ collision along the stadium billiard, and the vertical axis is a measure of the Euclidean distance between each collision point.

## 2.2 Periodic Orbits

Certain initial conditions in the circle and stadium billiard give rise to periodic orbits. For the circular billiard, there are many ways to create periodic orbits, one of which are shown below.




In this case, we have a period-four orbit generated by the initial condition $(0, 1/\sqrt{2})$. The stadium billiard exhibits similar properties to that of the circular billiards, particularly with period-four orbits. In this case, we observe that starting at an initial condition of $(0, 1/\sqrt{2})$ with initial $x$ velocity 1 and $y$ velocity 0 result in a similar period 4 orbit:




For other periodic orbits of the circle and the stadium, they both share similarities in the generation of period-two orbits. We can generate a period two orbit in the circular case by starting at the point $(0, 0)$ and choosing any $x$ or $y$ velocity, so long as they are not both zero. For period-two orbits in the stadium billiard, the only difference lies in the fact that there are multiple period-two orbits that oscillate between $y = 1$ and $y = -1$, whereas the circular billiard has only one such orbit.

## 2.3 Lyapunov Exponents

### 2.3.1 Numerical Computation of Lyapunov Exponents

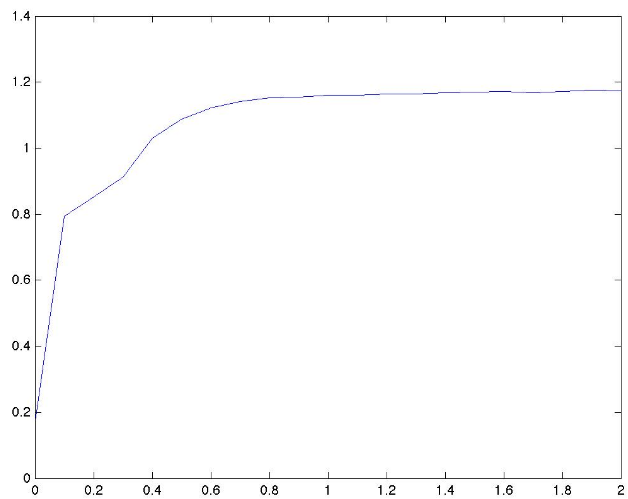
Calculating the Lyapunov exponents of each billiard system in question will help to explain the behavior of the system, and consequently whether the given system is chaotic.

To calculate a lower bound on the largest Lyapunov exponent, we measure the log of the growing difference as we apply the map thousands of times to close by initial points. This estimation gives us a lower bound for the largest Lyapunov exponent since the maximum log of the growing distance occurs in the eigen-direction for the eigen-value corresponding to the leading Lyapunov number.

Given that two points will not necessarily preserve their orientation under the mapping, the traditional method for calculating the Lyapunov exponent relies on constant orientation and normalization by the Gram-Schmidt process to calculate a Lyapunov number. Similarly, by not adjusting the direction, we cannot guarantee that the orientation of the points' orbits lie in the eigen–direction, but we do know that they cannot separate faster than they would in the eigen-direction. So this calculation reveals a lower bound on the Lyapunov exponent.

For the circular billiard, a lower bound on the largest Lyapunov exponent was obtained over thousands of iterations of the map between two nearby initial conditions. With many pairs of randomly chosen initial conditions from within the circle, we take the maximum value of the lower bounds Lyapunov exponents for many pairs of initial coordinates to find an approximation of the largest Lyapunov exponent. We were able to calculate a lowever bound of the Lyapunov Exponent approximately equal to zero for the circular billiard. Which does not reveal anything in particular about the behavior of the circular billiard.

The stadium billiard presents a more interesting case than the circular billiard because the horizontal side-length $a$ is just a variable in our calculation. That is, by letting a vary for $a \geq 0$, we obtain a range of Lyapunov exponents. Below, we plot Lyapunov exponents against values of $a$.




#### 2.3.2 Alternative Calculations of the Lyapunov Exponents

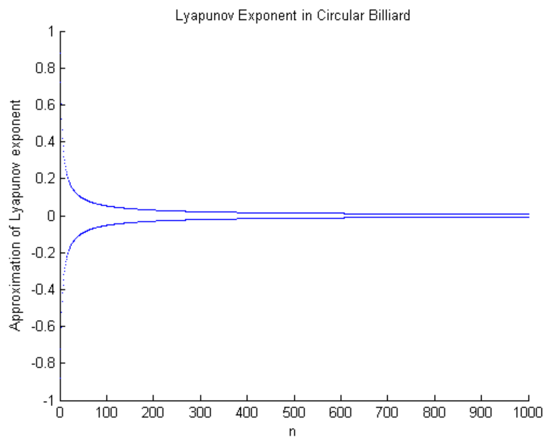
**Alternate Method of Calculating the Lyapunov Exponents of the Circular Billiard**

Instead of chosing points close by and measuring their growing separation as we apply the map, we can calculate the Lyapunov exponent using the matrix relation. In terms of $\Theta$ and $\Psi$ we have the relation

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \Theta_n \\ \Psi_n \end{pmatrix} = \begin{pmatrix} \Theta_{n+1} \\ \Psi_{n+1} \end{pmatrix}$$

To calculate the Lyapunov numbers of this map, we calculate the eigen-values of $\Phi^n (\Phi^n)^\tau$ which individually represent the squares of the $nth$ powers of the

Lyapunov numbers of the map. Thus, we can calculate the Lyapunov exponents using $h_k = \lim_{n\to\infty} \frac{1}{2n} log(r_k^n)$ [3]. Using Matlab, this is a simple calculation, revealing that our approximation of each the Lyapunov exponent approaches zero, so the circular billiard is non-chaotic.




# Part III
# Invariant Measure of the Map

Through the investigation of the circular billiard we found that a volume element in the phase space was preserved in the flow, but under what conditions does this invariance hold? To investigate whether a volume element in the phase space is preserved under the flow, we first answer the same question regarding a time-t map for arbitrarily small time.

The evaluation of a time-t map, $\Phi_t(x, y, \dot{x}, \dot{y})$, can be reduced to two cases by making $t$ to be arbitrarily small. In the first case no collision occurs during a time interval and in the second case, exactly one collision occurs. $\Phi_t$ can be reduced to these two cases since given any starting point in the phase space since the flow can be broken into several segments, each of which fall into one of the two previous cases. If a particular flow has more than one collision in a time, $t$, then either we can break the time interval into many subintervals where only one collision occurs. If we cannot break it into subintervals where only one collision occurs, we have reached an accumulation point where the trajectory tends towards a point as $t$ grows large. Instead of handling these cases, we restrict the following theorem to billiards where there are no accumulating points and where the boundary curves are well behaved through a series of assumptions made about the system.

# 3 Theorem

The flow $\Phi_\tau$ preserves the volume element $dx\,dy\,d\omega$

## 3.1 Assumptions[1]

1. The boundary of the billiard is a finite union of smooth compact curves with continuous third derivatives.

   If the boundary were composed of an infinite union of curves, we could easily violate the requirement that the third derivative be continuous by creating an infinitely fine set of points connected by the first order taylor series approximation of a function with a discontinuous third derivative. The other requirements restrict the boundary to parameterizable curves– an important step in analytically calculating the affect of the flow on a volume element.

2. The individual curves that compose the boundary only intersect eachother at their end points.

   Restricting this helps us to classify the intersection and corners.

3. The second derivative of each curve in the boundary is either identically zero or is never zero.

   Which helps to classify the boundaries as either focusing, diverging, or flat.

4. A billiard contains no cusps made by a focusing wall and a dispersing wall.

   According to Halpern's Theorem, Assumption 4 removes the remaining cases where a trajectory can accumulate. Such a trajectory has a limit as we allow the particle to travel and theoretically approaches the corner.

### 3.1.1 Proving the Theorem

To describe the two cases rigorously, we introduce variables to represent the state of a particle before and after the map is applied.

$x^{\pm}$ refers to the $x$ coordinates of a point before and after the map is applied

$y^{\pm}$ refers to the $y$ coordinates of a point before and after the map is applied

$\omega^{\pm}$ refers to the angles of the velocity against the $x$ axis before and after the map is applied. Since the speed can be set to one, we can treat it similarly to the velocity of the particle.

**Case: No Collisions** In the case where there are no collisions, the map can be expressed as the simple system of equations relating the input coordinates and velocity and the output coordinates and velocity.

$$x^+ = x^- + cos(\omega)$$

$$y^+ = y^- + sin(\omega)$$

$$\omega^+ = \omega^-(= \omega)$$

Differentiating reveals

$$dx^+ = dx^- - sin(\omega)td\omega$$

$$dy^+ = dy^- + cos(\omega)td\omega$$

$$d\omega^+ = d\omega^-$$

To compute the change in volume of an element $dx^-dy^-d\omega^-$ as it is passed through the map $\Phi_t$, we relate the volumes by the equation $dx^+dy^+d\omega^+ = det(\mathcal{J})dx^-dy^-d\omega^-$ where $\mathcal{J}$ is the Jacobian matrix of the mapping.

$$\begin{pmatrix} dx^+ \\ dy^+ \\ d\omega^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 & -sin(\omega) \\ 0 & 1 & cos(\omega) \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} dx^- \\ dy^- \\ d\omega^- \end{pmatrix}$$

$$\mathcal{J} = \begin{pmatrix} 1 & 0 & -sin(\omega) \\ 0 & 1 & cos(\omega) \\ 0 & 0 & 1 \end{pmatrix}$$

$$dx^+dy^+d\omega^+ = det(\mathcal{J})dx^-dy^-d\omega^- = dx^-dy^-d\omega^-$$

Thus, when there are no collisions, the time-t map preserves the volume of an element in the phase space.

**Case: Single Collision** In the case where there is a single collision, we introduce 'intermediate' variables to simplify the analysis.

Let

$\Gamma_i$ be the segment of the boundary on which the collision occurs

$(\bar{x}, \bar{y})$ be the location of the collision

$\gamma$ be the angle of the line tangent to $\Gamma_i$ at $(\bar{x}, \bar{y})$ off of the $x$ axis

$\psi$ be the angle of the velocity off of the line tangent to $\Gamma_i$ at $(\bar{x}, \bar{y})$

$r$ parametrize the arc length of $\Gamma_i$

$s^- + s^+ = t$ represent the time interval before the collision and the time interval afterward.

Using the system that these variables describe, we will first model the map $\Phi_t^- : (r, s^-, \psi) \longmapsto (x^-, y^-, \omega^-)$ and compare the volume elements $drds^-d\psi$ and $dx^-dy^-d\omega^-$ using the Jacobian of $\Phi_t^-$, $\mathcal{J}^-$Secondly, we will model the map $\Phi_t^+ : (r, s^+, \psi) \longmapsto (x^+, y^+, \omega^+)$ and compare the volume elements $drds^+d\psi$ and $dx^+dy^+d\omega^+$using the Jacobian of $\Phi_t^+$, $\mathcal{J}^+$.

**Backwards Map** $\Phi_t^- : (r, s^-, \psi) \longmapsto (x^-, y^-, \omega^-)$

$x^- = \bar{x} - s^-cos(\omega^-)$ Since $s^-cos(\omega^-)$ is the projection of the precollisional velocity on the $x$ axis scaled to equal the distance traveled precollision

$y^- = \bar{y} - s^-sin(\omega^-)$ Since $s^-sin(\omega^-)$ is the projection of the precollisional velocity on the $y$ axis scaled to equal the distance traveled precollision

$\omega^- = \gamma - \psi$

Differentiating reveals

$$dx^- = cos(\gamma)dr - cos(\omega^-)ds^- + s^-sin(\omega^-)d\omega^-$$

$$dy^- = sin(\gamma)dr - sin(\omega^-)ds^- - s^-sin(\omega^-)d\omega^-$$

$$d\omega^- = -\mathcal{K}dr - d\psi$$

$$d\bar{x} = cos(\gamma)dr$$

$$d\bar{y} = sin(\gamma)dr$$

$$d\gamma = -\mathcal{K}dr$$

Substituting the definition of $d\omega^-$ in resolves the system to

$$dx^- = [cos(\gamma) - \mathcal{K}s^-sin(\gamma - \psi)]dr + [-cos(\gamma - \psi)]ds^- + [-s^-sin(\gamma - \psi)]d\psi$$

$$dy^- = [sin(\gamma) - \mathcal{K}s^-cos(\gamma - \psi)]dr + [sin(\gamma - \psi)]ds^- + [s^-cos(\gamma - \psi)]d\psi$$

$$d\omega^- = -\mathcal{K}dr - d\psi$$

The differential equation that follows is thus

$$\begin{pmatrix} dx^- \\ dy^- \\ d\omega^- \end{pmatrix} = \begin{bmatrix} cos(\gamma) - \mathcal{K}s^- sin(\gamma-\psi) & -cos(\gamma-\psi) & -s^- sin(\gamma-\psi) \\ sin(\gamma) - \mathcal{K}s^- cos(\gamma-\psi) & sin(\gamma-\psi) & s^- cos(\gamma-\psi) \\ -\mathcal{K} & 0 & -1 \end{bmatrix} \times \begin{pmatrix} dr \\ ds^- \\ d\psi \end{pmatrix}$$

And the determinant $dx^- dy^- d\omega^- = det(\mathcal{J}^-) dr\, ds^- d\psi = -sin(\psi) dr\, ds^- d\psi$

**Forwards Map** $\Phi_t^+ : (r, s^+, \psi) \longmapsto (x^+, y^+, \omega^+)$

$x^+ = \bar{x} + s^+ cos(\omega^+)$ Since $s^+ cos(\omega^+)$ is the projection of the precollisional velocity on the $x$ axis scaled to equal the distance traveled postcollision

$y^+ = \bar{y} + s^+ sin(\omega^+)$Since $s^+ sin(\omega^+)$ is the projection of the precollisional velocity on the $y$ axis scaled to equal the distance traveled postcollision

$\omega^+ = \gamma + \psi$

Differentiating reveals

$dx^+ = cos(\gamma)dr + cos(\omega^+)ds^+ - s^+ sin(\omega^+)d\omega^+$

$dy^+ = sin(\gamma)dr + sin(\omega^+)ds^+ + s^+ sin(\omega^+)d\omega^+$

$d\omega^+ = -\mathcal{K}dr + d\psi$

$d\overline{x} = cos(\gamma)dr$

$d\overline{y} = sin(\gamma)dr$

$d\gamma = -\mathcal{K}dr$

Substituting the definition of $d\omega^+$ in resolves the system to

$dx^+ = [cos(\gamma) + \mathcal{K}s^+ sin(\gamma+\psi)]dr + [cos(\gamma+\psi)]ds^+ + [-s^+ sin(\gamma+\psi)]d\psi$

$dy^+ = [sin(\gamma) - \mathcal{K}s^+ cos(\gamma+\psi)]dr + [sin(\gamma+\psi)]ds^+ + [s^+ cos(\gamma+\psi)]d\psi$

$d\omega^+ = -\mathcal{K}dr + d\psi$

The differential equation that follows is thus

$$\begin{pmatrix} dx^+ \\ dy^+ \\ d\omega^+ \end{pmatrix} = \begin{bmatrix} cos(\gamma) + \mathcal{K}s^+ sin(\gamma+\psi) & cos(\gamma+\psi) & -s^+ sin(\gamma+\psi) \\ sin(\gamma) - \mathcal{K}s^+ cos(\gamma+\psi) & sin(\gamma+\psi) & s^+ cos(\gamma+\psi) \\ -\mathcal{K} & 0 & 1 \end{bmatrix} \times \begin{pmatrix} dr \\ ds^+ \\ d\psi \end{pmatrix}$$

And the determinant $dx^+ dy^+ d\omega^+ = det(\mathcal{J}^+) dr\, ds^+ d\psi = sin(\psi) dr\, ds^+ d\psi$

Now we can relate the volumes by the original definition $s^- + s^+ = t$, leading to $ds^+ = -ds^-$. Thus

$$dx^- dy^- d\omega^- = -sin(\psi) dr\, ds^- d\psi = sin(\psi) dr\, ds^+ d\psi = dx^+ dy^+ d\omega^+$$

So for any billiard that satisfies our assumptions, we can take a volume element in the phase space and trace its flow as a series of time-t maps each of which preserves the volume. Following from this, we see that the volume is preserved over the general flow.

# Part IV

# The Effect of Changing a Billiard on the Flow

In this investigation, we limitted analysis to only considering the possible effects of changing size of the circular billiard durring a flow to change the behavior. First we consider whether it is possible to change the size of the a billiard to change a non-periodic flow to a periodic flow.

Consider a quasi-periodic orbit of a particle in the circular billiard. Where $\psi$ represents the angle between the trajectory and the tangent to the circle and $\theta$ represents the angle of a collision point measured from the positive $x$ axis, we find the relations

$$\theta_n = \theta_{n-1} + 2\psi\,(mod\, 2\pi)$$

$$\psi_n = \psi_{n-1}$$

So we can draw a triangle with vertices at $\theta_n$, $\theta_{n-1}$, and the origin and use the law of sines, to find $sin\frac{\pi}{2} = \frac{sin(\pi/2-\psi)}{x}$ where $x$ is length of the perpendicular to the chord connecting $\theta_n$ and $\theta_{n-1}$. Thus, we find $x = sin(\frac{\pi}{2} - \psi) = cos(\psi)$. So the inner "caustic" circle that each chord lies tangent to is distance $cos(\psi)$ from the origin.

To create a familiar periodic orbit, we extend the circle to have some radius $r$ at which $\psi'$, the angle between the tragectory and the tangent to the new circle will be $\pi/3$ to create an equilateral triangle. Thus, using the same relations in the new circle, we construct a similar right triangle with hypotenuse of length $r$ and vertices at the origin, a point on the "caustic" circle, and a point on the circumference of the new circle. Thus, using the law of sines, we have $\frac{sin(\frac{\pi}{2}-\psi')}{cos(\psi)} = 1/r$ which shows $r = cos(\psi)/cos(\psi') = 2cos(\psi)$. We are allowed to change the size of the billiard since for at least one point in its trajectory, the particle is $cos(\psi)$ distance from the origin (and $cos(\psi) \neq 0$ since the only quasi-periodic orbits have $\psi$irrational with respect to $\pi$).

To convert from a periodic orbit to a quasi-periodic orbit, we need only preform these operations in reverse, allowing a period-3 orbit in a billiard of radius $2cos(\psi)$ to devolve into a non-periodic orbit as we change the radius back to 1 while it passes through a point distance $cos(\psi)$ from the origin.

## References


[1] Chernov, Nikolai, Markarian Roberto. *Chaotic Billiards.* Providence: American Mathematical Society, 2006. Print, 19-25

[2] Tabachnikov, Serge. *Geometry and Billiards.* Providence: American Mathematical Society, 2005. Print.

[3] Alligood, Kathleen, Sauer, Tim, Yorke, James. *Chaos: An Introduction to Dynamical Systems.* New York: Textbooks in Mathematical Sciences, 1996. Print

[4] "Billiards and Chaos Theory." Serendip's Exchange. Web. 27 Nov. 2009. <http://serendip.brynmawr.edu/chaos/home.html>.