EXERCISES FOR WEEK 3

Solutions to these problems are due on Wednesday, October 13.

1. Suppose that scientists measure the total yearly precipitation at a certain point for 100 years. On average, how many of those years will have record high precipitation? (For the purposes of this problem, please assume that the weather data is uncorrelated from year to year, with no long-term trends.)

2. Give a heuristic argument based on the previous problem for why the harmonic series diverges.

An *inversion* in the permutation π is a pair of indices $i < j$ such that $\pi(i) > \pi(j)$. The *Eulerian number* $I(n,k)$ denotes the number of permutations of $[n]$ with k inversions.

3. (Easy) Prove that $I(n,k) = I(n, \binom{n}{2} - k)$.

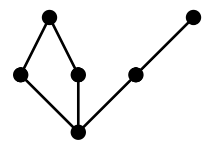
4. Define the polynomial $P_n(x) = \sum I(n,k)x^k$, where the sum is over all k between 0 and $\binom{n}{2}$. Prove that P_n satisfies

$$P_n(x) = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}).$$

5. (Easy) Prove that the Eulerian numbers are unimodal. (The Eulerian numbers are actually log-concave, but this is more difficult to prove.)

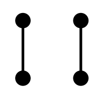
6. Prove that every poset with n elements contains either a chain or antichain with at least $\sqrt{n}$ elements. (This is a generalization of a famous theorem of Erdős and Szekeres.)

7. Compute the Möbius function for the poset P whose Hasse diagram appears below. Please express your answer as an upper-trangular matrix.



An *interval order* is a poset defined on closed intervals of real numbers, with the ordering $[a,b] < [c,d]$ if and only if $b < c$, that is, $[a,b] < [c,d]$ if and only if $[a,b]$ lies entirely to the left of $[c,d]$.

8. Prove that any poset isomorphic to an interval order cannot contain four elements w, x, y, z which satisfy $w < x$ and $y < z$ with w and x both incomparable to y and z. In pictures, this says that an interval order cannot contain the poset $\mathbf{2}+\mathbf{2}$ shown below.



9. *Extra credit:* Prove the converse to the previous problem, that every poset that does not contain $\mathbf{2}+\mathbf{2}$ is isomorphic to an interval order.

10. *Extra credit:* A *unit interval order* is defined like an interval order, but the intervals must all be of the same length (normally taken to be 1, but this doesn't matter). What can you say about unit interval orders?