EXERCISES FOR WEEK 5

Solutions to these problems are due on Wednesday, October 28.

1. Find the generating function for the sequence $\{a_n\}$ given by $a_0 = 0$ and

$$a_n = 2a_{n-1} + \binom{n}{2}$$

for all $n \geqslant 1$.

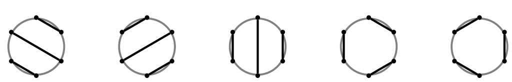
2. Find an explicit formula for the nth term of the sequence in the previous exercise.

3. Prove using Newton's Binomial Theorem that the coefficient of x^n in

$$\frac{1-\sqrt{1-4x}}{2x}$$

really is $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

4. Fix $2n$ equally spaced points on a circle. Show that the number of ways to join these points in pairs so that the resulting n line segments do not intersect equals the n^{th} Catalan number. For example, here are the 5 ways to do this when $n = 3$:



5. Prove, by any means you like *except* for citing Exercise 8, that the sequence $\{n^d\}$ has a rational generating function for all positive integers d.

A *descent* in the permutation π is an index i for which $\pi(i) > \pi(i+1)$. The *Eulerian number* $A(n,k)$ denotes the number of permutations of $[n]$ with $k-1$ descents. For example, when $n = 3$, we have the following:

k	permutations with $k-1$ descents	$A(3,k)$
1	123	1
2	132, 213, 231, 312	4
3	321	1

As we have done with various other bivariate sequences, we define the *Eulerian polynomials* by

$$A_n(x) = A(n,1)x + A(n,2)x^2 + \cdots + A(n,n)x^n.$$

6. (Easy) Prove that $A(n,k) = A(n, n+1-k)$.

7. Prove that

$$A(n,k) = kA(n-1,k) + (n-k+1)A(n-1,k-1).$$

8. Prove, by induction on d, that the generating function for the sequence $\{n^d\}$ is

$$\frac{A_d(x)}{(1-x)^{d+1}}.$$

9. *(Extra credit)* Give a combinatorial explanation for the result of the previous exercise.