Appendix A:

Equations of motion in General Three-Body Problem

$$\vec{F} = \begin{bmatrix} \dot{\vec{x}}_1 \\ \dot{\vec{p}}_1 \\ \dot{\vec{x}}_2 \\ \dot{\vec{p}}_2 \\ \dot{\vec{x}}_3 \\ \dot{\vec{p}}_3 \end{bmatrix} = \begin{bmatrix} \vec{p}_1 \\ \frac{m_2}{\|\vec{x}_2 - \vec{x}_1\|^3}(\vec{x}_2 - \vec{x}_1) + \frac{m_3}{\|\vec{x}_3 - \vec{x}_1\|}(\vec{x}_3 - \vec{x}_1) \\ \vec{p}_2 \\ \frac{m_1}{\|\vec{x}_1 - \vec{x}_2\|^3}(\vec{x}_1 - \vec{x}_2) + \frac{m_3}{\|\vec{x}_3 - \vec{x}_2\|^3}(\vec{x}_3 - \vec{x}_2) \\ \vec{p}_3 \\ \frac{m_1}{\|\vec{x}_1 - \vec{x}_3\|^3}(\vec{x}_1 - \vec{x}_3) + \frac{m_2}{\|\vec{x}_2 - \vec{x}_3\|^3}(\vec{x}_2 - \vec{x}_3) \end{bmatrix}$$

Appendix B:

Derivation of the Lagrange - Jacobi identity: *

In the center of mass coordinate system, the moment of of inertia of the three-body system is:

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Or in the Lagrangian form:

$$I = \frac{m_1 m_2 m_3}{M}\left(\frac{r_{21}^2}{m_3} + \frac{r_{13}^2}{m_2} + \frac{r_{32}^2}{m_1}\right)$$

where:

$$M = m_1 + m_2 + m_3$$

$$r_{21} = \|\vec{r}_2 - \vec{r}_1\|$$

$$r_{13} = \|\vec{r}_1 - \vec{r}_3\|$$

$$r_{32} = \|\vec{r}_3 - \vec{r}_2\|$$

Differentiating I twice with respect to time gives

$$\dot{I} = 2m_1 \vec{r}_1 \cdot \dot{\vec{r}}_1 + 2m_2 \vec{r}_2 \cdot \dot{\vec{r}}_2 + 2m_3 \vec{r}_3 \cdot \dot{\vec{r}}_3$$

$$\ddot{I} = 2m_1 (v_1^2 + \vec{r}_1 \cdot \ddot{\vec{r}}_1) + 2m_2 (v_2^2 + \vec{r}_2 \cdot \ddot{\vec{r}}_2) + 2m_3 (v_3^2 + \vec{r}_3 \cdot \ddot{\vec{r}}_3)$$

$$\frac{1}{2}\ddot{I} = 2T + V = 2E - V$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2, \quad E = T + V, \quad V = -G\sum_{j=1}^{n}\sum_{k=j+1}^{n} \frac{m_j m_k}{r_{jk}}$$

* From Valtonen p32 - p33

Appendix C:

Translate Equations of Motion to Rotating Inertial Frame:

To go to the rotating inertial reference frame we first scale the problem to depend on a single parameter, μ.

- Mass of planet = μ
  Mass of Sun = $1-\mu$

$$\mu = \frac{\text{mass of planet}}{\text{total mass}} = \frac{m}{m+M}$$

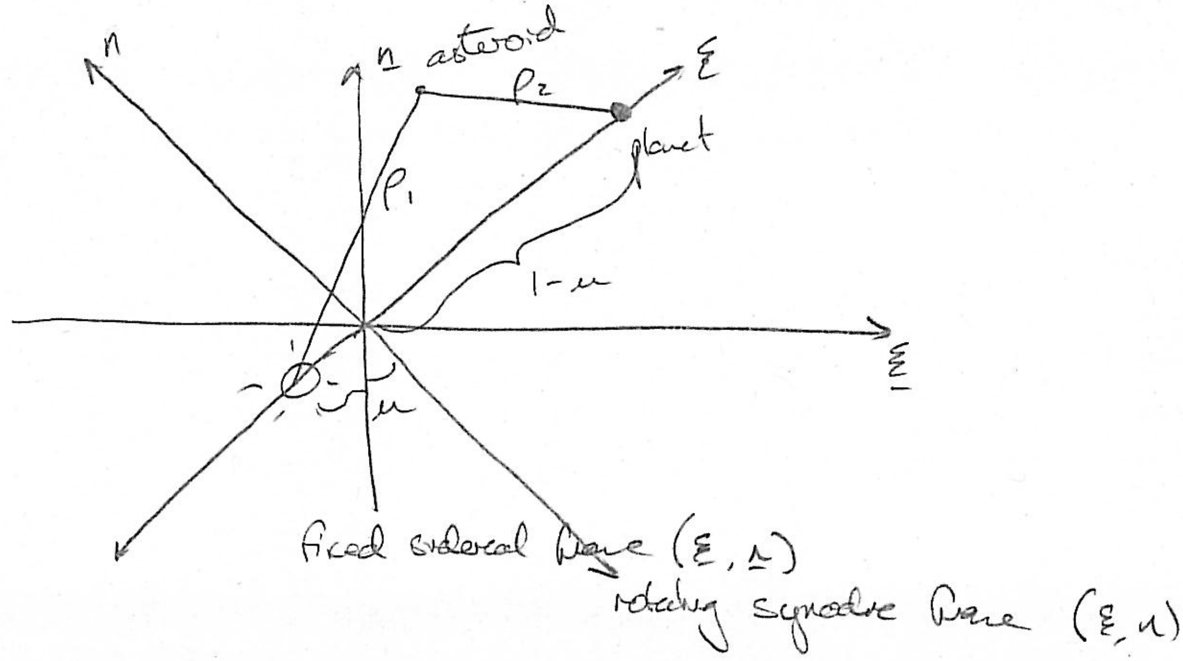
- distance between primaries (planet and Sun) = 1
  - distance from center of mass to planet = $1-\mu$
  - distance from center of mass to Sun = μ
- unit of time chosen so that mean motion of primaries is $n = 1$. hence mean anomaly = time
- From these it follows that the gravitational constant, $G = 1$.

To describe the rotating inertial frame, we first start in the fixed sidereal frame: $\underline{\xi}, \underline{\eta}$. In this frame the equations of motion are:

$$\text{Sun} \begin{cases} \underline{\xi} = -\mu\cos t \\ \underline{\eta} = -\mu\sin t \end{cases}$$

$$\text{Planet} \begin{cases} \underline{\xi} = (1-\mu)\cos t \\ \underline{\eta} = (1-\mu)\sin t \end{cases}$$

But, we want to translate to the rotating, synodic frame ξ, η where the primaries are stationary.



The Hamiltonian of the asteroid in the fixed frame is:

$$\underline{H} = \tfrac{1}{2}\left(p_{\underline{\xi}}^2 + p_{\underline{\eta}}^2\right) - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}$$

where

$$\rho_1 = \left((\underline{\xi} + \mu\cos t)^2 + (\underline{\eta} + \mu\sin t)^2\right)^{\frac{1}{2}}$$

$$\rho_2 = \left((\underline{\xi} - (1-\mu)\cos t)^2 + (\underline{\eta} - (1-\mu)\sin t)^2\right)^{\frac{1}{2}}$$

The fixed frame $\rightarrow$ rotating frame are related by the following equations:

$$\underline{\xi} = \xi\cos t - \eta\sin t$$

$$\underline{\eta} = \xi\sin t + \eta\cos t$$

The transformation can be accomplished using a generating function:

$$F = F(p_{\underline{\xi}}, p_{\underline{\eta}}, \xi, \eta)$$

$$= -(\xi\cos t - \eta\sin t)p_{\underline{\xi}} - (\xi\sin t + \eta\cos t)p_{\underline{\eta}}$$

The new momenta are:

$$P_\xi = -\frac{\partial F}{\partial \xi} = p_\xi \cos t + p_\eta \sin t$$

$$P_\eta = -\frac{\partial F}{\partial \eta} = -p_\xi \sin t + p_\eta \cos t$$

The new Hamiltonian is:

$$H = \underline{H} + \frac{\partial F}{\partial t} = \frac{1}{2}(P_\xi^2 + P_\eta^2) - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}$$

$$+ \xi p_\xi \sin t + \eta p_\xi \cos t + \xi p_\eta \cos t + \eta p_\eta \sin t$$

$$= \frac{1}{2}(P_\xi^2 + P_\eta^2) - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} + \xi(-P_\eta) + \eta P_\xi$$

where:

$$\rho_1^2 = (\xi + \mu)^2 + \eta^2$$

$$\rho_2^2 = (\xi - (1-\mu))^2 + \eta^2$$

The equations of motion in the synodic frame are now:

$$\dot{\xi} = \frac{\partial H}{\partial P_\xi} = P_\xi + \eta$$

$$\dot{\eta} = \frac{\partial H}{\partial P_\eta} = P_\eta - \xi$$

$$\dot{P}_\xi = -\frac{\partial H}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) + P_\eta = \frac{\partial}{\partial \xi}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) + \dot{\eta} + \xi$$

$$\dot{P}_\eta = -\frac{\partial H}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) - P_\xi = \frac{\partial}{\partial \eta}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) - \dot{\xi} + \eta$$

$$\dot{P}_\xi = \ddot{\xi} - \dot{\eta}, \quad \dot{P}_\eta = \ddot{\eta} + \dot{\xi}$$

So:

$$\ddot{\xi} - \dot{\eta} = \dot{\eta} + \xi + \frac{\partial}{\partial \xi}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)$$

$$\ddot{\eta} + \dot{\xi} = -\dot{\xi} + \eta + \frac{\partial}{\partial \eta}\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)$$

These equations simplify to:

$$\ddot{\xi} = 2\dot{\eta} + \xi - (1-\mu)(\xi+\mu)\left[(\xi+\mu)^2 - \eta^2\right]^{-3/2}$$
$$\qquad - \mu\left(\xi - (1-\mu)\right)\left[(\xi - (1-\mu))^2 + \eta^2\right]^{-3/2}$$

$$\ddot{\eta} = -2\dot{\xi} + \eta - \eta(1-\mu)\left[(\xi+\mu)^2 + \eta^2\right]^{-3/2}$$
$$\qquad - \eta\mu\left[(\xi - (1-\mu))^2 + \eta^2\right]^{-3/2}$$

These equations form the following system of coupled first order equations

$$\dot{\xi} = p$$
$$\dot{p} = 2q + \xi - (1-\mu)(\xi+\mu)\left[(\xi+\mu)^2 - \eta^2\right]^{-3/2}$$
$$\qquad - \mu\left(\xi - (1-\mu)\right)\left[(\xi - (1-\mu))^2 + \eta^2\right]^{-3/2}$$

$$\dot{\eta} = q$$
$$\dot{q} = -2p + \eta - \eta(1-\mu)\left[(\xi+\mu)^2 + \eta^2\right]^{-3/2}$$
$$\qquad - \eta\mu\left[(\xi - (1-\mu))^2 + \eta^2\right]^{-3/2}$$

q ?

From Valtonen p123 - 125

Appendix D

Lagrange Points: Locations and Stabilities

Location:

Lagrange points occur where $\dot{\xi} = \dot{\eta} = 0$
This occurs when $\dfrac{\partial \Omega}{\partial \xi} = \dfrac{\partial \Omega}{\partial \eta} = 0$ where

but Ω is not H the Hamiltonian — why not?

$$\Omega = \tfrac{1}{2}(\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}$$

ρ_1 & ρ_2 defined in Appendix C

because

$$\ddot{\xi} - 2\dot{\eta} = \frac{\partial \Omega}{\partial \xi} \quad \text{and} \quad \ddot{\eta} + 2\dot{\xi} = \frac{\partial \Omega}{\partial \eta}$$

Calculating these derivatives we get a pair of equations for the Lagrange equation

$$\frac{\partial \Omega}{\partial \xi} = \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi - (1-\mu))}{\rho_2^3} = 0 \quad (5.17)$$

$$\frac{\partial \Omega}{\partial \eta} = \eta - \frac{(1-\mu)\eta}{\rho_1^3} - \frac{\mu\eta}{\rho_2^3} = 0 \quad (5.18)$$

The second equation has the trivial solution where $\eta = 0$. The first equation then gives:

$$\xi - \frac{(1-\mu)(\xi+\mu)}{[(\xi+\mu)^2]^{3/2}} - \frac{\mu(\xi - (1-\mu))}{[(\xi - (1-\mu))^2]^{-3/2}} = 0$$

This has the solution:

$$\xi - \frac{1-\mu}{(\xi+\mu)^2} - \frac{\mu}{(\xi-(1-\mu))^2} = 0 \qquad 1-\mu < \xi$$

$$\xi - \frac{1-\mu}{(\xi+\mu)^2} + \frac{\mu}{(\xi-(1-\mu))^2} = 0 \qquad -\mu < \xi < 1-\mu$$

$$\xi + \frac{1-\mu}{(\xi+\mu)^2} + \frac{\mu}{(\xi-(1-\mu))^2} = 0 \qquad \xi < -\mu$$

These can be solved numerically.

For $\eta \neq 0$ we have the second equation

$$1 - \frac{(1-\mu)}{\rho_1^{\,3}} - \frac{\mu}{\rho_2^{\,3}} = 0. \qquad (5.18)$$

Multiplying by $\xi + \mu$ and subtracting the first equation (5.17)

$$\frac{\mu}{\rho_2^{\,3}} - \mu = 0$$

therefore $\rho_2 = 1$

Multiplying (5.18) by $\xi - (1-\mu)$ and subtracting (5.17) we get

$$1 - \mu - \frac{1-\mu}{\rho_1^{\,3}} = 0$$

~~therefore~~ $\rho_1 = 1$

Since $\rho_1 = 1$ and $\rho_2 = 1$ then the points are those on the equilateral triangle with the points of the primaries:

$$(\xi - (1-\mu))^2 + \eta^2 = 1$$

$$(\xi + \mu)^2 = \eta^2 = 1$$

$\therefore \quad \xi = \tfrac{1}{2} - \mu, \quad \eta = \pm \sqrt{3}/2$

Stability

We can explore the stability perturbing a body near a Lagrangian point.

Let (ξ_0, η_0) be any Lagrangian point and (x, y) the position of the body relative to this point:

$$x = \xi - \xi_0$$
$$y = \eta - \eta_0$$

Concentrating on the neighbourhood about the Lagrangian point we can linearly approximate the derivatives of Ω

$$\frac{\partial \Omega}{\partial \xi} = \frac{\partial \Omega}{\partial \eta} = 0$$

$$\frac{\partial \Omega}{\partial \xi} \approx x \frac{\partial^2 \Omega}{\partial \xi^2} + y \frac{\partial^2 \Omega}{\partial \xi \partial \eta}$$

$$\frac{\partial \Omega}{\partial \eta} \approx x \frac{\partial^2 \Omega}{\partial \xi \partial \eta} + y \frac{\partial^2 \Omega}{\partial \eta^2}$$

So the equations of motion are

$$\ddot{x} - 2\dot{y} = x \frac{\partial^2 \Omega}{\partial \xi^2} + y \frac{\partial^2 \Omega}{\partial \xi \partial \eta}$$
$$\ddot{y} + 2\dot{x} = x \frac{\partial^2 \Omega}{\partial \xi \partial \eta} + y \frac{\partial^2 \Omega}{\partial \eta^2}$$

Considering the three Lagrangian Points L1, L2, L3, the derivatives of Ω are:

$$\left.\begin{aligned} \frac{\partial^2 \Omega}{\partial \xi^2} &= 1 + 2\alpha \\ \frac{\partial^2 \Omega}{\partial \eta^2} &= 1 - \alpha \\ \frac{\partial^2 \Omega}{\partial \xi \partial \eta} &= 0 \end{aligned}\right\} \quad \text{where } \alpha = \frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}$$

as $\eta = 0$ for L1, L2 and L3

The equations of motion are now:

$$\ddot{x} - 2\dot{y} = x(1+2\alpha)$$
$$\ddot{y} + 2\dot{x} = y(1-\alpha)$$

We can study their trajectories to determine stability:

$$x = Ae^{\omega t}$$
$$y = Be^{\omega t}$$

where A, B, and ω are constants

If $Re(\omega) \neq 0$ then the solutions grow without limit and are unstable.

If $Re(\omega) = 0$ then the solutions are stable.

Substituting in the solution and performing some algebra we see:

$$A\omega^2 - 2B\omega = A(1+2\alpha)$$
$$B\omega^2 + 2A\omega = B(1-\alpha)$$

Eliminating A & B and performing more algebra we get

$$\omega^4 + \omega^2(2-\alpha) + (1+2\alpha)(1-\alpha) = 0$$

If $Re(\omega) = 0$ then there must be two negative solutions for ω^2.
The product of these roots must be positive, therefore:

$$(1+2\alpha)(1-\alpha) > 0$$

therefore: $\alpha < 1$

But the Lagrangian points must satisfy (5.17) from Appendix C

$$\frac{\partial \Omega}{\partial \xi} = \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-(1-\mu))}{\rho_2^2} = 0$$

Rearranging the terms

$$\xi - \xi\left[\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right] - \frac{\mu(1-\mu)}{\rho_1^3} + \frac{\mu(1-\mu)}{\rho_2^3} = 0$$

or $\quad \xi(1-\mu) - \mu(1-\mu)\left[\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right] = 0$

from which we see

$$1-\mu = \frac{\mu(1-\mu)}{\xi}\left[\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right]$$

Therefore for all Lagrangian points on ξ axis, like L1, L2, L3 we're considering, the bracketed expression and ξ have opposite signs. The right hand side must, therefore be negative, so:

$$\alpha > 1.$$

But this contradicts our previous requirement. Therefore we cannot find purely imaginary solutions for ω for L1, L2, L3. Therefore L1, L2, L3 must be unstable

For L4, however, we see that the linearized equations of motion are

$$\ddot{x} - 2\dot{y} = \frac{3}{4}x + \frac{3\sqrt{3}}{4}(1-2\mu)y$$

$$\ddot{y} + 2\dot{x} = \frac{9}{4}y + \frac{3\sqrt{3}}{4}(1-2\mu)x$$

And therefore the trial solution

$$x = Ae^{\omega t}$$
$$y = Be^{\omega t}$$

gives:

$$\omega^4 + \omega^2 + \frac{27}{4}\mu(1-\mu) = 0$$

Because

$$\omega_1^2\omega_2^2 = \frac{27}{4}\mu(1-\mu) > 0$$

The possible real roots have the same sign, furthermore since:

$$\omega_1^2 + \omega_2^2 = -1 < 0$$

Both roots ω_i^2 must be negative. Since we require that

$\omega^4 + \omega^2 + \frac{27}{4}\mu(1-\mu) = 0$ has real roots and thus

ω is imaginary

<u>or</u>:

$$27\mu^2 - 27\mu + 1 > 0$$

This happens when

$$\mu < \frac{1}{2} - \sqrt{\frac{23}{108}} \approx .0385$$

So L_4 and L_5 have stable orbits when $\mu < .0385$.

Appendix E: Df of Three-Body Problem in Rotating Inertial Frame

Equations of Motion in Rotating Inertial Frame

$$\vec{F} = \begin{pmatrix} \dot{\xi} \\ \dot{p} \\ \dot{\eta} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} p \\ 2q + \xi - (1-\mu)(\xi+\mu)\left[(\xi+\mu)^2 + \eta^2\right]^{-3/2} - \mu(\xi - 1 + \mu)\left[(\xi - 1 + \mu)^2 + \eta^2\right]^{-3/2} \\ q \\ -2p + \eta - \eta(1-\mu)\left[(\xi+\mu)^2 + \eta^2\right]^{-3/2} - \eta\mu\left[(\xi - 1 + \mu)^2 + \eta^2\right]^{-3/2} \end{pmatrix}$$

$$\overrightarrow{Df} = \begin{pmatrix} \frac{\partial \dot{\xi}}{\partial \xi} & \frac{\partial \dot{\xi}}{\partial p} & \frac{\partial \dot{\xi}}{\partial n} & \frac{\partial \dot{\xi}}{\partial q} \\ \frac{\partial \dot{p}}{\partial \xi} & \frac{\partial \dot{p}}{\partial p} & \frac{\partial \dot{p}}{\partial n} & \frac{\partial \dot{p}}{\partial q} \\ \frac{\partial \dot{n}}{\partial \xi} & \frac{\partial \dot{n}}{\partial p} & \frac{\partial \dot{n}}{\partial n} & \frac{\partial \dot{n}}{\partial q} \\ \frac{\partial \dot{q}}{\partial \xi} & \frac{\partial \dot{q}}{\partial p} & \frac{\partial \dot{q}}{\partial n} & \frac{\partial \dot{q}}{\partial q} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial \dot{p}}{\partial \xi} & 0 & \frac{\partial \dot{p}}{\partial n} & 2 \\ 0 & 0 & 0 & 1 \\ \frac{\partial \dot{q}}{\partial \xi} & -2 & \frac{\partial \dot{q}}{\partial n} & 0 \end{pmatrix}$$

$$\dot{p} = 2q + \xi - (1-\mu)(\xi+\mu)\left[(\xi+\mu)^2 + n^2\right]^{-3/2} - \mu(\xi-1+\mu)\left[(\xi-1+\mu)^2 + n^2\right)^{-3/2}$$

$$\frac{\partial \dot{p}}{\partial \xi} = 1 - \Big\{ (1-\mu)(1)\left[(\xi+\mu)^2 + n^2\right]^{-3/2} + (1-\mu)(\xi+\mu)(-3/2)\left[(\xi+\mu)^2+n^2\right]^{-5/2} 2(\xi+\mu)(1) \Big\}$$
$$- \Big\{ \mu(1)\left[(\xi-1+\mu)^2 + n^2\right]^{-3/2} + \mu(\xi-1+\mu)(-3/2)\left[(\xi-1+\mu)^2+n^2\right]^{-5/2} 2(\xi-1+\mu)(1)$$

$$\frac{\partial \dot{p}}{\partial n} = -(1-\mu)(\xi+\mu)(-3/2)\left[(\xi+\mu)^2+n^2\right]^{-5/2}(2n) - \mu(\xi-1+\mu)(-3/2)\left[(\xi-1+\mu)^2+n^2\right]^{-5/2}(2n)$$

$$\dot{q} = -2p + n - n(1-\mu)\left[(\xi+\mu)^2+n^2\right]^{-3/2} - n\mu\left[(\xi-1+\mu)^2+n^2\right]^{-3/2}$$

$$\frac{\partial \dot{q}}{\partial \xi} = -n(1-\mu)(-3/2)\left[(\xi+\mu)^2+n^2\right]^{-5/2} 2(\xi+\mu) - n\mu(-3/2)\left[(\xi-1+\mu)^2+n^2\right]^{-5/2} 2(\xi-1+\mu)$$

$$\frac{\partial \dot{q}}{\partial n} = 1 - \Big\{ (1-\mu)\left[(\xi+\mu)^2+n^2\right]^{-3/2} + n(1-\mu)(-3/2)\left[(\xi+\mu)^2+n^2\right]^{-5/2} 2n \Big\}$$
$$- \Big\{ \mu\left[(\xi-1+\mu)^2+n^2\right]^{-3/2} + n\mu(-3/2)\left[(\xi-1+\mu)^2+n^2\right]^{-5/2}(2n) \Big\}$$