Discrete Chaotic Dynamical Systems in Economic Models

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Abstract

In this paper we will examine three simple economic models from a mathematical perspective, and prove the existence of chaos in each. We will analyze a supply and demand model, a model for endogenous fluctuations in a monetary economy, and a model for the relationship between unemployment and inflation, and show how each model can exhibit chaotic behavior for reasonable values of the relevant parameters. No extensive knowledge of economics is assumed; all economic terms are defined and explained.

1 A Simple Supply and Demand Model

This section addresses a simple model of supply and demand, and how for certain parameter values, even this elementary model can display chaotic behavior. First, we define the notion of a supply and demand model. Note that the model that follows is from Zhang [5, p.178-83].

1.1 What is a Supply and Demand Model?

Supply and demand models are among the simplest classes of economic models. Simply stated, as the name implies, a supply and demand model is an economic model of the market demand and market supply for a given good or service. In particular, a supply and demand model is an effective means of modeling how market forces determine the price, the quantity supplied by producers, and the quantity demanded by consumers, of a good or service. At the market-clearing equilibrium, we have the condition that the demand equals the supply.

1.2 Deriving the Zhang Supply and Demand Model

What follows is an explanation of the basic assumptions inherent in this specific model, following from Zhang. First, we consider that there is a time lag in supply, which results because producers must decide how much to produce before prices are observed in the market. This is reflected in the model by the fact that quantity supplied will be given

as a function of the expected price, and not the actual price. Note that expected price simply refers to the mathematical expected value of the price.

Next, we assume that the supply curve exhibits an S-shape, as is standard. Start-up costs and fixed costs of production for producers will limit supply, and force the quantity supplied to increase slowly at low prices. Likewise, supply will increase slowly at high prices due to constraints on the amount a producer is physically able to produce. Observing that the arctan function exhibits such a shape, we may then express the quantity supplied at a time t, $q^s(t)$, as a function of the expected price at a time t, $p^e(t)$, by

$$q^s(t) = \arctan(\mu p^e(t)). \tag{1.1}$$

Supply is symmetric about the origin and has an inflection point there, following from properties of the arctan function. As noted in Figure 1, the steepness of the S -shape curve is determined by the parameter μ.

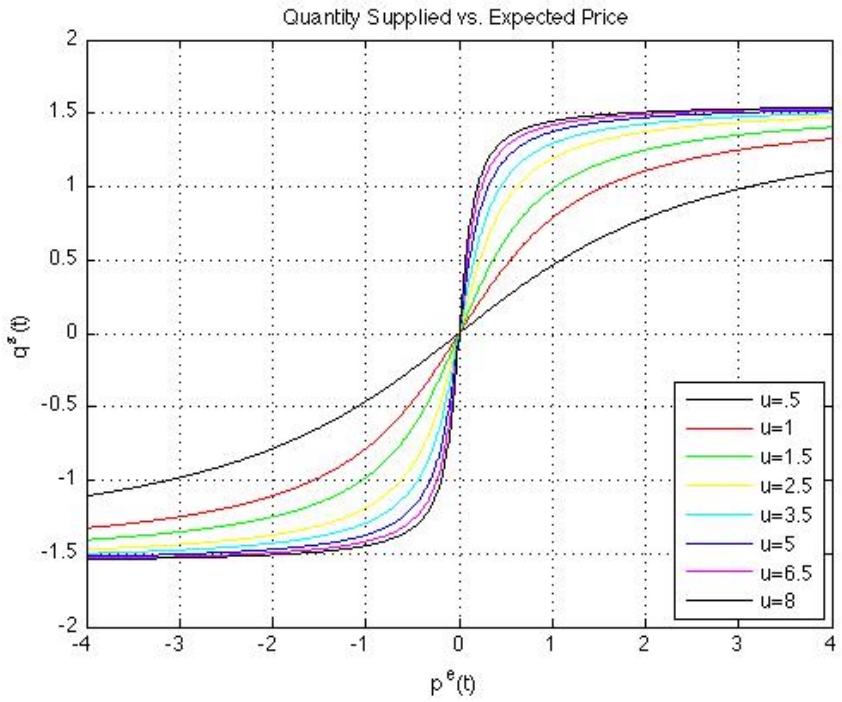


Figure 1: The relationship between supply and expected price

Note that high values of μ correspond to a very steep curve near the origin that quickly flattens out, while low values correspond to a less steep curve near the origin that takes longer to flatten.

Moving on to demand, we make the assumption that the demand curve is linear, for simplicity. Then, we can represent the quantity demanded at a time t, $q^d(t)$, as a linear function of the actual price at a time t, $p(t)$, by

$$q^d(t) = a - bp(t), b > 0 \tag{1.2}$$

with parameters a and b.

Now, we assume that the expected price, $p^e(t+1)$, at a time $t+1$ is given as a function of the actual price at t, $p(t)$, and the expected price at t, $p^e(t)$. That is, the price expectation satisfies

$$p^e(t+1) = \lambda p(t) + (1-\lambda)p^e(t) \tag{1.3}$$

where λ is a parameter.

At market equilibrium, we have that the quantity supplied equals the quantity demanded, or $q^s(t) = q^d(t)$. Thus, setting equation (1.1) equal to equation (1.2) and simplifying, we obtain the following expression for price, in terms of expected price

$$p(t) = \frac{a}{b} - \frac{\arctan(\mu p^e(t)}{b}.$$

Substituting the above equality into equation (1.3) and simplifying, we can then express price expectation by the difference equation

$$p^e(t+1) = (1-\lambda)p^e(t) + \frac{a\lambda}{b} - \frac{\lambda \arctan(\mu p^e(t)}{b} \equiv f(p^e(t)). \tag{1.4}$$

1.3 Numerical Analysis

We demonstrate the dynamic behavior of this model numerically. For the remainder of this section, we fix $\lambda = 0.3$ and $b = 0.25$. We now investigate bifurcation diagrams for this model, allowing a to be a bifurcation parameter as we fix different values for μ.

Consider the case where $\mu = 1$. Figure 2 below depicts the bifurcation diagram, where $a \in [-1.4, 1.4]$.

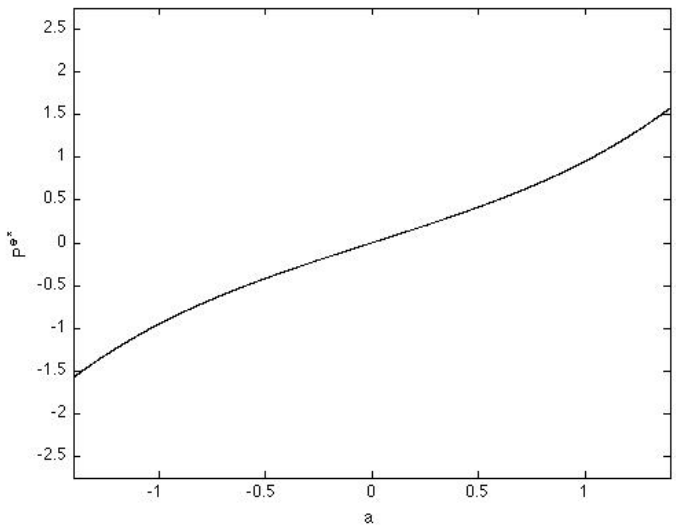


Figure 2: Bifurcation Diagram for $\mu = 1$ and $a \in [-1.4, 1.4]$

Note that for these values, the map f as defined in equation (1.4) contains a unique fixed point for all values of a, as evidenced by Figure 2.

Next, consider the case where we increase the value of μ to 3. The corresponding bifurcation diagram with a as the bifurcation parameter is given below in Figure 3.

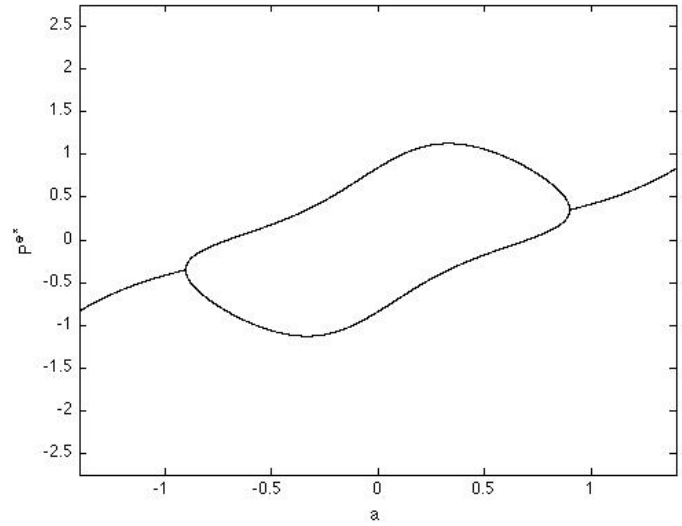


Figure 3: Bifurcation Diagram for $\mu = 3$ and $a \in [-1.4, 1.4]$

For low values of a, the map f once again contains a single unique fixed point. Then, at roughly $a = -0.9$, a period-doubling bifurcation occurs. For values of a larger than -0.9, there are now two fixed points for the map. This stable period-2 orbits persists until roughly $a = 0.9$. At this point, a period-halving bifurcation occurs, and for values of a greater than 0.9, there is once again a single, unique fixed point for the map, which corresponds to a unique stable equilibrium.

Now, we increase the value of μ, and specify that $\mu = 3.5$. Figure 4 depicts the corresponding bifurcation diagram for this scenario.

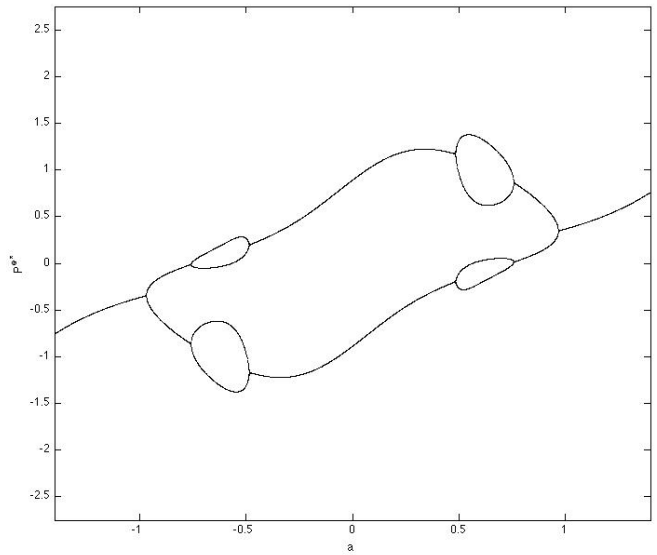
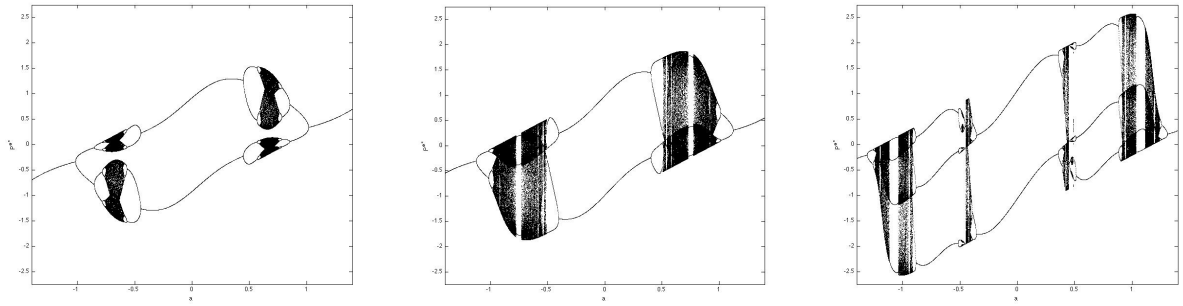


Figure 4: Bifurcation Diagram for $\mu = 3.5$ and $a \in [-1.4, 1.4]$

As in Figure 3, there are period-doubling and period-halving bifurcations at approximately $a = -0.9$ and $a = 0.9$, respectively. However, additional period-doubling bifurcations that occur within this diagram. Figure 4 thus shows the existence of a period-4

orbit for certain values of a, which eventually turns back into a period-2 orbit and finally a stable equilibrium with a single, unique fixed point.

We conclude this section by examining the bifurcation diagrams for larger values of μ. Figure 5 displays the bifurcation diagrams for μ values of 4, 5.7, and 15. Note the presence of chaos within the period four orbits in Figure 5a. In Figure 5b, there clearly exist period three orbits within a certain window of values for the parameter a. Since the existence of a period three orbit in a continuous map guarantees orbits of all periods, as well as sensitive dependence, we know that Figure 5b also exhibits chaos [1, p.32-5]. Finally, as we increase μ to 15, the bifurcation diagram becomes much more complicated, but because a period three orbit still exists, we are guaranteed chaos.



(a) $\mu = 4$ (b) $\mu = 5.7$ (c) $\mu = 15$

Figure 5: Bifurcation Diagrams

The reader is directed to the Appendix for the source code to reproduce these plots in *Matlab*. Additionally, the appendix contains the code to create a bifurcation video, which shows how the bifurcation diagrams for the bifurcation parameter a changes as we slowly increase the value of μ from 0 to 15.

2 A Model of a Monetary Economy

This section identifies and analyzes endogenous fluctuations in a monetary economy, first proposed by Matsuyama [3]. That is, the model presented in this section examines how internal forces within a monetary economy are able to bring about irregular variations in the economy. The goal of this section is to rigorously prove that, for specific values of some parameters, the model must exhibit chaos. Before we present the model itself, though, first we present enough background economics to make sense of this model.

2.1 Basic Economics

In this model, the assumption is made that the monetary economy in question is inhabited by an agent with an infinite lifetime and perfect foresight. Within economics, the term *agent* is simply used to refer to a decision maker within a model, and can be used to indicate a specific person, household, firm, or even government. In particular, the agent in this model is assumed to have an infinite lifetime, and to possess *perfect foresight*, or the ability to predict future prices perfectly.

The goal of the agent within the model is to maximize his *utility*. Within economics, utility is simply a measure of consumer satisfaction. In this specific model, we assume that the agent derives satisfaction from both the consumption of goods and the possession of money. Hence, the agent's utility function will be given as a function of consumption and real balance. Note that *real balance* simply refers to the actual amount of money the agent possesses, adjusted for inflation.

2.2 Derivation of the Model

A fundamental assumption of this model is that the agent will maximize his total utility. However, since the agent is assumed to be infinitely lived, he will have a utility function for each of the infinite number of periods that he is alive. Hence, he will maximize not his specific utility in one period t, but rather the sum of his utility over all periods. So, the agent maximizes

$$\sum_{t=0}^{\infty} \beta^t U(c(t), m(t))$$

where, as previously discussed, his utility U in the current period, t, is expressed as a function of his current consumption, $c(t)$, and real balance, $m(t)$. In this equation, $\beta \in (0, 1)$ is the discount factor, a parameter that determines how heavily the agent discounts his utility in the future time periods. That is, β describes how much less the agent cares about his utility in future periods compared to the amount he cares about his utility in the current period.

However, clearly the agent must have some constraints on his consumption and real balance, or else his utility would always be trivially infinite. We express this constraint through the following flow budget constraint, given $M(-1)$:

$$M(t) = P(t)(y - c(t)) + H(t) + M(t-1).$$

This equation expresses the agent's limited budget in the period t. In this equation y is the constant endowment of perishable consumption goods, meaning y denotes a parameter describing the constant level of a consumption good that the agent has already been given. $M(t)$ is simply the agent's nominal money holdings in the period t, where nominal money holdings refers to the amount of money the agent possesses unadjusted for inflation; hence, the agent holds this money in name only. We can relate $M(t)$ to the agent's real balance $m(t)$ by

$$m(t) = \frac{M(t)}{P(t)}$$

where $P(t)$ is the current price level. The agent considers this price level during every period, $\{P(t)\}_{t=0}^{\infty}$, to be independent of his own money holdings, meaning $P(t)$ never depends on $M(t)$ or $m(t)$. Additionally, we specify that at the beginning of each period t, the agent receives $H(t)$ units of paper money from the government, an event considered independent of the agent's previous money holdings. In effect, $H(t)$ represents a "helicopter drop" of money to the agent in every period.

We assume that the money supply in the economy grows at the rate μ, where $\mu > \beta$. Hence, by the definition of μ we have that $M(t) = \mu M(t-1)$. Then, by observing that

we can express $H(t)$ as the difference in the nominal money holdings during periods $t-1$ and t, we may use the previous equation to obtain

$$H(t) = (\mu - 1)M(t-1)$$

Hence, the markets will clear when we have

$$M(t) = \mu^t M_0,\ c(t) = y \ \ \forall\ t.$$

Following from Matsuyama, we observe that an equilibrium point of this economy can be given by a nonnegative sequence of real balances that satisfy

$$\beta U_c(y, m(t+1))m(t+1) = \mu m(t)[U_c(y, m(t)) - U_m(y, m(t))], \tag{2.1}$$

where U_c and U_m denote the respective partial derivatives of U. In addition, for an equilibrium point to occur we require that the transversality condition

$$\lim_{t\to\infty} \beta^t U_c(y, m(t))m(t) = 0$$

be satisfied. A transversality condition in an economic model is simply a condition on variables inherent to the model. Since it typically appears only for optimization problems in which there is an infinite horizon, it generally takes the form of a limit of some quantity as time approaches infinity.

The steady state of this system is simply given by $m(t) = m^*$. However, since we also have that $M(t) = \mu^t M_0$ and $m(t) = M(t)/P(t)$, this then implies that the steady state can be rewritten as

$$P(t) = \mu^t \frac{M_0}{m^*} \ \ \forall\ t.$$

Note that $m^* > 0$, m^* must exist uniquely, and that m^* must satisfy

$$(\mu - \beta)U_c(y, m^*) = \mu U_m(y, m^*),$$

where the above is obtained by substituting $m(t+1) = m(t) = m^*$ into equation (2.1).

Following Matsuyama, we make the simplifying assumption that the utility function is of the form

$$U(c, m) = \begin{cases} -\frac{[g(c)m]^{-(1+\eta)}}{1+\eta} & \text{for } \eta \neq -1, \\ \log g(c) + \log m & \text{otherwise} \end{cases} \tag{2.2}$$

where g is simply some arbitrary function of c satisfying $g > 0, g' > 0$, and

$$\sup\left[\frac{gg''}{g'^2}\right] < 1.$$

Additionally, η is a parameter satisfying

$$(\eta + 2)\left[2 - \sup\left(\frac{gg''}{g'^2}\right)\right] > 1.$$

The parameter η has an economic interpretation, since η relates to the elasticity of intertemporal substitution of real balances, σ, through

$$\sigma = (\eta + 2)^{-}1.$$

The elasticity of intertemporal substitution of real balances is a term that describes the responsiveness of the growth rate of real balances to the real interest rate, or the rate of interest an investor expects to receive after allowing for inflation.

Substitution of equation (2.2) into equation (2.1) yields

$$p(t+1) = (1+\delta)^{1/\eta} p(t)(1-p(t))^{1/\eta} \equiv F(p(t)) \quad \forall\, t. \tag{2.3}$$

We define the new parameter δ and the new variable $p(t)$ by

$$\delta \equiv \frac{\mu}{\beta} - 1 > 0, \; p(t) \equiv \frac{g(y)}{g'(y)m(t)}, \quad p(t) \in (0,1).$$

For this new variable p, the previous transversality condition now becomes

$$\lim_{t\to\infty} \beta^t p^\eta(t) = 0.$$

We can also determine the unique steady state in terms of these new parameters:

$$p^* = \frac{\delta}{1+\delta}.$$

We rule out the case of $\eta = 0$ for simplicity. For the remainder of this section, we require that $\eta > 0$. The special case where $\eta = 1$ is a well-known case, since substituting $\eta = 1$ into equation (2.3) yields

$$F(p(t)) = (1+\delta)p(t)(1-p(t)), \; p(t) \in (0,1). \tag{2.4}$$

This is simply the well-studied logistic map, which is analyzed in most basic texts on chaos [1, p.17-31]. It contains periodic, aperiodic, and chaotic solutions to the given system.

2.3 Proving the Existence of Chaos

The remainder of this section is dedicated to proving the existence of period three orbits, which guarantees both orbits of all periods and an uncountably infinite set of points displaying sensitive dependence. The proof that the existence of a period three orbit implies chaos can be found in texts on chaos [1, p.32-5].

At this point we cite five properties of the function F. The proof of these properties is straightforward and contained in Matsuyama.

1. $F(0) = F(1) = 0$
2. F has a single peak at $\tilde{p} = \frac{\eta}{1+\eta}$. F is strictly increasing on $[0, \tilde{p})$ and strictly decreasing on $(\tilde{p}, 1]$
3. $F'(0) = (1+\delta)^{1/\eta} > 1$
4. $F'(p^*) = 1 - \frac{\delta}{\eta}$

5. Define $\Delta(\eta)$ by $\Delta(\eta) \equiv \eta^{-\eta}(1+\eta)^{1+\eta} - 1$. If $\delta \leq (>)1$, then $F(\tilde{p}) \leq (>)1$. F maps $[0,1]$ onto itself if $\delta \leq \Delta(\eta)$, and F maps $(0,1)$ into itself if $\delta < \Delta(\eta)$. Finally, the function $\Delta(\eta)$ defined on $(0,\infty)$ is strictly increasing, and

$$\lim_{\eta \to \infty} \Delta(\eta) = 0, \ \Delta(\eta) > 0, \quad \forall \ \eta.$$

Next, we cite a proposition from Matsuyama that will aid in our eventual proof of the existence of chaos in the model.

Proposition 2.1

1. If $0 < \delta \leq 2\eta$, then for all $p_0 \in (0,1)$,

$$\lim_{t \to \infty} F^t(p_0) = p^*$$

2. If $0 < 2\eta \leq \delta$, then a period-2 orbit must exist, and the set $p_0 \in (0,1)$ such that $F^t(p_0)$ converges, denoted N, is at most a countable set. Additionally, if $2\eta \leq \delta < \Delta(\eta)$, then the set of initial prices that lead to equilibrium points along which the price level will fluctuate forever, denoted N^*, is of full Lebesgue measure.

At this point, we are finally ready to prove the following proposition, which guarantees the existence of chaos in the model:

Proposition 2.2 For any $\eta > 0$, there exists a value $\Delta^*(\eta)$ satisfying $2\eta < \Delta^*(\eta) < \Delta(\eta)$, such that a period-3 orbit of F exists if $\delta > \Delta^*(\eta)$.

Proof: We define the following function $G(p)$ by:

$$G(p) \equiv \left(\frac{F^3(p)}{p} \right)^{\eta}.$$

Recall that since p^* is the steady state that, by definition, $F(p^*) = p^*$. This implies that $F^3(p^*) = p^*$ and so $G(p^*) = 1$. Consider some $p_0 \in (0, \tilde{p})$ that is close to 0. Since $F'(0) > 1$ and F is strictly increasing on $[0, \tilde{p})$, we have that $F(p_0) > p_0$ for values of p_0 near 0. But then this implies that $F^3(p_0) > p_0$, and so $\frac{F^3(p_0)}{p_0} > 1$ and $G(p_0) > 1$. Taking the limit as $p_0 \to 0^+$ then yields $G(0^+) > 1$.

We have that $G(0^+) > 1$ and $G(p^*) = 1$. Hence, by the Intermediate Value Theorem, it suffices to show that there exists some $p_c \in (0, \tilde{p})$ such that $G(p_c) < 1$. To see this, note that $G(p_c) < 1$ implies there exists some $p' \in (0, \tilde{p})$ such that $G(p') = 1$ by the Intermediate Value Theorem. But, $G(p') = 1$ implies that $\left(\frac{F^3(p')}{p'} \right)^{\eta} = 1$. Since we assume $\eta > 0$ as stated previously, we have that $F^3(p') = p'$ which suggests that p' is a point in a period-3 orbit of F. Clearly, p' is not a fixed point of F since the only fixed points are 0 and p^*, so p' must correspond to a period-3 orbit. Therefore, if we can show that there exists some $p_c \in (0, \tilde{p})$ such that $G(p_c) < 1$, the proof is complete.

Now, if $\delta > \Delta(\eta)$ then by Property 5 for F we have that $F(\tilde{p}) > 1$. From Property 1 of F, we know $F(0) = 0$. Since F is strictly increasing on $[0, \tilde{p})$, then by the Intermediate Value Theorem we have that there must exist some $p_c \in (0, p^*)$ such that $F(p_c) = 1$. But, $F(p_c) = 1$ implies $F^2(p_c) = F^3(p_c) = 0$, and so we have that $G(p_c) = 0 < 1$. Clearly, for

$\delta > \Delta(\eta)$ we have found a p_c with the required property; all that remains is to show that there exists some $\Delta^*(\eta)$ satisfying $2\eta < \Delta^*(\eta) < \Delta(\eta)$ where, if $\delta > \Delta^*(\eta)$, there still exists such a p_c.

Consider the case where $\delta < \Delta(\eta)$. Now, recall that for $\delta > \Delta(\eta)$ we have that there exists some $p_c \in (0, p^*)$ such that $G(p_c) = 0$. Hence, by the continuity of G on δ, if we specify that $\delta < \Delta(\eta)$, then there must exist some $p_c \in (0, p^*)$ such that $G(p_c) < 1$, since if there exists $p_c \in (0, p^*)$ such that $G(p_c) = 0$ for $\delta > \Delta(\eta)$, then for some values of $\delta < \Delta(\eta)$ there must exist $p_c \in (0, p^*)$ such that $G(p_c) < 1$, although we do not necessarily have $G(p_c) = 0$. At some value of $\delta < \Delta(\eta)$, however, we do not necessarily have that there exists some $p_c \in (0, p^*)$ such that $G(p_c) < 1$. Denote the threshold value of δ where we have that $G(p_0) \geq 1$ for any $p_0 \in (0, p^*)$ by $\Delta^*(\eta)$, so that for $\Delta^*(\eta) < \delta < \Delta(\eta)$ we have that there exists some $p_c \in (0, p^*)$ such that $G(p_c) < 1$, while for $\delta < \Delta^*(\eta)$ we do not have some $p_c \in (0, p^*)$ such that $G(p_c) < 1$.

The only thing that remains to be shown is that $2\eta < \Delta^*(\eta)$. This follows directly from Statement 1 of Proposition 2.1. If $0 < \delta \leq 2\eta$, we have that $\forall\, p_0 \in (0, 1)$, $\lim_{t\to\infty} F^t(p_0) = p^*$, meaning that there cannot exist a $p_c \in (0, p^*)$ such that $G(p_c) < 1$ for $\delta \leq 2\eta$. This is because the condition that $G(p_c) < 1$ implies the existence of orbits of all periods, but Proposition 2.1 forces all orbits to converge to the fixed point. Hence, we have that $2\eta < \Delta^*(\eta)$. This completes our proof. $\square$

Therefore, we have shown that for any $\eta > 0$, there exists a value $\Delta^*(\eta)$ satisfying $2\eta < \Delta^*(\eta) < \Delta(\eta)$, such that a period-3 orbit of F exists if $\delta > \Delta^*(\eta)$. Thus, this model exhibits chaos for certain values of the parameter δ.

3 Chaos in a Model of Unemployment and Inflation

This section explores a model of unemployment and inflation, as originally proposed by Neugart [4]. It is a model of worker flow, meaning that it shows how unemployment changes in discrete time as some workers become unemployed and others find jobs. First, we will derive the model showing how the levels of unemployment and inflation in a period $t + 1$ depends on the levels of unemployment and inflation in the period t. Then, we numerically explore various aspects of the model.

3.1 Derivation of the Model

3.1.1 Deriving the Unemployment Model

We begin with the basic assumption that the unemployment level will change only if the inflow and outflow rate of the unemployment sector differ. Thus, we are able to express the unemployment U_t by the equation

$$U_{t+1} - U_t = i(L - U_t) - o_t U_t. \tag{3.1}$$

The parameter i, where we specify $i > 0$, denotes the inflow rate of workers that are entering the unemployment sector. People enter unemployment because of structural shifts in the economy, causing their jobs to be reallocated elsewhere and thereby forcing the worker into unemployment. Note that the parameter i is *exogenous*, meaning that its value is predetermined and unrelated to any of the other quantities present in the model. For the time being, we assume that i is constant for simplicity, although when numerically

investigating the model we will change the value of i. L simply refers to the labor force, or the total number of people that are able to work, regardless of their employment status, and for simplicity, we specify that $L = 1$. Finally, o_t is the outflow rate from unemployment, meaning it is simply the rate of workers leaving the unemployment sector. We define o_t as the fraction of jobs that come to the market at a time t to the total number of people seeking jobs. That is,

$$o_t = \frac{J_t}{U_t + d(1 - U_t)}. \tag{3.2}$$

The parameter d describes the number of currently employed workers who are searching for jobs as a constant fraction of total employed workers, and we specify that $0 < d < 1$. In addition, J_t denotes job creations. Note that this formulation of the outflow rate tells us two important qualitative things about the overall unemployment level. First, for a given number of jobs in the economy, an unemployed worker is less likely to leave unemployment and find work if there is an increasing number of other workers who are also unemployed and seeking jobs. This makes sense, and is supported by the model through the fact that o_t will clearly decrease if we increase U_t and hold everything else constant. Second, it is also less likely that an unemployed worker will be able to leave unemployment and find a job if they are forced to compete with a higher number of already employed workers also currently searching for jobs. Again, this basic intuition is supported in the model by the simple fact that an increase in d will cause o_t to decrease.

Now, let J_t be given by

$$J_t = J_s + \gamma(m - \pi_t). \tag{3.3}$$

In this formulation, J_s is a parameter denoting the job creation due to the structural characteristics of the economy, and the $\gamma(m - \pi_t)$ term describes the cyclical component of job creations. We have γ as a positive parameter, and m is the exogenous money growth rate, which is simply the growth rate in the total amount of money available in the economy at a given time. In addition, π_t refers to the specific inflation rate at t. Although an intuitive term, recall that *inflation rate* simply refers to the rate at which the general level of prices of goods and services increases in the economy. Thus, the above formulation reveals that the total number of jobs created in a period depends on both the constant, structural characteristics of the economy and on the cyclical term, which itself depends on fluctuating money growth and inflation rates. Note that if the inflation rate exceeds the money growth rate, then there will intuitively be a decrease in the total supply of money in the economy, leading to a loss in jobs. This is in accordance with equation (3.3), since $\pi_t > m$ implies that J_t will decrease. Conversely, if the inflation rate does not exceed the money growth rate, then there will be an increase in the total supply of money in the economy, leading to an increase in jobs; this also follows from equation (3.3) since $m > \pi_t$ implies that J_t will increase.

We may substitute equation (3.3) into (3.2) in order to express the outflow rate from unemployment as

$$o_t = \frac{J_s + \gamma(m - \pi_t)}{U_t + d(1 - U_t)}. \tag{3.4}$$

At this point, we rearrange equation (3.1) to yield

$$U_{t+1} = U_t + i(L - U_t) - o_t U_t$$

Substituting equation (3.4), and the fact that $L = 1$, into the above we finally have an equation for unemployment in the period $t + 1$ in terms of unemployment and inflation during the period t:

$$\begin{aligned} U_{t+1} &= U_t + i(1 - U_t) - U_t \frac{J_s + \gamma(m - \pi_t)}{U_t + d(1 - U_t)} \\ &\equiv f(U_t, \pi_t). \end{aligned} \tag{3.5}$$

3.1.2 Deriving the Inflation Model

Following the model proposed in Neugart, we suppose that future inflation rates will be governed solely by the expected inflation rate, and by the wage gap $\frac{w_{b,t} - w_p}{w_p}$. That is, we define the inflation rate π_t by

$$\pi_t = \frac{1}{\delta} \left(\pi_t^e + \frac{w_{b,t} - w_p}{w_p} \right). \tag{3.6}$$

$w_{b,t}$ is simply the wage, adjusted for inflation, that workers would bargain with employers for, while w_p refers to the wage, adjusted for inflation, that is determined solely by price. That is, $w_{b,t}$ is the bargained real wage, while w_p is the price determined real wage. The expected inflation rate during t is represented by π_t^e. We make the assumption that firms may only change their prices by some fraction of nominal wage during any given time period t, where nominal wage simply refers to wage unadjusted for inflation. This assumption reflects the requirement that $\delta > 1$. Our reasoning behind having inflation rates partly driven by the previously defined wage gap is very intuitive, and is as follows.

If unemployment rates are low, available labor is low and workers are in a good position, causing bargained real wage to exceed price determined real wage. This inequality will cause firms to raise prices, and hence increase the inflation rate, in order to try to capture their share of the output. Conversely, if unemployment is high, workers are not in a position to bargain and so price determined real wage will exceed bargained real wage. This will result in a decline in prices, and so a falling inflation rate, as claims on output fall short of what they were previously.

Following Neugart, we express the price determined real wage by

$$w_p = (1 - \mu)y. \tag{3.7}$$

The parameter y is the constant marginal labor productivity, defined to be a constant that represents the gain to productivity from hiring an additional worker. For simplicity, we normalize to the value $y = 1$. The parameter $\mu \geq 0$ is the inverse of the demand elasticity, also known as the fixed mark-up, and it relates to a firm's ability to price a good over its cost.

Typically, the bargained real wage is determined in a way that takes into account the value that a worker derives from having a job and not being unemployed, and the value that a firm derives from having a job filled. It makes sense that the bargained real wage is a decreasing function of the unemployment rate, since at higher unemployment rates it is easier for firms to fill jobs and more difficult for workers to acquire jobs. Hence, at high levels of unemployment most of the bargaining power lies with the firms, implying a lower bargained real wage than at lower levels of unemployment where labor is more scarce. We will express bargained real wage by the simple function

$$w_{b,t} = 1 - (1 - b)U_t. \tag{3.8}$$

In the case that there is no unemployment, we would then logically have that workers bargain for a real wage that equals their marginal labor productivity, which in this instance has been specified by $y = 1$. Conversely, when all workers are unemployed, we have that the bargained real wage equals b, which is the reservation wage. This parameter $0 < b < 1$, then, is intuitively determined by unemployment benefits or the unemployment assistance program of an economy, since it relates what the bargained real wage will be in the case of all workers being unemployed.

We define the expected inflation rate in the period $t + 1$ as a weighted average of the actual inflation rate and the expected inflation rate during the previous period. That is,

$$\pi^e_{t+1} = a\pi_t + (1 - a)\pi^e_t, \tag{3.9}$$

where a is a parameter that determines the relative weights of the actual inflation rate and the expected inflation rate in the weighted average.

Now, we substitute equations (3.7) and (3.8) into (3.6) to yield

$$\pi_t = \frac{1}{\delta}\left(\pi^e_t + \frac{\mu - (1 - b)U_t}{1 - \mu}\right). \tag{3.10}$$

First, let us shift time forward for the above so we have an expression for π_{t+1}:

$$\pi_{t+1} = \frac{1}{\delta}\left(\pi^e_{t+1} + \frac{\mu - (1 - b)U_{t+1}}{1 - \mu}\right). \tag{3.11}$$

Additionally, we solve equation (3.10) for π^e_t:

$$\pi^e_t = \delta\pi_t - \frac{\mu - (1 - b)U_t}{1 - \mu}.$$

Substituting the above equation into (3.9), and then substituting the resulting expression for π^e_{t+1} back into equation (3.11), we finally have an expression for π_{t+1} in terms of π_t and U_t:

$$\begin{aligned} \pi_{t+1} &= \frac{1}{\delta}\left(\frac{\mu}{1 - \mu} + a\pi_t + (1 - a)\left(\delta\pi_t - \frac{\mu - (1 - b)U_t}{1 - \mu}\right) - \frac{1 - b}{1 - \mu}f(U_t, \pi_t)\right) \\ &\equiv g(U_t, \pi_t) \end{aligned} \tag{3.12}$$

where we have made the additional substitution $U_{t+1} \equiv f(U_t, \pi_t)$. Hence we have our system.

3.2 Analysis of the Model

3.2.1 Analytical

For an equilibrium state (U^*, π^*) we know by definition that $\pi_t = \pi_{t+1} = \pi^*$ and $U_t = U_{t+1} = U^*$. Note that we make the assumption that in the steady state, the inflation rate will be equal to the real money growth rate, $\pi^* = m$. It makes intuitive sense that in an equilibrium, the inflation rate in the economy equals the growth rate of the real money supply.

From these simple facts, we can determine a value for J_s. We make the substitutions that $U_t = U_{t+1} = U^*$ and $\pi_t = \pi_{t+1} = \pi^* = m$ into equation (3.5) and then solve for J_s:

$$
\begin{aligned}
U^* &= U^* + i(1-U^*) - U^* \frac{J_s}{U^* + d(1-U^*)} \\
J_s &= \frac{i(1-U^*)(U^* + d(1-U^*))}{U^*}
\end{aligned}
$$

By substituting this expression for J_s into equation (3.12), and once again making the substitutions that $U_t = U_{t+1} = U^*$ and $\pi_t = \pi_{t+1} = \pi^* = m$, we can solve to find U^* in terms of parameters. This algebra then yields the unique steady state p_e:

$$
p_e = (U^*, \pi^*) = \left(\frac{\mu - m(\delta - 1)(1-\mu)}{1-b}, m \right) \tag{3.13}
$$

To analyze stability of the steady state, we take the Jacobian of our system of coupled equation, and then evaluate at p_e. Omitting the algebra, the Jacobian matrix J for this system is:

$$
J = \begin{pmatrix} \frac{\partial f}{\partial U} & \frac{\partial f}{\partial \pi} \\ \frac{\partial g}{\partial U} & \frac{\partial g}{\partial \pi} \end{pmatrix}
$$

$$
\begin{aligned}
\left.\frac{\partial f}{\partial U}\right|_{p_e} &= 1 - i\left(1 + d\frac{1-U^*}{U^*(U^* + d(1-U^*))}\right) \\
&\equiv j_{11} \\
\left.\frac{\partial f}{\partial \pi}\right|_{p_e} &= \gamma \frac{U^*}{U^* + d(1-U^*)} \\
&\equiv j_{12} \\
\left.\frac{\partial g}{\partial U}\right|_{p_e} &= \frac{1-b}{\delta(1-\mu)}(1 - a - j_{11}) \\
&\equiv j_{21} \\
\left.\frac{\partial g}{\partial \pi}\right|_{p_e} &= \frac{1}{\delta}\left(a + \delta(1-a) - j_{12}\frac{1-b}{1-\mu}\right) \\
&\equiv j_{22}
\end{aligned}
$$

Now that we have the Jacobian, we can calculate its eigenvalues, thereby determining the stability of our steady state equilibrium. By definition, the steady state is stable when $|\lambda_{1.2}| < 1$ and unstable when $|\lambda_1| > 1$ or $|\lambda_2| > 1$. Using the four terms of the Jacobian above, we can explicitly express the eigenvalues of the Jacobian by:

$$
\lambda_{1,2} = \frac{j_{11} + j_{22}}{2} \pm \frac{1}{2}\sqrt{(j_{11} - j_{22})^2 + 4 j_{12} j_{21}}
$$

3.2.2 Numerical

For the numerical analysis of this system, we define specific values for the parameters in the model, as proposed by Neugart [4], and use them for several numerical examples. For the remainder of this section, we define

$$a = .5 \qquad b = .5 \qquad d = .01 \qquad m = .03$$
$$\mu = .04 \qquad \delta = 2 \qquad \gamma = .5$$

unless otherwise stated. Substitution of the above values into equation (3.13) yields a numerical value for the steady state of the system: $(U^*, \pi^*) = (.0224, .03)$.

We now test the stability of the system for various values of the parameter i. First, let $i = 0.12$. This produces the Jacobian

$$J \approx \begin{pmatrix} -0.748 & 0.348 \\ 0.325 & 0.659 \end{pmatrix}$$

which has eigenvalues:

$$\begin{aligned} \lambda_1 &= -0.8239 \\ \lambda_2 &= 0.7356. \end{aligned}$$

Since $|\lambda_{1,2}| < 1$, the steady state is at a stable equilibrium given this value for i. Next, we let $i = .16$, yielding

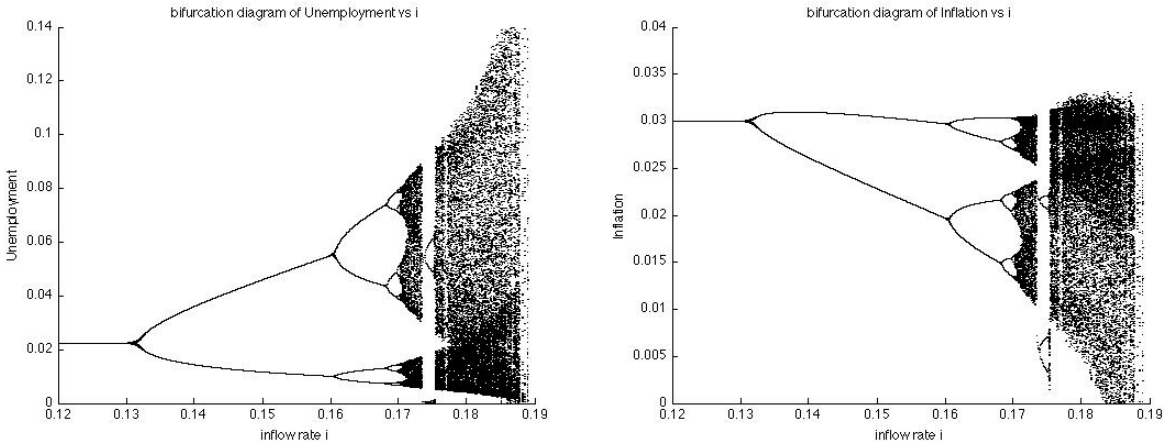
$$J \approx \begin{pmatrix} -1.330 & 0.348 \\ 0.477 & 0.659 \end{pmatrix}$$

with eigenvalues:

$$\begin{aligned} \lambda_1 &= -1.4104 \\ \lambda_2 &= 0.7395. \end{aligned}$$

Since $|\lambda_1| > 1$, we have that the steady state is unstable for $i = .16$. Hence, it is clear that for some value of i between 0.12 and 0.16, the system loses stability. It is straightforward to verify numerically that the negative eigenvalue achieves a value of -1 when $i = 0.131992$. Hence, at this value of i the steady state loses stability.

The following bifurcation diagrams for the system depict the equilibrium values for Unemployment and Inflation (U^* and π^*, respectively) for different values of the parameter i.



(a) U^* vs i (b) π^* vs i

Figure 6: Bifurcation Diagrams

Figure 6 supports all of the results we previously derived. For low values of i, we have the single unique steady state $(U^*, \pi^*) = (.0224, .03)$. At $i = 0.131992$ we observe that the single steady state becomes unstable and a period-doubling bifurcation occurs in both bifurcation diagrams of Figure 6. Hence, for $i > 0.131992$ we no longer have a stable fixed point for the system. Instead, this fixed point is now unstable, replaced by a stable period-2 orbit. A period doubling cascade occurs at $i \approx 0.17$. This *period doubling bifurcation route to chaos* implies that for large values of i, the model must exhibit chaos. For the remainder of this section, we fix $i = 0.18$; hence, for the rest of this section, orbits of any period are possible.

Given this coupled system of equations for unemployment and inflation, as t tends to infinity any initial value is either eventually periodic to the fixed point $(U, \pi) \approx$ (-.1832, .1371), or is sucked into an orbit, often times an orbit inside the chaotic attractor, shown in Figure 7. If it does not end up inside the chaotic attractor or at the fixed point, it will end up in some sort of periodic orbit; note that since $i = 0.18$, orbits of all periods are possible, from Figure 6. However, the remarkable thing about a chaotic orbit is that, after running 50,000 iterations on the system, not a single point is repeated. That is, given an initial condition that is sucked into the attractor, it will never visit the same point in the attractor twice. Hence, the attractor appears to be dense in this given area of the plane, although a proof is not obvious.

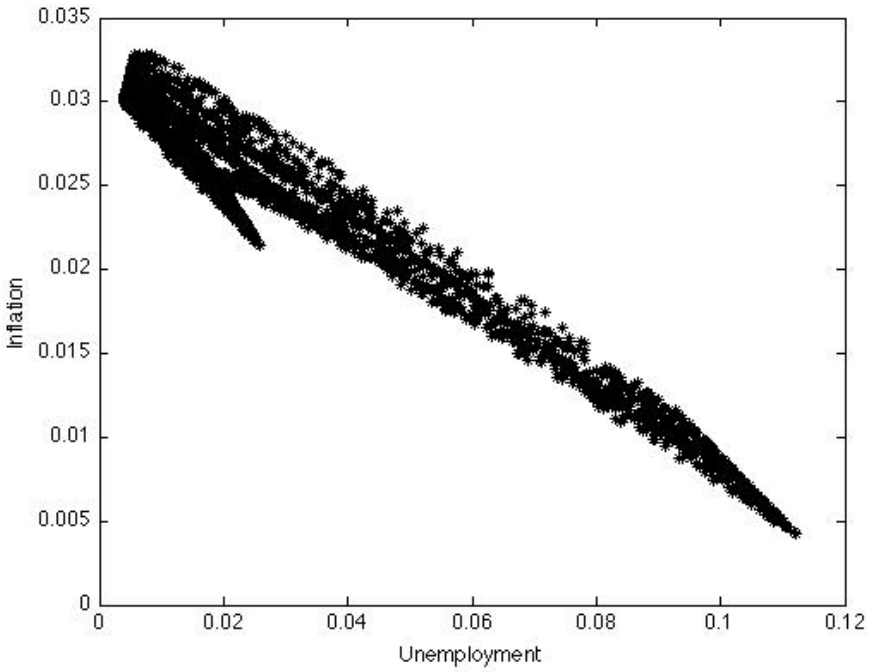


Figure 7: Chaotic Attractor, Initial Condition $(0.1, 0.37)$

On a related note, it is very straightforward to show the existence of sensitive dependence on initial conditions for the model, using what we have already shown. Consider initial conditions $a_0 = (.2, .15)$ and $b_0 = (.2, .15 + 10^{-10})$. Although these initial conditions only differ by 10^{-10} in one coordinate, as t tends to infinity, these initial conditions diverge from each other, eventually converging to entirely different orbits. The difference between a_0^k and b_0^k, where k is the number of iterations, is shown in Figure 8 with respect to U and π. At around 60 iterations, the difference stabilizes to that of any two

arbitrary points in the chaotic orbit, showing the divergence of points initially arbitrarily close. Furthermore, many iterations later, a_0 actually regains stability in a period-11 orbit within the chaotic attractor, while b_0 remains in the attractor. Plotted next to each other in Figure 9 are the iterations of a_0 and b_0, where the final period-11 orbit of a_0 is shown in white points.

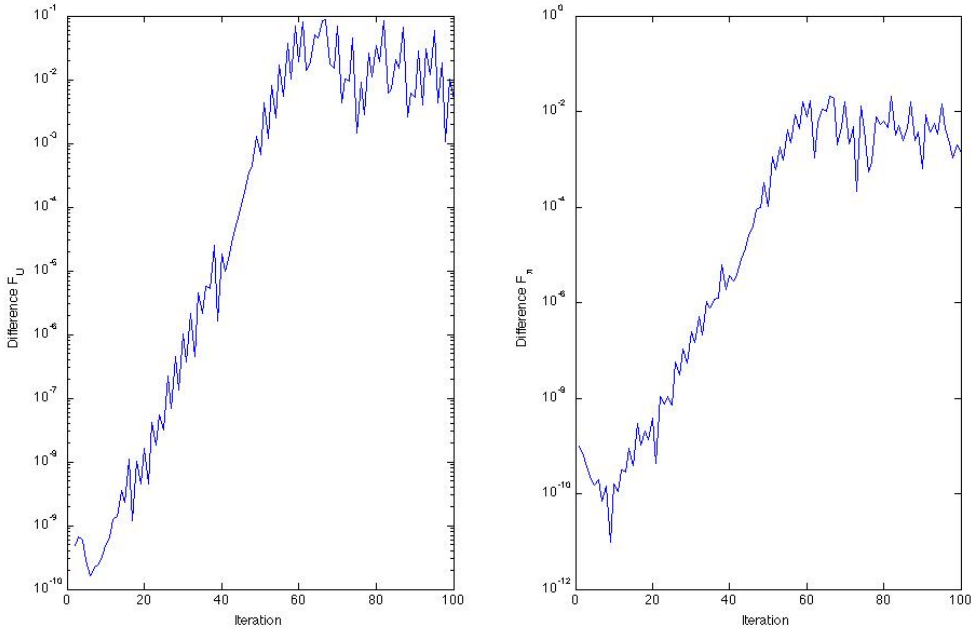


Figure 8: Difference in Orbits of a_0,b_0 after given number of iterations

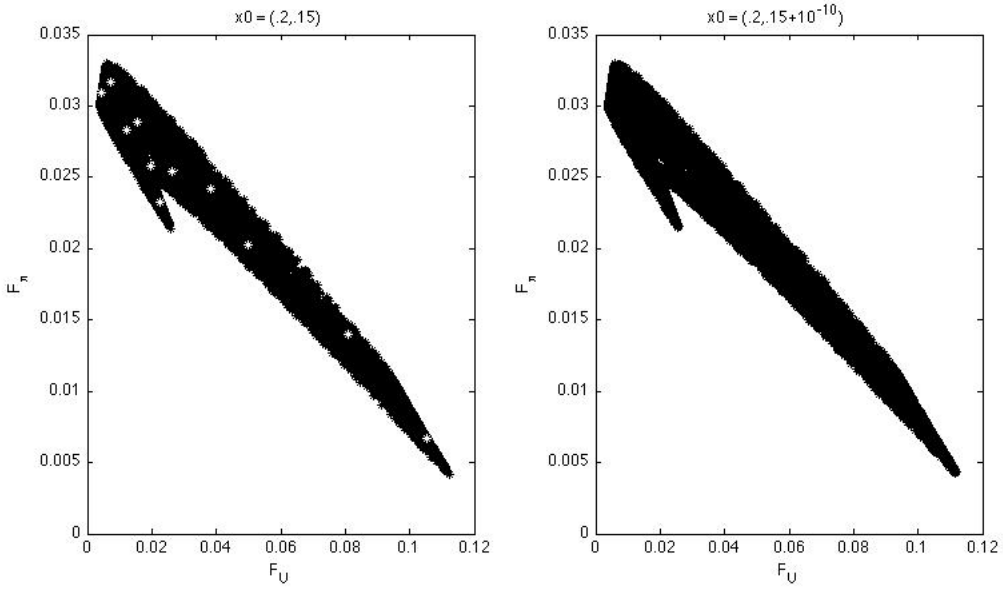


Figure 9: Orbit of a_0 at left, b_0 at right

Recall that since we have fixed $i = 0.18$, the system clearly exhibits chaos. This implies the existence of orbits of every period; these orbits appear to always occur within

the attractor. For example, note the period-11 orbit inside the attractor in Figure 9. However, the basin of a periodic orbit within a chaotic attractor is difficult to analyze. In this case, it is the set of all points that eventually map to the given period-11 orbit. In general, it is not easy or computationally feasible to determine the basin of a periodic orbit for some given period; this makes sense since we are dealing with a seemingly dense attractor that contains orbits of all periods. With exception of the fixed point and chaotic attractor, the basin of any of these orbits is what is known as a riddled basin. A riddled basin is a basin such that any arbitrarily small ϵ-disk contains a non-zero area of points which converge to different orbits.

Although it is infeasible to produce a riddled basin, we can produce a basin for the fixed point, as shown in Figure 10. This basin shows the end behavior of initial conditions. Black points are attracted to the fixed point, while white points are sucked into the chaotic attractor or into an orbit of period-2 or greater.

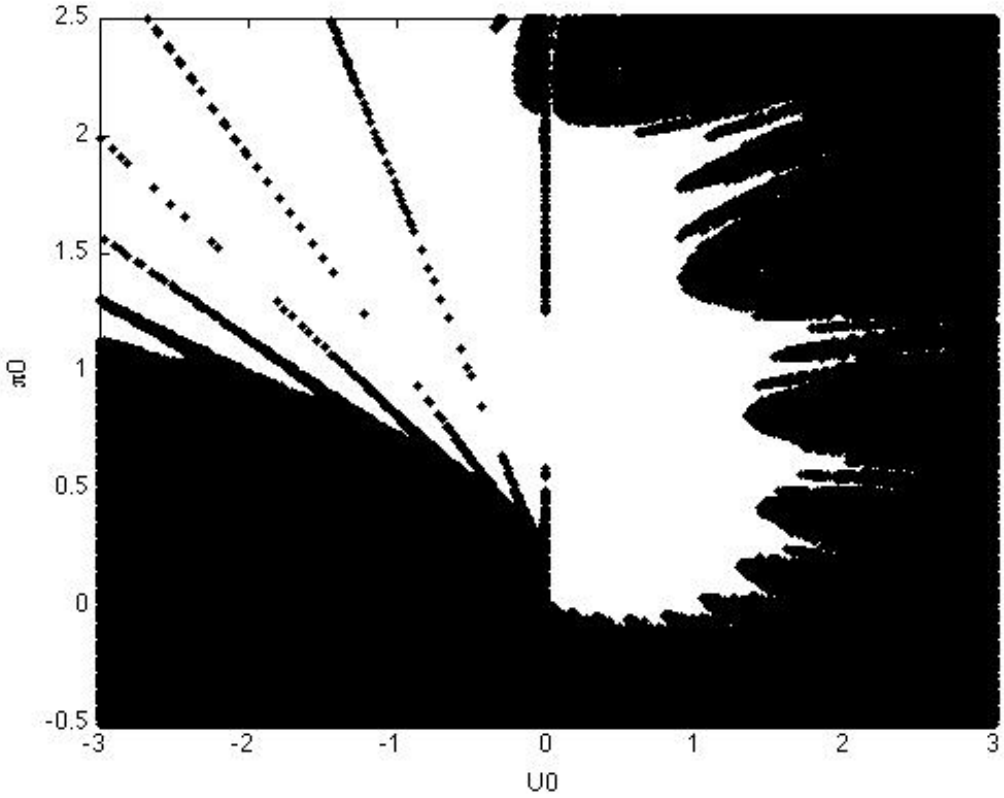


Figure 10: The Basin

4 Conclusions

4.1 Conclusions from the Supply and Demand Model

In Section 1, we determined that for given values of the parameters μ and a, the model exhibits chaotic behavior. Recall that μ was the parameter that determined the steepness of the supply curve, and a determined the vertical intercept, or intercept on the quantity axis, of the demand curve. Since the steepness of the supply curve determines its elasticity, this implies that if the supply curve is very elastic near the origin and inelastic

far from the origin, then choosing certain values for a will yield chaotic behavior. Note that an elastic supply curve implies that small changes in price yield large changes in quantity, while an inelastic supply curve implies that small changes in price only yield small changes in quantity.

More important than this specific conclusion is the fact that a very simple model of supply and demand was able to exhibit chaotic behavior. For reasonable values of the parameters, even this elementary model of supply and demand exhibits chaos.

4.2 Conclusions from the Model of a Monetary Economy

In Section 2, we determined that for large enough values of δ, this model exhibits chaos. Since δ is a derived parameter, where $\delta \equiv \frac{\mu}{\beta} - 1 > 0$, this implies that the model will exhibit chaos for large enough values of μ and/or small enough values of β. Recall that μ is the growth rate of the money supply and β is the discount factor. Hence, if the agent severely discounts his future utility, if the money supply grows at a very large rate, or both, the model will exhibit chaos. In short, in this section we show that a very high growth rate of the money supply in an economy can cause the economy to exhibit erratic, chaotic behavior, even in the absence of external forces.

4.3 Conclusions from the Unemployment and Inflation Model

Section 3 involves the most complicated model, and as such the analysis of this model was more in depth than in the previous two sections. Ultimately, we chose reasonable values of the parameters, following from Neugart [4], and drew conclusions based off of the model given these parameters. Holding all other values constant, we showed through the bifurcation diagrams in Figure 6 that high values of i result in chaos. Setting i to a value in this range (again following Neugart, we chose $i = 0.18$) proved the existence of a chaotic attractor, and hence guaranteed chaos for this economic model.

It is interesting to observe that the chaotic attractor derived from this model seems to resemble the classical Phillips Curve from economics. Simply stated, the Phillips Curve is a historical inverse relationship between inflation and unemployment. However, as we observed in our numerical investigations, although the attractor seems to exhibit such an inverse relationship, the fact that chaotic orbits in the attractor move around erratically defies the existence of any stable tradeoff. Despite the resemblance to the Phillips curve, irregular fluctuations due to the presence of a chaotic attractor make long run behavior unpredictable.

4.4 Final Thoughts

The purpose of this paper is not to examine the implications of chaos in economic models. Rather, it is simply to verify the existence of chaos in them. The models that we examined are relevant to very different subfields of economics, but all clearly demonstrated chaotic dynamics for reasonable parameters. Chaos is prevalent in models throughout economics, and the role of chaos should not be discounted when analyzing economic models.

Appendices

A Matlab Code

A.1 Video/Bifurcation Code

This is the Matlab code used to produce a timelapse video of bifurcation diagrams for Section 1. Note, specific pieces of code used to create the video file (i.e. getframe, aviobj, etc.) were removed to allow for individual bifurcation plots to be made. .

```
%Code to create a timelapse movie of Bifurcation diagrams for a given
%function while varying a parameter within the function.
clear

l = .3;
b = .25;

pe = @(a,u,p) (1-l)*p+(a*l/b)-(l*atan(u*p)/b);
aviobj = avifile('bifur.avi');

for i=0:300

  u = 0+i*.05;
  r = 10000; % level of rounding
  p = 200; % max points to plot
  N = 3000; % number of values
  M = 500; % number of iterations
  a = -1.4; % starting value
  b = 1.4; % final value
  dt = linspace(a,b,N);

  for j = 1:N

      t=dt(j);
      x=zeros(M,1);
      x(1) = 0.5;

      for k = 2:M
          x(k) = pe(t,u,x(k-1));
      end

      out{j} = unique(round(r*x(end-p:end))); % output unique values
  end

  data = [];
  for l = 1:length(dt)
      n = length(out{l});
      data = [data; dt(l)*ones(n,1),out{l}];
```

```
  end

  data(:,2) = data(:,2)/r;
  figure(1);
  plot(data(:,1),data(:,2),'k.','markers',1); % plot bifurcation diagram
  axis tight
  set(gcf,'color','white')
  axis([-1.4 1.4 -2.75 2.75])
  F(i+1) = getframe;

end

aviobj = addframe(aviobj,F); % add set of frames F to .avi
aviobj = close(aviobj);
```

A.2 Attractor Code

This is Matlab code used to iterate the system of Section 3 given a set of initial conditions and plot the last 5,000 iterations. It also calculates the number of points that are repeated over 230,000 iterations.

```
%Plot iterations of a coupled system of equations F with parameters given by
%a,b,d,de,m,me,g,i and JS, and given an initial value y0.
clear

a = .5; b = .5;
de = 2; d = .01;
m = .03; me = .04;
g = .5; i = .18;
Us=(me-m*(de-1)*(1-me)) / (1-b);
Js=i*(1-Us)*(Us+d*(1-Us))/Us;

F = @(y) [y(1,:)+i*(1-y(1,:))-y(1,:)*(Js+g*(m-y(2,:)))/(y(1,:)+d*(1-y(1,:)));...
   1/de*(me/(1-me)+a*y(2,:)+(1-a)*(de*y(2,:)-(me-(1-b)*y(1,:))/(1-me))-(1-b)/...
   (1-me)*(y(1,:)+i*(1-y(1,:))-y(1,:)*(Js+g*(m-y(2,:)))/(y(1,:)+d*(1-y(1,:)))))];

z(:,1) = [-1.189;1.159];

for j=2:30000
   z(1:2,j) = F(z(1:2,j-1));
end

figure;
plot(z(1,25000:30000),z(2,25000:30000),'r*')
xlabel('Unemployment');
ylabel('Inflation');

A = z';
```

```
B = unique(A,'rows');
repeats = length(A)-length(B) % calculate the number of repeated points
```

A.3 Attracting Basin Code

This is Matlab code which plots the basin of the chaotic attractor of Section 3.

```
%Code which plots all points over a given (x,y) interval which converge
%to (-.1832,.1371) in black, and leaves all points which converge to other
%orbits in white.
clear

span = -3:.01:3; % x-interval
n = length(span);
span2 = -.5:.01:2.5; % y-interval
n2 = length(span2);
err = .001; % guaranteed convergence within this range

a = .5; b = .5;
de = 2; d = .01;
m = .03; me = .04;
g = .5; i = .18;
Us=(me-m*(de-1)*(1-me)) / (1-b);
Js=i*(1-Us)*(Us+d*(1-Us))/Us;

F = @(y) [y(1,:)+i*(1-y(1,:))-y(1,:)*(Js+g*(m-y(2,:)))/(y(1,:)+d*(1-y(1,:)));...
   1/de*(me/(1-me)+a*y(2,:)+(1-a)*(de*y(2,:)-(me-(1-b)*y(1,:))/(1-me))-(1-b)/...
   (1-me)*(y(1,:)+i*(1-y(1,:))-y(1,:)*(Js+g*(m-y(2,:)))/(y(1,:)+d*(1-y(1,:)))))];

l = 0;

for j=1:n
   for k=1:n2
       l = l+1;
       y(:,1) = [span(j);span2(k)];

       for i=2:2000
           y(1:2,i) = F(y(1:2,i-1));

           e1 = abs(y(1,i)+.1832);
           e2 = abs(y(2,i)-.1371);

           if e1 < err && e2 < err % output values which converge
               out(l) = y(1,1);
               out2(l) = y(2,1);
               break;
           end

       end
```

```
   end
end

figure
plot(out(:),out2(:),'black.')
title('Points Converging to Chaotic Attractor');
ylabel('\pi0');
xlabel('U0');
```

A.4 Bifurcation Diagram Code

```
%Code that can be used to plot bifurcation diagrams for the model in
%section 3 of unemployment and inflation.

clear

a=0.5;
del=2;
b=0.5;
gam=0.5;
d=0.01;
m=0.03;
mu=0.04;
Us=(mu-m.*(del-1).*(1-mu))/(1-b);

g=0.5;
me=0.04;
de=2;

N=300; %number of iterations to run for each value of the parameter

y = zeros(2,N+1);
ns= zeros(1,N+1);
figure; hold on;
axis([0.12 0.19 0 0.14]);
y(1,1)=0.01;
y(2,1)=0.01;

for i=.12:.0001:.19 %test i across a range of values
    Js=i.*(1-Us).*(Us+d.*(1-Us))/Us;
    for n=1:N
        ns(1,n)=i;

        y(1,n+1) = y(1,n)+i*(1-y(1,n))-y(1,n)*(Js+g*(m-y(2,n)))/...
        (y(1,n)+d*(1-y(1,n))); %U(t+1)
```

```
        y(2,n+1) = 1/de*(me/(1-me)+a*y(2,n)+(1-a)*(de*y(2,n)-(me-(1-b)*y(1,n))/...
        (1-me))-(1-b)/ (1-me)*(y(1,n)+i*(1-y(1,n))-y(1,n)*(Js+g*(m-y(2,n)))/...
        (y(1,n)+d*(1-y(1,n)))); %Pi(t+1)


    plot(ns(1,1:150),y(1,150:299), 'kx','markers',1) %plot for this parameter value
    hold on;
    end
end

    title('bifurcation diagram of Unemployment vs i');
    xlabel('inflow rate i');
    ylabel('Unemployment');
```

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[5] Wei-Bin Zhang, *Discrete Dynamical Systems, Bifurcations, and Chaos in Economics*. Elsevier, Boston, 2006, pp. 178-83, 213-19, 331-36.