1. Week 1

1.1. Lecture 1: Binomial Coefficients, Unimodality, Log-Concavity

Recall that the number of k-element subsets of an n-element set is denoted by $\binom{n}{k}$. This is read "n choose k." To refresh your memory, the binomial coefficients have the following formula and recurrence.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Of course, this recurrence shows the connection between the binomial coefficients and Pascal's Triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

The binomial coefficients satisfy a vast number of interesting identities, for example, the following

$$\begin{aligned}
\sum \binom{n}{k} &= 2^n, \\
\sum k \binom{n}{k} &= n2^{n-1}, \\
\sum \binom{n}{k}^2 &= \binom{2n}{n}.
\end{aligned}$$

It would be a good exercise to try to remember (or reinvent, or simply invent) proofs for these three. Generally there are three ways to prove any binomial coefficient identity: combinatorially, with the formula or recurrence, or via the Binomial Theorem:

Binomial Theorem. For all non-negative integers n,

$$(x+y)^n = \sum \binom{n}{k} x^k y^{n-k}.$$

Proof. Simply multiply the left-handside:

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}.$$

To get an $x^k y^{n-k}$ term in this product, we must choose precisely k xs and $n-k$ ys. Therefore the coefficient of $x^k y^{n-k}$ is the number of ways to do this, $\binom{n}{k}$. □

Notice that Pascal's Triangle increases to the maximum and then decreases. More generally, we say that the sequence $\{a_k\}$ is *unimodal* if for some integer m,

$$\begin{array}{lcll} a_{k-1} & \leqslant & a_k & \text{if } k \leqslant m \text{, and} \\ a_k & \geqslant & a_{k+1} & \text{if } k \geqslant m. \end{array}$$

Theorem 1. For every n, the sequence $\left\{\binom{n}{k}\right\}$ is unimodal.

Proof. Dividing $\binom{n}{k+1}$ by $\binom{n}{k}$,

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n-k}{k+1},$$

we see that the sequence is increasing if $n-k \geqslant k+1$, i.e., if $2k+1 \leqslant n$, and decreasing if $2k+1 \geqslant n$.
□

Let us now extend our definition a bit, and say that the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is *unimodal* if the sequence $\{a_0, a_1, \ldots, a_n\}$ is unimodal. So, we've just seen that the polynomial

$$(x+1) = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$$

is unimodal for all n. But clearly, this is not the only example of a unimodal polynomial. It would be nice to have some conditions which would imply that polynomials are unimodal without expanding them out.

First, we need to introduce a strongly property. The sequence $\{a_k\}$ is *log-concave* if

$$a_k^2 \geqslant a_{k-1}a_{k+1}$$

for all (relevant) k. This property gets its name from the fact that if $\{a_k\}$ is log-concave, then the sequence $\{\log a_k\}$ is concave, i.e., it satisfies

$$\log a_k \geqslant \frac{\log a_{k-1} + \log a_{k+1}}{2}.$$

As with unimodal, we also extend this definition to polynomials, and will say that the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is *log-concave* if the sequence $\{a_0, a_1, \ldots, a_n\}$ is log-concave.

Theorem 2. If the sequence $\{a_k\}$ is log-concave, then it is unimodal.

Proof. Rearranging the defining inequality for log-concavity, we see that

$$\frac{a_k}{a_{k-1}} \geqslant \frac{a_{k+1}}{a_k},$$

so the ratio of consecutive terms is decreasing. Until the ratios decrease below 1, the sequence is increasing, and after this point, the sequence is decreasing, so it is unimodal. □

Theorem 3. For every n, the sequence $\left\{\binom{n}{k}\right\}$ is log-concave.

Theorem 4. We want

$$\binom{n}{k}\binom{n}{k} \geqslant \binom{n}{k-1}\binom{n}{k+1}.$$

After expressing all the binomial coefficients in terms of factorials and simplifying, this reduces to

$$\frac{1}{k(n-k)} \geqslant \frac{1}{(k+1)(n-k+1)},$$

which is clearly true.

Our goal now is to prove a famous sufficient condition for unimodality, due to Newton.

Newton's Real Roots Theorem. If the polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

has positive (> 0) coefficients and all of its roots are real, then it is log-concave, and hence unimodal.

How could a condition on real roots have any relation to unimodality? The only polynomials most of us know how to solve (other than the linear ones) are quadratics, so let's look at the case where

$$p(x) = ax^2 + bx + c.$$

For $p(x)$ to be log-concave means that

$$b^2 \geqslant ac,$$

while for it to have real roots, we need that its discriminant (the quantity under the radical in the quadratic formula) must be nonnegative:

$$b^2 - 4ac \geqslant 0.$$

Clearly if $b^2 - 4ac \geqslant 0$ then $b^2 \geqslant ac$, so indeed, Newton's Real Roots Theorem holds for quadratics.

In the general proof though, we're going to need to show that $a_k^2 \geqslant a_{k-1}a_{k+1}$, so we need somehow to reduce a degree n polynomial to a quadratic. To do so, we will take derivatives, making use of the following result.

Lemma 5. If the polynomial $p(x)$ has all real roots, then its derivative has all real roots as well.

Proof. Suppose that the degree of $p(x)$ is n, so having all real roots means that it has n real roots, counting multiplicity, and that we would like to show that its derivative has $n-1$ real roots. Suppose that the roots of $p(x)$ are $r_1 < r_2 < \cdots < r_\ell$, where the root r_i occurs with multiplicity m_i. We know that its derivative has a root at r_i of multiplicity $m_i - 1$, so we only need to find the other $\ell - 1$ roots that get us up to $n-1$. Rolle's Theorem does this for us, because it says that between any two roots $p(r_i) = p(r_{i+1}) = 0$, the derivative $p'(x)$ must have a root. Since there are $\ell - 1$ pairs of consecutive roots, we have found all $n-1$ roots and we know they are all real. $\square$

The proof of Newton's Real Roots Theorem now follows from some creative shifting around of the coefficients of $p(x)$.

Proof of Newton's Real Roots Theorem. Choose k between 1 and $n-1$. We would like to prove that $a_k^2 \geqslant a_{k-1}a_{k+1}$. Our strategy is to manipulate $p(x)$ until we have a quadratic with precisely these three coefficients (plus some factorials), and then we will observe that the discriminant must be nonnegative. In order to get rid of the coefficients $a_0, a_1, \ldots, a_{k-2}$, we first define

$$q(x) = \frac{d^{k-1}}{dx^{k-1}}p(x) = (k-1)!a_{k-1} + k!a_k x + \frac{(k+1)!}{2}a_{k+1}x^2 + \cdots.$$

Note that $q(x)$ has real roots by iterating Lemma 5. Now we would like to get rid of the coefficients $a_{k+2}, a_{k+3}, \ldots, a_n$, but taking derivatives won't help us there. Instead, we flip the coefficients around:

$$\begin{aligned} r(x) &= x^{n-k+1}p\left(1/x\right), \\ &= x^{n-k+1}\left((k-1)!a_{k-1} + k!a_k\left(\frac{1}{x}\right) + \frac{(k+1)!}{2}a_{k+1}\left(\frac{1}{x}\right)^2 + \cdots\right), \\ &= (k-1)!a_{k-1}x^{n-k+1} + k!a_k x^{n-k} + \frac{(k+1)!}{2}a_{k+1}x^{n-k-1} + \cdots. \end{aligned}$$

Note that $r(x)$ is a polynomial because the degree of $q(x)$ was $n-k+1$. Also note that the roots of $r(x)$ are the reciprocals of the roots of $q(x)$, so they are all real. Now we can use the derivative trick once more to get rid of the coefficients $a_{k+2}, a_{k+3}, \ldots, a_n$:

$$s(x) = \frac{d^{n-k-1}}{dx^{n-k-1}} r(x) = \frac{(k-1)!(n-k+1)}{2} a_{k-1}x^2 + k!(n-k)!a_k x + \frac{(k+1)!(n-k-1)}{2} a_{k+1}.$$

Noticing the similarity between the coefficients of $s(x)$ and binomial coefficients, we simplify this as

$$s(x) = \frac{n!}{2}\left(\frac{a_{k-1}}{\binom{n}{k-1}}x^2 + 2\frac{a_k}{\binom{n}{k}}x + \frac{a_{k+1}}{\binom{n}{k+1}}\right).$$

Since we got $s(x)$ by taking derivatives of $r(x)$, we know that it too must have all real roots. Therefore we know that its discriminant must be nonnegative:

$$4\left(\frac{a_k}{\binom{n}{k}}\right)^2 - 4\frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}} \geqslant 0.$$

This shows that the sequence $\left\{a_k/\binom{n}{k}\right\}$ is log-concave. This is actually stronger than what we wanted. Notice that if $\{b_k\}$ and $\{c_k\}$ are both log-concave, then $b_k^2 \geqslant b_{k-1}b_{k+1}$ and $c_k^2 \geqslant c_{k-1}c_{k+2}$, so clearly $(b_kc_k)^2 \geqslant (b_{k-1}c_{k-1})(b_{k+1}c_{k+1})$. That is, the product $\{b_kc_k\}$ is also log-concave. Therefore, since we know that $\left\{a_k/\binom{n}{k}\right\}$ is log-concave, and we know that $\left\{\binom{n}{k}\right\}$ is log-concave, we may conclude that their product, $\{a_k\}$, is log-concave. $\square$