

## EXERCISES FOR HOMEWORK #6

Solutions to these problems are due on Wednesday, November 11.

1. How many maximal independent sets (MISes) does a path with  $n$  vertices have? (Give a formula or recurrence.)

2. Define

$$g(n) = \begin{cases} 3^k & \text{if } n = 3k, \\ 4 \cdot 3^{k-1} & \text{if } n = 3k + 1, \\ 2 \cdot 3^k & \text{if } n = 3k + 2. \end{cases}$$

We proved in class that  $g(n)$  is the maximum number of MISes that a graph on  $n$  vertices can have. Give a proof from first principles (i.e., without using this MIS fact) that  $g(n)$  is the greatest number that can be expressed as the product of nonnegative integers whose sum is  $n$ .

3. What happens if we replace "integer" in the previous problem with "real number", i.e., what is the greatest real number that can be expressed as the product of nonnegative real numbers whose sum is  $n$ ?

A *separating family of sets over  $[m]$*  is a collection  $\mathcal{S}$  of subsets of  $[m]$  with the property that for every

$i, j \in [m]$ , there are *disjoint* sets  $S, T \in \mathcal{S}$  with  $i \in S$  and  $j \in T$ . In 1973, Gyula Katona raised the question of determining

$$f(m) = \min \left\{ n \ : \ \begin{array}{l} \text{there is a separating family} \\ \text{over } [m] \text{ with } n \text{ sets.} \end{array} \right\}.$$

This question laid open for 9 years until it was solved by Mao-Cheng. Exercises 4–6 present a simpler proof, using the Moon-Moser theorem on MISes (which was proved in 1965).

4. Suppose that the graph  $G$  has  $n$  vertices and  $m$  MISes. Show how to use  $G$  to construct a separating family of  $[m]$  with  $n$  sets.

5. Suppose that the separating family  $\mathcal{S}$  of  $[m]$  has  $n$  sets. Show how to use  $\mathcal{S}$  to construct a graph with  $n$  vertices and  $m$  MISes.

6. Prove that

$$\begin{aligned} f(m) &= \min\{n : g(n) \geq m\}, \text{ and} \\ g(n) &= \max\{m : f(m) \leq n\}. \end{aligned}$$

This proves that  $f \circ g$  is the identity, thereby giving a formula for  $f(m)$ . What is this formula?