Stability Analysis of Spherically Symmetric Solutions to the Klein Gordon Equation

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1 Introduction

From the classical wave and heat equations to more quantized equations such as those of Schrödiner and Klein Gordon, differential equations are usually the first step in the study of physical systems. For the purpose of this project we study the Klein Gordon Equation. If the reader wishes to forgo the physics discussion, then the following definition would suffice.[1]

Definition 1:More generally, a scar field can be a map ϕ: $\mathbb{M} \rightarrow \mathbb{N}$ where M and N are open, connected, and non empty subsets of a manifold $\mathcal{U}$. For the purpose of this project a scalar field ϕ will be a map ϕ: $\mathbb{R}^n \rightarrow \mathbb{C}$.

Equipped with the definition of what a scalar field. We move to its equation of motion.[2]

2 Equations of Motion

We are working in a 'usual' space time, one time and three spatial dimensions. The action for a real scalar field in (3 space + 1 time) dimensions is given by the following integral.

[1]See Appendix for a more physical definition
[2]We leave the derivation of the equations of motion Appendix.

The integral is expressed using Einstein notation.[3]

$$S[\phi] = \int d^4x \Big[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)\Big] \tag{2.1}$$

The equation of motion given by the Lagrangian is then the Klein Gordon equation. The real equation looks different, as it contains contributions from the constants: c and $\hbar$.We have set all of these constants equal to 1. The resulting equation is:

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = -\frac{\partial V(\phi)}{\partial \phi} \tag{2.2}$$

Note that 2.2 is valid for any choice of potential V. The solutions that we are interested in are called oscillons. The latter are unstable (albeit long lived) spherically symmetric solutions to 2.2. We chose to call these solutions "bubbles" from now on. The potential we consider for our analysis is the following:

$$V_L(\phi) = (\phi + 1)^2 \tag{2.3}$$

For physical reasons we impose the following conditions to our solutions.[4] Note that ' indicated derivate in space and an over dot a derivative in time.

$$\phi(r \to \infty, \quad t) = \phi_0, \quad \phi'(0,t) = 0, \quad \dot{\phi}(r,0) = 0 \tag{2.4}$$

The first condition assures that oscillons decay to the vacuum solution at $r \to \infty$.[5] The last two conditions account for an oscillon bubble to start at rest and make sure that behavior at the origin is non-singular.[6]

For 2.2 the way to a solution is simple and can be solved by separation of variables i.e. $\phi = R(r)T(t)$ (with separation constant $-k^2$ or wave number if computing the Fourier transform), where the second order partial differential equation can be expressed as follows:

$$\nabla^2 R + k^2 R = 0 \rightarrow \qquad R = Ae^{\pm \mathbf{k} \cdot \mathbf{r}} \tag{2.5}$$

$$\ddot{T} + (2 + k^2)T = 0 \rightarrow \qquad T = e^{\pm t\sqrt{k^2+2}} \tag{2.6}$$

[3]See J. Hartle's 'Gravity: An Introduction to Einstein's General Relativity' Chapter 6
[4][5]
[5]$\phi_0$can be thought of as a constant
[6][5]

Since we are interested in stability of the bubbles we write 2.5-2.6 as a system of first order ODEs as such:

$$\begin{aligned} R' &= S \\ S' &= -\frac{2}{r} - k^2 R \end{aligned} \tag{2.7}$$

$$\begin{aligned} \dot{T} &= W \\ \dot{W} &= -(2 + k^2)T \end{aligned} \tag{2.8}$$

The two systems have the respective Jacobian matrices:

$$\begin{bmatrix} 0 & 1 \\ -k^2 & -\frac{2}{r} \end{bmatrix} \& \begin{bmatrix} 0 & W \\ -(2+k^2) & 0 \end{bmatrix}$$

The first Jacobian matrix is for the spatial component of our equation and has the following eigenvalues $\lambda = -\frac{1}{r} \pm \sqrt{r^{-2} - 4k^2}$ which have $\mathfrak{Re}(\lambda) < 0 \; \forall r > 0$.Moreover, the eigenvalues gain a non positive imaginary component when $r < k^{-2}$ signifying oscillatory behavior after that point. The temporal Jacobian matrix has purely imaginary eigenvalues. We can see this behavior via the phase diagrams and evolution plots for the temporal and spatial solutions.

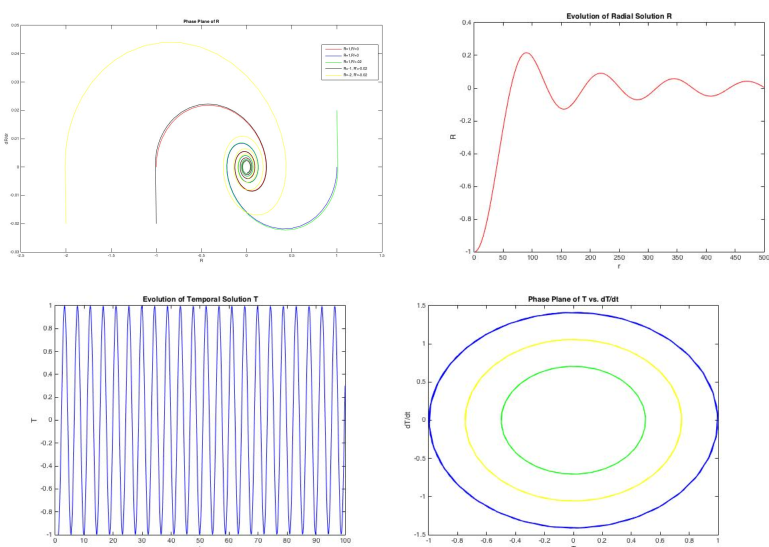


Figure 2.1: Evolution and Phase Planes for Solutions

3 Equations of Motion With A Twist!

Now comes the case of most interest albeit no nearly as simple as the previous case. Now we must solve a more complicated version of the K.G.E, one of the form:[7]

$$\ddot{\phi} + 3\frac{\dot{a}(t)}{a(t)}\dot{\phi} = \frac{\nabla^2\phi}{a(t)^2} - 2\phi - 2 \tag{3.1}$$

The reader may realize that I have replaced the generic form of the partial derivative on the right hand side by the linear potential from before. This case is more interesting than before as the value of $3\frac{\dot{a(t)}}{a(t)}$ is acting as a damping term for the equation of motion. The definition of $a(t)$ is related to physics so we give the following definitions to help:

Defintion 2: The function $a(t)$ is a map $a(t) : \mathbb{R}_+ \to \mathbb{R}$ that affects the metric of the region $\mathcal{U}$. We require that $a(t)$ be only once differentiable.We define for convenience $H := \frac{\dot{a(t)}}{a(t)}$

We are interested in the form of $a(t)$ that agrees with modern observations so we chose $a(t) = e^{Ht}$. This choice of $a(t)$ assures that H is just a constant so easier for stability analysis.As before we solve this equation via separation of variables. We are left with:

$$\begin{aligned} \nabla^2 R + k^2 R = 0 \to \qquad & R = Ae^{\pm\mathbf{k}\cdot\mathbf{r}} \\ \ddot{T} + 3H\dot{T} + \left(2 + \frac{k^2}{a(t)^2}\right)T = 0 \to \qquad & T = B(k,H)e^{\frac{-3}{2}t}\mathcal{J}_{\pm\xi}\left(\frac{ke^{-Ht}}{H}\right) \end{aligned} \tag{3.2}$$

Where $\mathcal{J}$ is the Bessel function of the first kind with order $\xi = \frac{\sqrt{9H^2-8}}{H}$. We know the stability of the spatial part from the previous section.[8] Consider the following system and following Jacobian matrices:

$$\begin{gathered} \dot{T} = W \\ \dot{W} = -3HW - (2+k^2)T \\ \begin{bmatrix} 0 & 1 \\ -(2+\frac{k^2}{a(t)^2}) & -3H \end{bmatrix} \end{gathered} \tag{3.3}$$

The matrix has the following eigenvalues:

$$\lambda = \frac{-3H \pm \sqrt{9H^2 - 8 - \frac{k^2}{a(t)^2}}}{2} \tag{3.4}$$

[7] [4]

[8] Discussion of the solutions will be outside the scope of this project. Solution obtained via Mathematica

We now examine the values of λ to figure out the behavior of the system. It is worthwhile to study what happens at $t = \epsilon$ where $0 < \epsilon << 1$ and at $t \to \infty$. We see:

$$\lim_{t\to\infty} \mathrm{Im}(\lambda) \to \infty \tag{3.5}$$

$$\lim_{a(t)\to 1} \lambda = \pm i\sqrt{2+k^2} \tag{3.6}$$

This shows the long time stability of the system resembles a over-damped oscillator, which is right in line with what we required from our boundary conditions. The second condition shows that for small times, the system resembles our undamped system. The following plots show what we have found analytically:

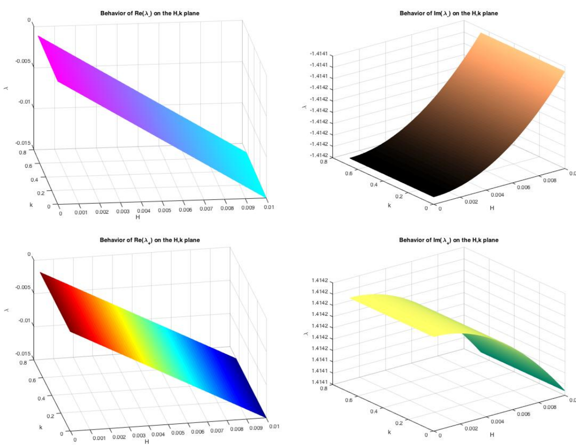


Figure 3.1: Evolution of λ on the H,k plane. The choice of H and k to plot are explained in sources 4 and 5. They depend on the physical constraints of the system.

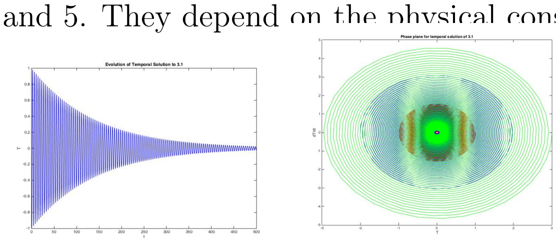


Figure 3.2: Solution to 3.3 for fixed H&k and Corresponding Phase Plane

From Figure 3.1 we see that the real part of $\lambda_\pm$ will always be negative and that an increase in H will make the value of $\mathfrak{Re}(\lambda_\pm)$ more negative (coinciding with our damping analogy earlier). Moreover, we ca also see that regardless of choice of k and H, the value of $Im(\lambda_\pm) \neq 0$ thus guaranteeing oscillatory behavior. Finally, Figure 3.2 shows evolution of $T(t)$ along with the phase plane of T and $\frac{dT}{dt}$ and verifies the global asymptotic behavior of solutions (the different phase plane colors represent different orbits).

4 Appendix

This section is optional and it is intended for the reader that wishes to know more about the physics of the project. This appendix is ordered in the following way. Derivation of the Klein Gordon equation, physical interpretation of oscillons, sources used, and larger resolution versions of the plots included in the body.

4.1 Derivation of Klein Gordon Equation

Like in lecture, our equations of motions are derived from the entity known as the Lagrangian. However, we are considering a scalar field not a single particle as we have seen in lecture. Therefore, to accurately get the equations of motion we cannot simply derive the Lagrangian of just one particle. We must take the Lagrangian density, $\mathcal{L}$ (the Lagrangian equivalent for many particle systems). For our field $\phi(\vec{r}, t)$ the Lagrangian density is the following:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \mathrm{m}^2 \phi^2 \tag{4.1}$$

The first term on the l.f.s. is the inverse metric that describes the spacetime of the scalar field.[9] Think of it as a matrix that describes how locations in space and time are related. This is where the equation in 3.1 came from and explain the difference between it and the simpler equation in section 2.The indices on the partial derivative signs just indicate what variable we are differentiating.[10] The fact that ϕ is time and space dependence explains where the double derivate in time and the following Laplace operator comes from. The resulting equation is simply:

$$\mathcal{L} = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla \phi)^2 \right) - \frac{1}{2} \mathrm{m}^2 \phi^2 \tag{4.2}$$

This form of $\mathcal{L}$ should look familiar to the reader since we can split it up into the classic $L = T - V$ where $T \rightharpoonup \frac{1}{2}\dot{\phi}^2$ and $V \rightharpoondown \frac{1}{2}((\nabla\phi)^2 + \mathrm{m}^2\phi^2)$.[11] From here we can compute the usual Euler-Lagrange equations of motion which give:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= -\mathrm{m}^2 \phi \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= (\dot{\phi}, -\nabla \phi) \end{aligned} \tag{4.3}$$

[9]See Hartle's Gravity

[10]See Hartle's Gravity

[11]I cannot use the equal sign as the true potential and kinetic energy require an integral over all the particles in our field, so technically the energies are really integrals

This gives us the Klein Gordon equation with potential $V(\phi) = \mathrm{m}^2\phi^2$ but of course we can generalize it to any potential as we did for this project.

4.2 Physical Interpretation

In this project we have used the word oscillon and bubbles to represent spherically symmetric solutions to the Klein Gordon equation. This type of solutions are important in physics as they help understand the status of the universe at early times. We have also have used the term scalar field without giving it much physical meaning. In cosmology and other areas of physics, we look for symmetry when examining systems. The idea of a scalar field follows from a desire to have a field, that gives every point in space a number (hence scalar field) that does not change under coordinate transformation. An example of this is temperature and electric potential. No matter what coordinate we use, the temperature at a point in space stays the same. The reason we examine oscillons in scalar field theories is due to their longevity and instability. Such traits are of interest in high energy physics where the solutions act as a buffer between energy phases. This is why certain constants in the project like k and H were chosen. The values are not arbitrary as if H is too large, we get too fast of an expansion and oscillons decay immediately and k needs to be smaller or approximately equal to the effective radius of our bubbles. The results showing that these bubbles are long lived and that their longevity is dependent on the expansion of the universe will be used for further work by the author.

4.3 Sources Used

1. Alligood, Kathleen T., and Tim Sauer. Chaos: An Introduction to Dynamical Systems. New York: Springer, 1997. 284-299. Print.

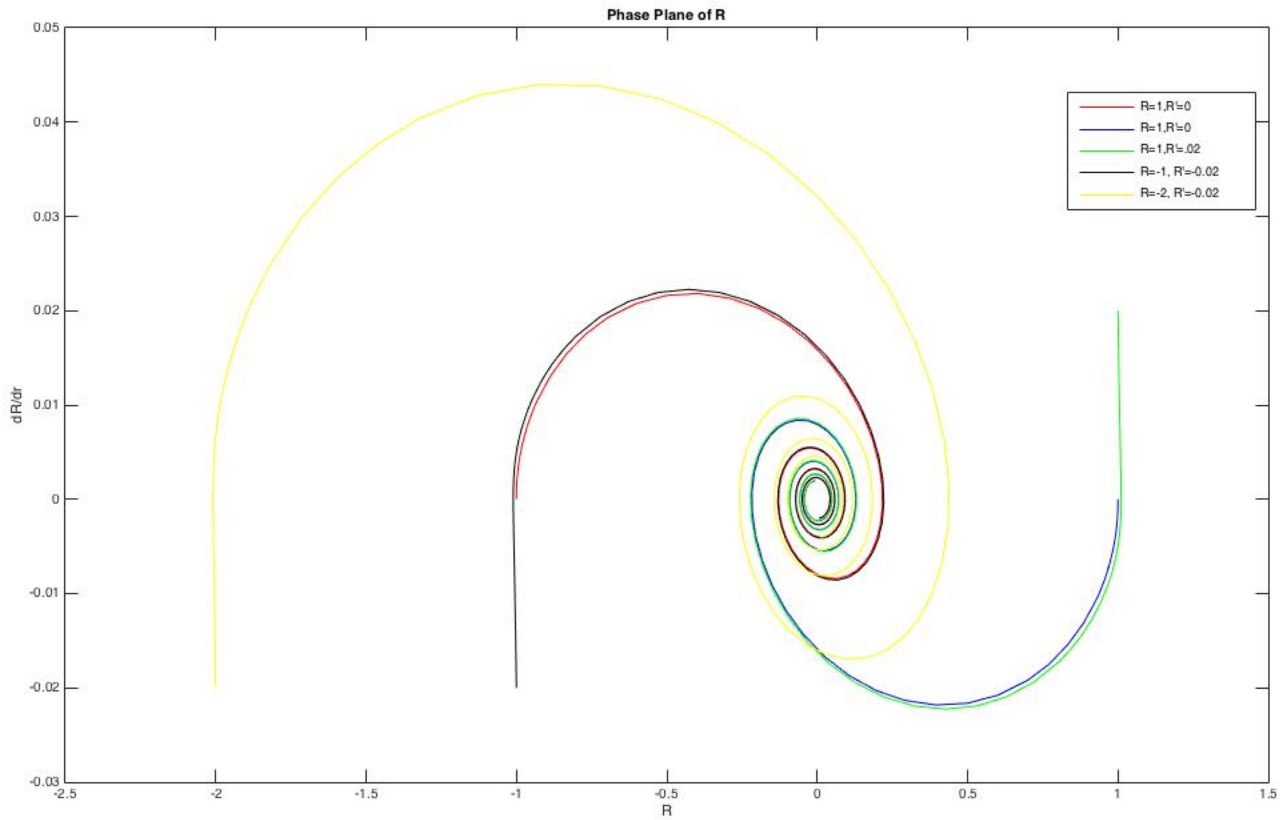
2. Hartle, J. B. "Gravity As A Geometry." Gravity: An Introduction to Einstein's General Relativity. San Francisco: Addison-Wesley, 2003. Print.

3. Schwartz, Matthew Dean. "Classical Field Theory." Quantum Field Theory and the Standard Model. 29-31. Print.

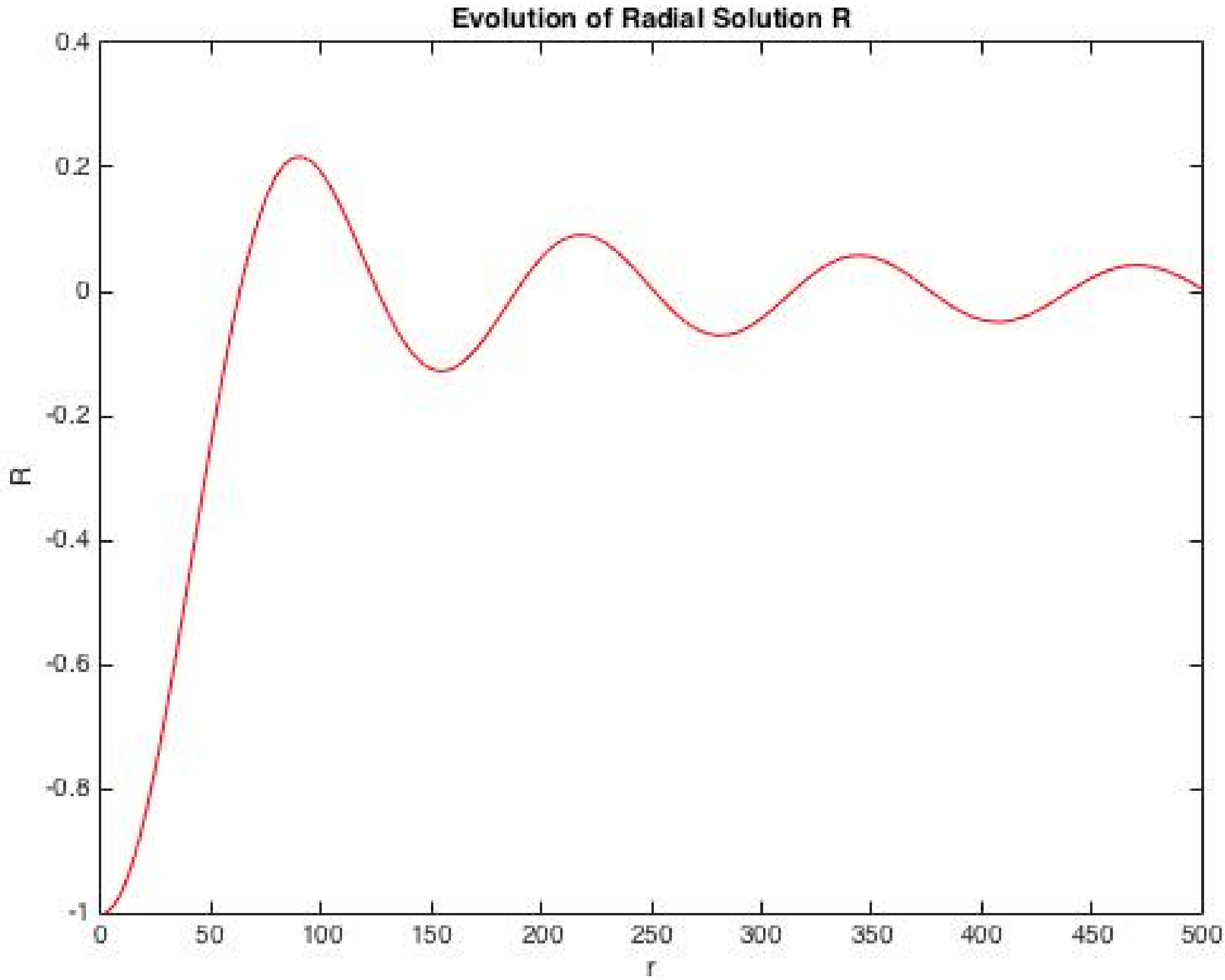
4. Long-lived time-dependent remnants during cosmological symmetry breaking: From inflation to the electroweak scale:Marcelo Gleiser, Noah Graham, and Nikitas Stamatopoulos; 16 August 2010 Phys. Rev. D 82, 04351

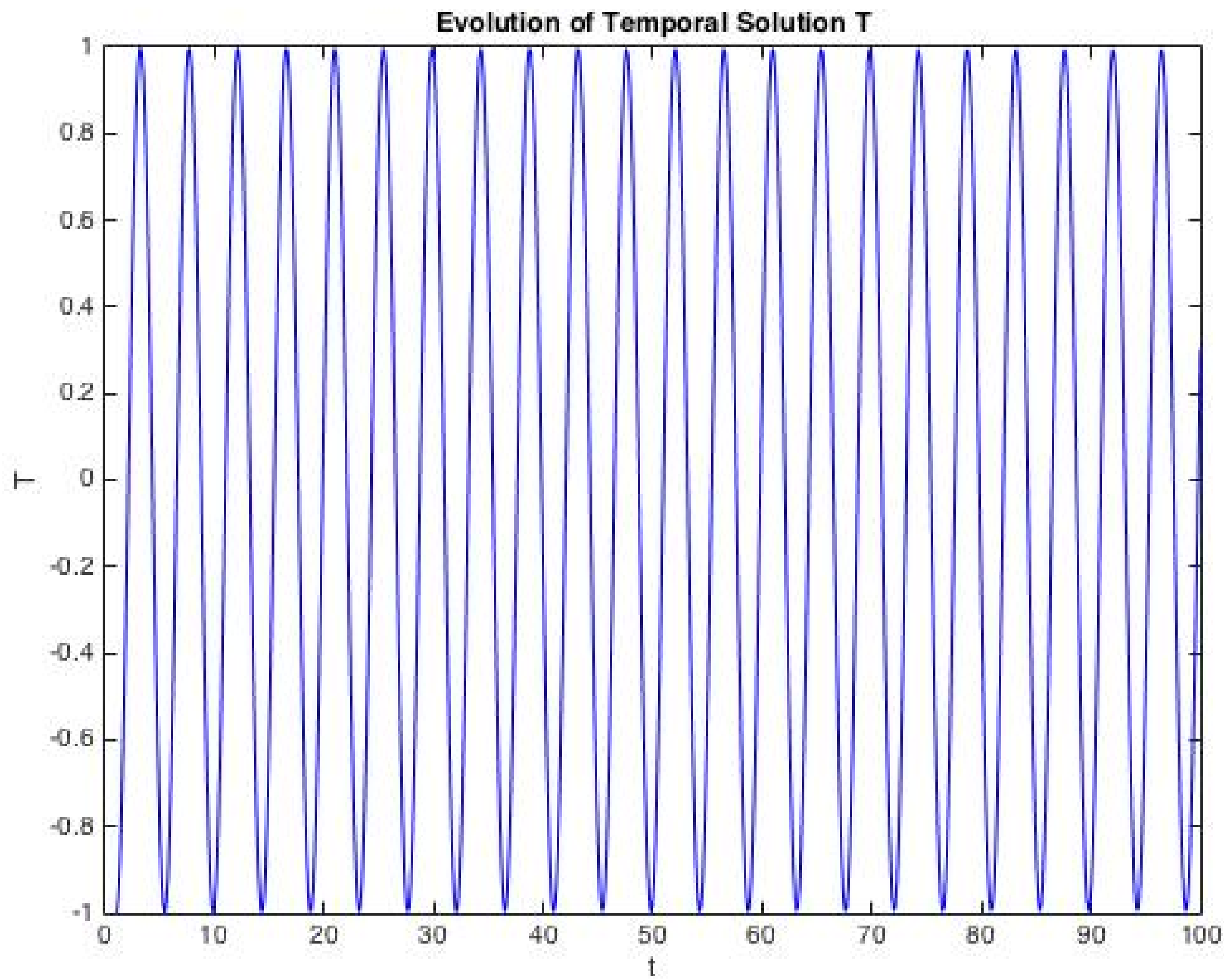
5. Oscillons: Resonant configurations during bubble collapse:E. J. Copeland, M. Gleiser, and H.-R. Müller;5 August 1995 Phys. Rev. D 52, 1920

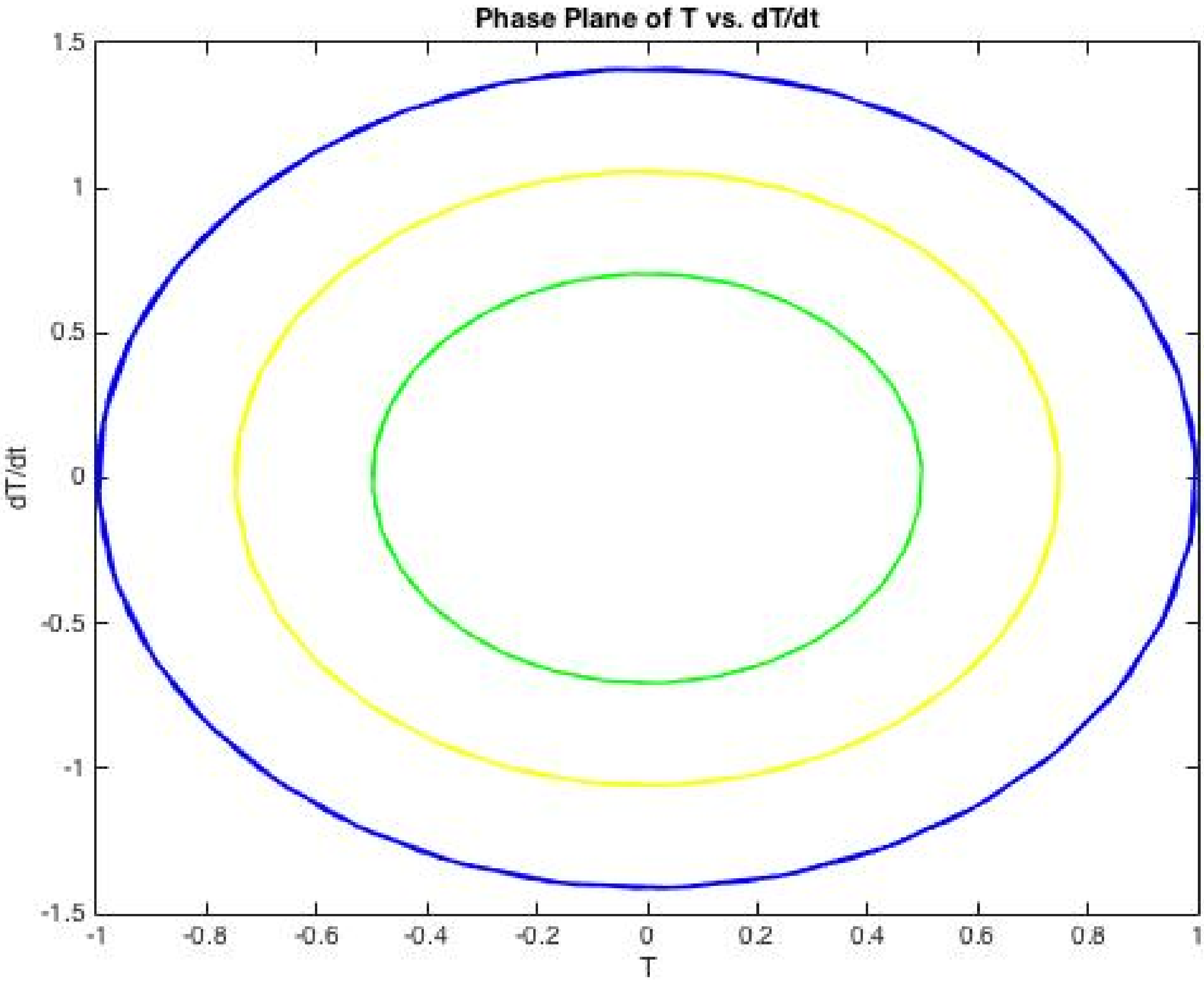
4.4 Plots



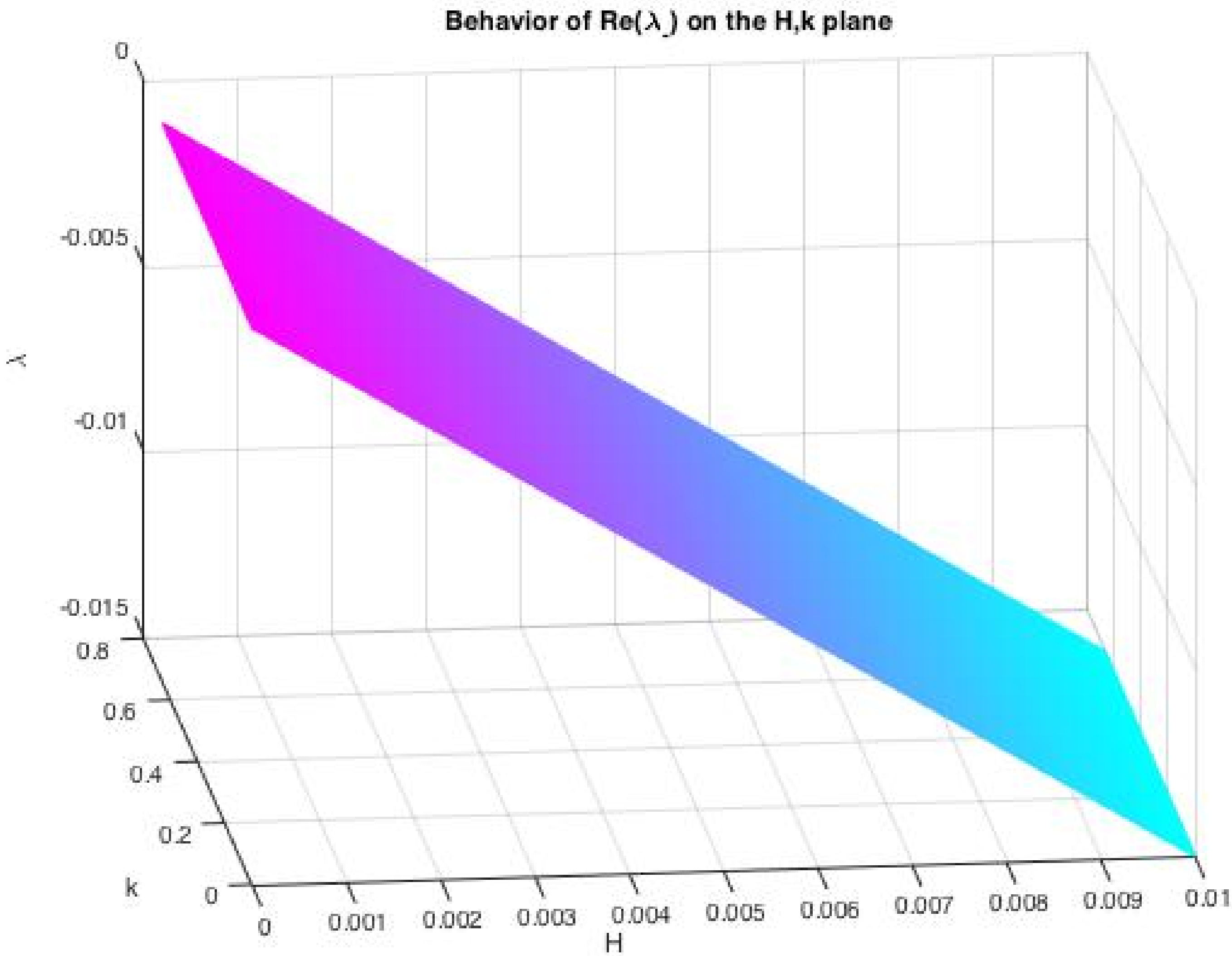
Phase Plane of R
R=1,R'=0
R=1,R'=0
R=1,R'=.02
R=-1, R'=-0.02
R=-2, R'=-0.02
dR/dr
R
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5

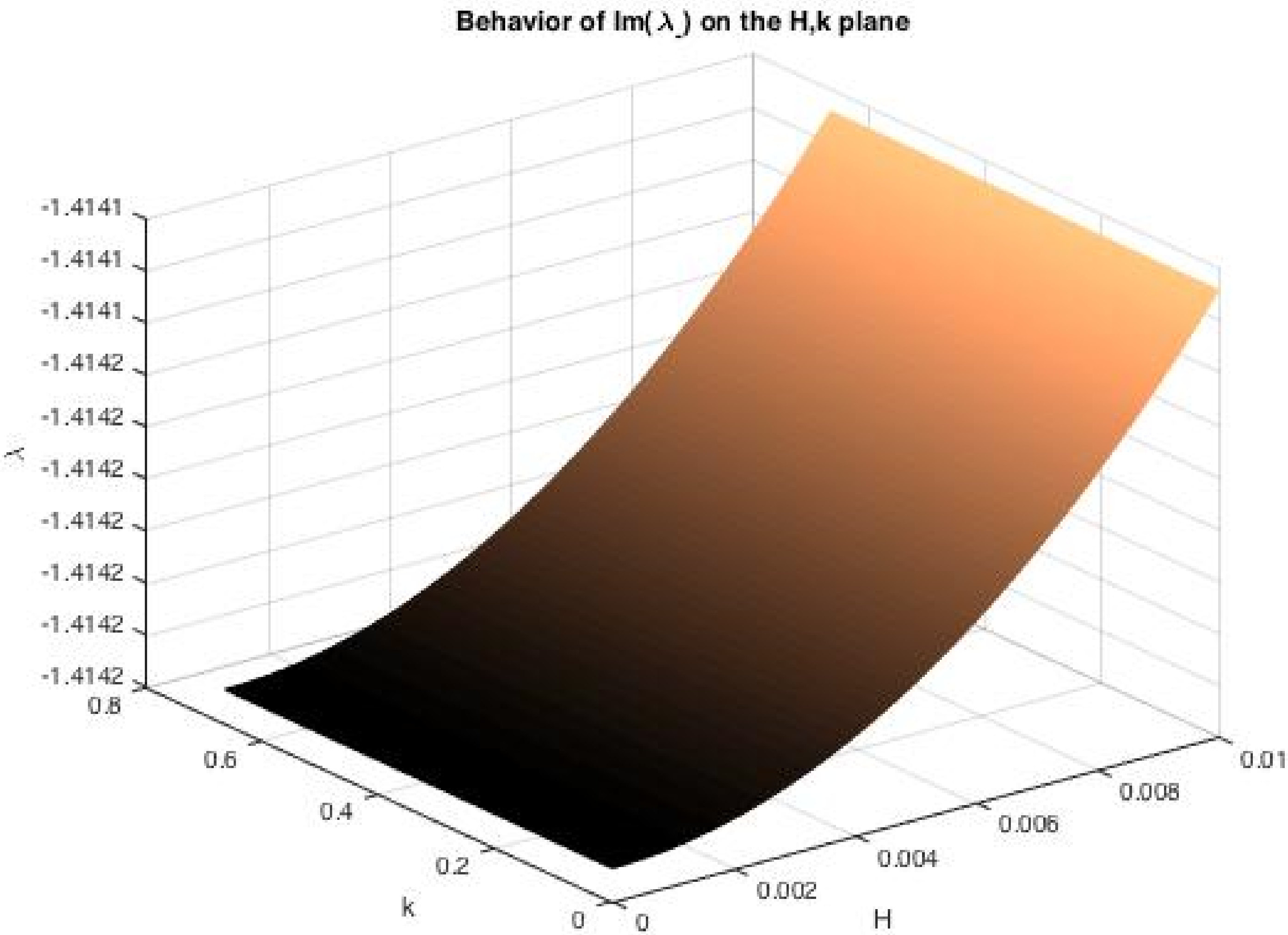

Evolution of Radial Solution R
R
r
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
50
100
150
200
250
300
350
400
450
500


Evolution of Temporal Solution T
T
t
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
10
20
30
40
50
60
70
80
90
100

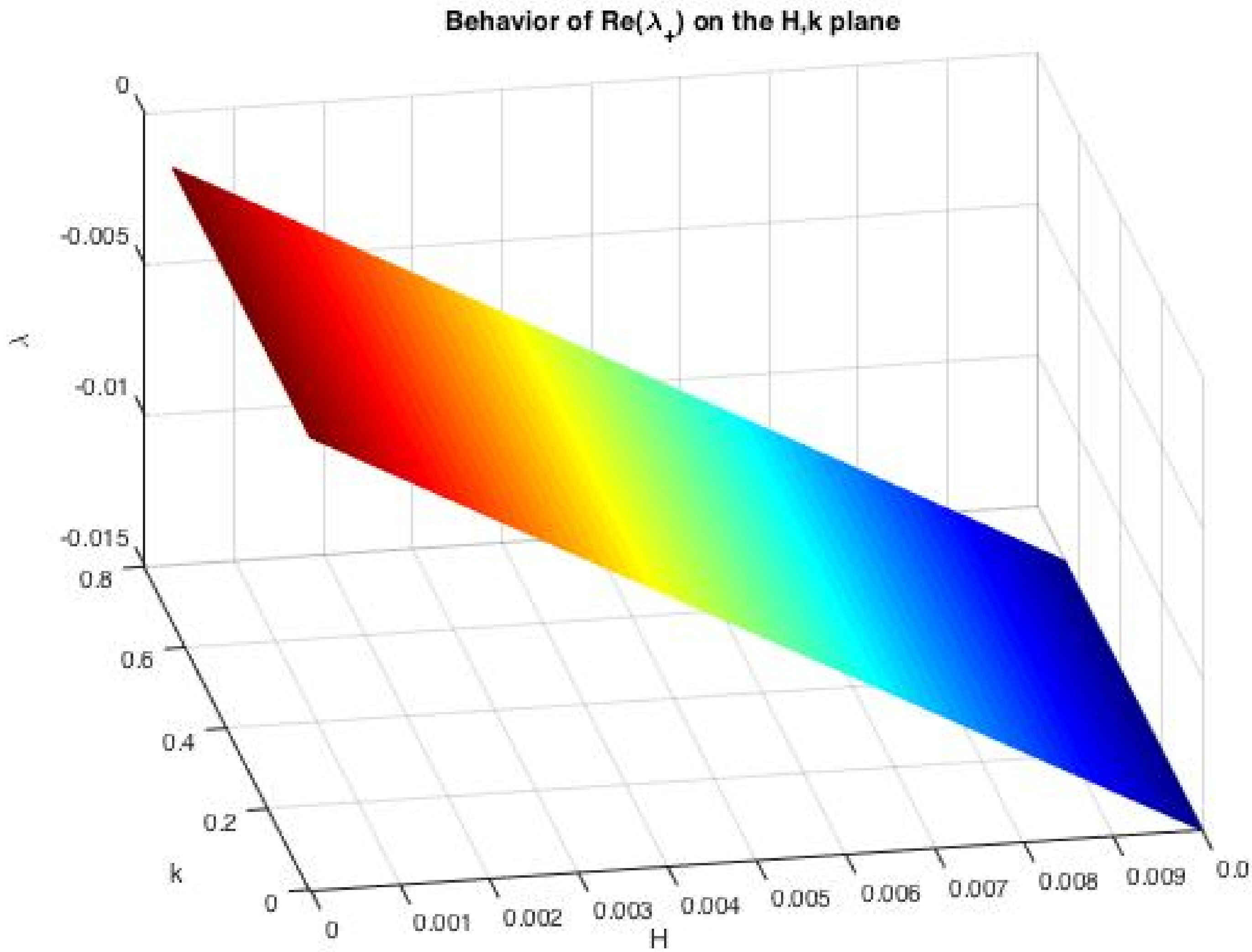


Phase Plane of T vs. dT/dt
1.5
1
0.5
0
-0.5
-1
-1.5
dT/dt
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
T

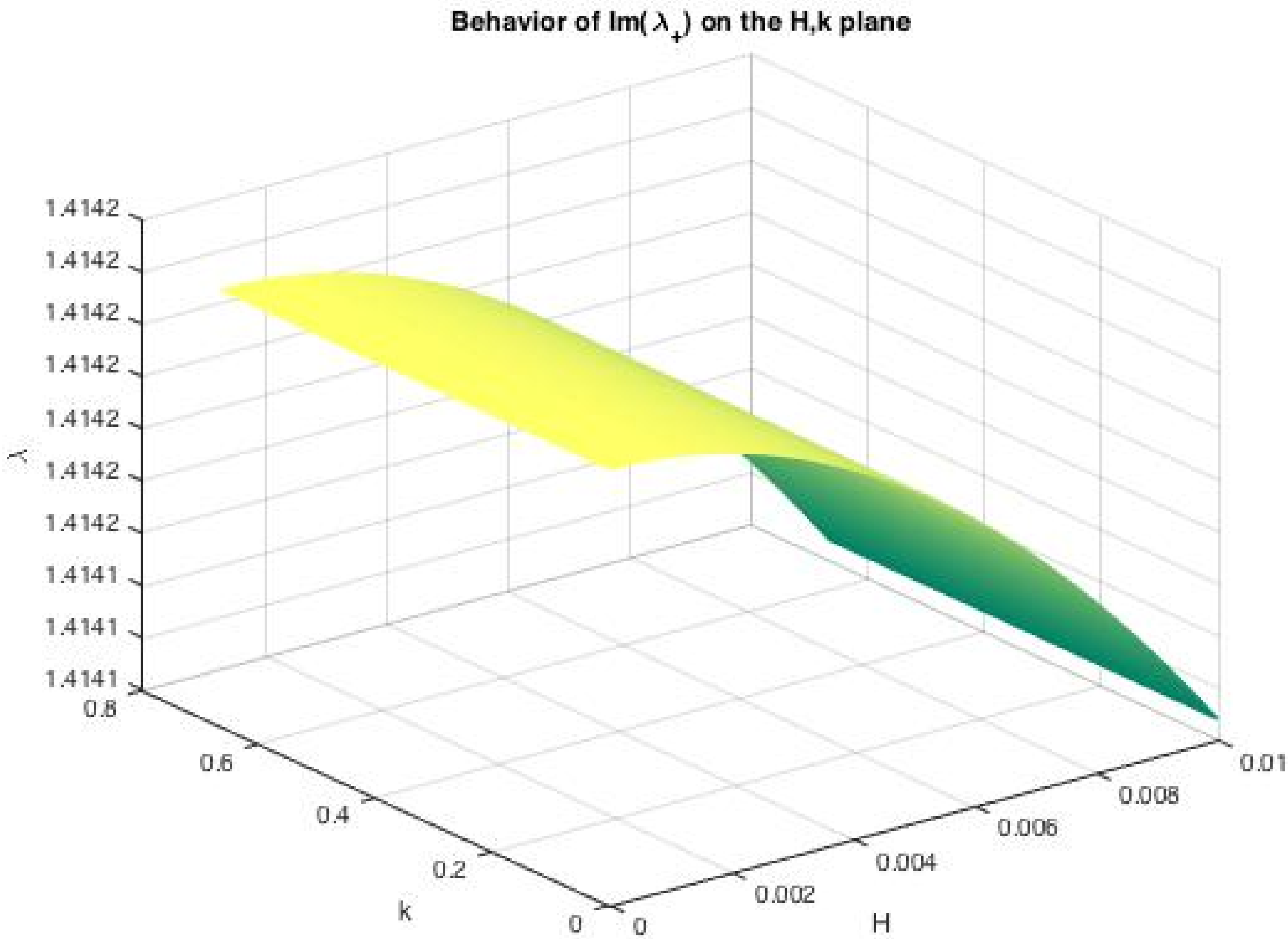

Behavior of Re(λ_) on the H,k plane
0
-0.005
-0.01
-0.015
λ
0.8
0.6
0.4
0.2
0
k
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01
H



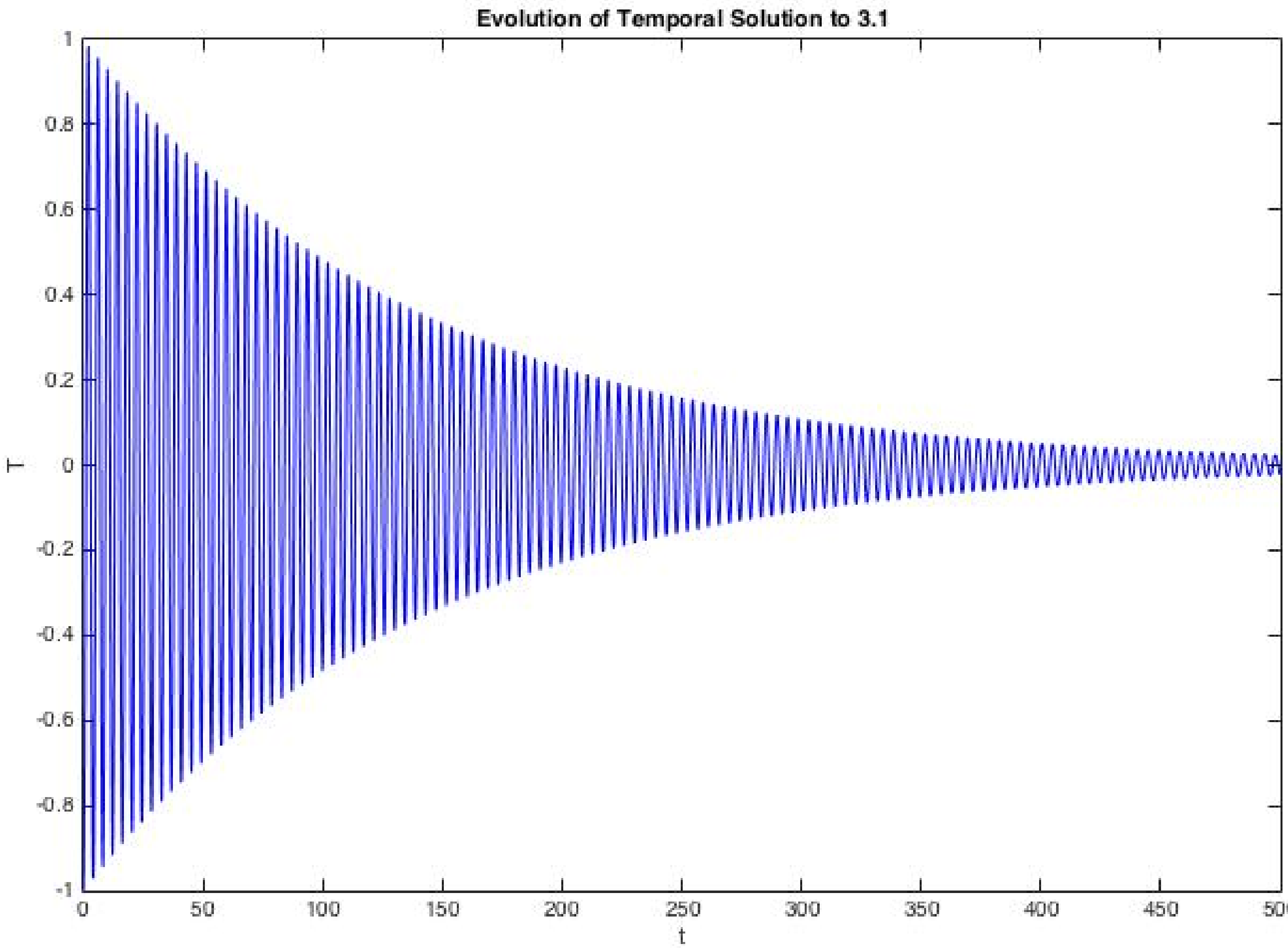
Behavior of Im(λ_) on the H,k plane
-1.4141
-1.4141
-1.4141
-1.4142
-1.4142
-1.4142
-1.4142
-1.4142
-1.4142
-1.4142
λ
0.8
0.6
0.4
0.2
0
k
0
0.002
0.004
0.006
0.008
0.01
H



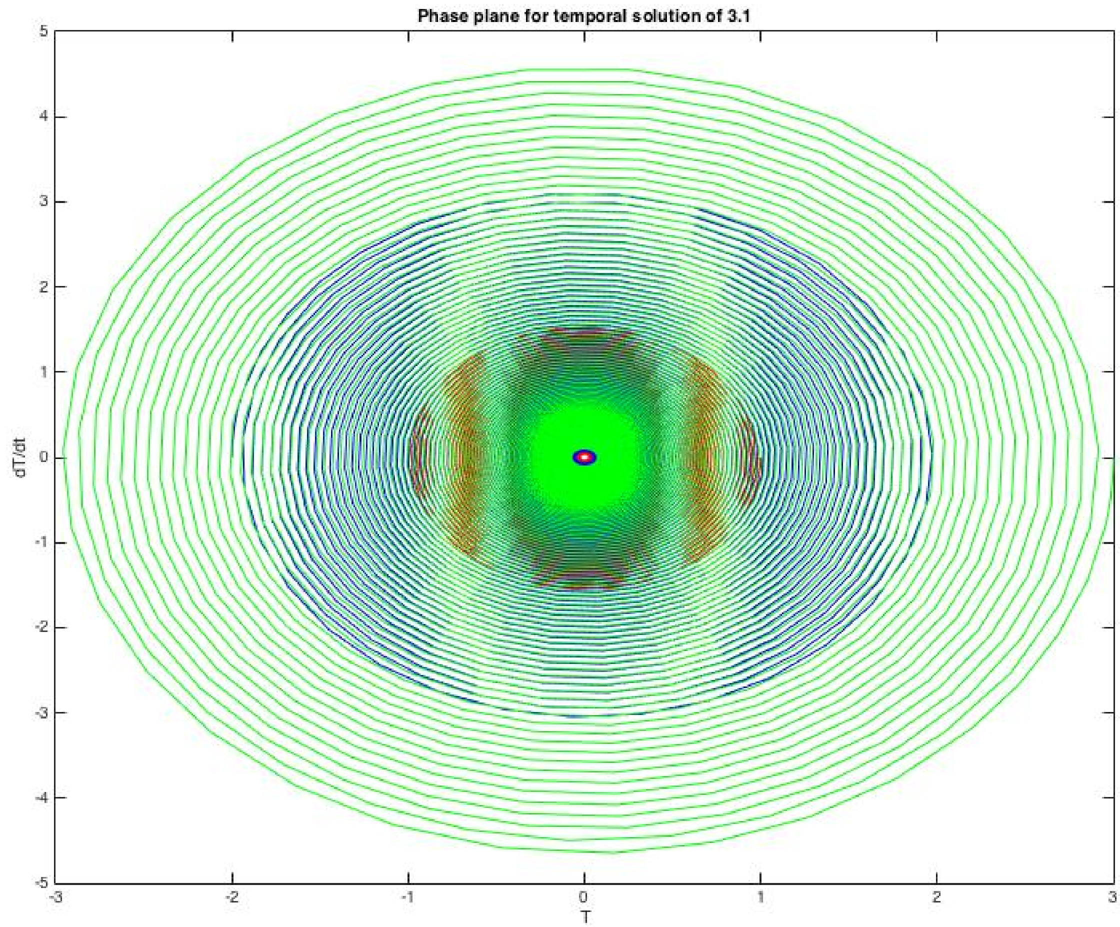
Behavior of Re(λ+) on the H,k plane
0
-0.005
-0.01
-0.015
λ
0.8
0.6
0.4
0.2
0
k
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01
H



Behavior of Im(λ_+) on the H,k plane
1.4142
1.4142
1.4142
1.4142
1.4142
1.4142
1.4142
1.4141
1.4141
1.4141
λ
0.8
0.6
0.4
0.2
0
k
0
0.002
0.004
0.006
0.008
0.01
H



Evolution of Temporal Solution to 3.1
T
t
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
50
100
150
200
250
300
350
400
450
500


Phase plane for temporal solution of 3.1
dT/dt
T
5
4
3
2
1
0
-1
-2
-3
-4
-5
-3
-2
-1
0
1
2
3