Last lecture:

The Principle of Mobiüs Inversion

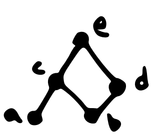
Let P be a locally finite poset, and $f, g : P \to R$ (some ring).

If
$$g(x) = \sum_{y \geq x} f(y)$$
for all $x \in P$, then
$$f(x) = \sum_{y \geq x} \mu(x,y)\, g(y),$$
where μ is the Mobiüs function of P.

Other direction:

If
$$g(x) = \sum_{y \leq x} f(y)$$
for all $x \in P$, then
$$f(x) = \sum_{y \leq x} \mu(y,x)\, g(y).$$

The Mobiüs function is defined by
$$\mu(x,x) = 1,$$
$$\mu(x,y) = 0 \text{ if } x \nleq y,$$
and for $x < y$,
$$\sum_{x \leq z \leq y} \mu(x,z) = 0.$$
This last equation shows how to define μ recursively, by
$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,y).$$

Ex: $P =$ 

$$\mu = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & & 0 & \\ 0 & 1 & & & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now fill in the rest.

Ex: Let $P = (\mathbb{N}, \leq)$. Then...

$\mu(0,0) = 1$

$\mu(0,1) = -1$

$\mu(0,2) = 0$

$\mu(0,3) = 0$

$\vdots$

In general,
$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -1 & \text{if } x = y-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: By definition, $\mu(x,x) = 1$ and $\mu(x,y) = 0$ if $x \nleq y$, so it suffices to check that
$$\sum_{x \leq z \leq y} \mu(x,z) = 0$$
if $x < y$. But this is clear, since this sum reduces to $\mu(x,x) + \mu(x,x+1)$. ■

Note: If $g(n) = \sum_{i \leq n} f(i)$, then

$$f(n) = \sum_{i \leq n} \mu(i,n)\, g(i) = -g(n-1) + g(n) = g(n) - g(n-1).$$

Ex 16.19: Let $P = (2^{[n]}, \leq)$. Then:

$$\mu(S,T) = \begin{cases} (-1)^{|T-S|} & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We must verify that this formula satisfies

$\mu(S,S) = 1$ (clearly true),
$\mu(S,T) = 0$ if $S \not\subseteq T$ (clearly true), and

$$\sum_{S \subseteq Z \subseteq T} \mu(S,Z) = 0 \text{ if } S \subsetneq T.$$

This last equation reduces to

$$\sum_{i=0}^{|T-S|} (-1)^i \binom{|T-S|}{i} = 0,$$

which follows from the Binomial Thm. ∎

Ex 16.20 Let P be the set of positive integers under the divisor order. Then:

$$\mu(x,y) = \begin{cases} (-1)^t & \text{if } \frac{y}{x} \text{ is the product of } t \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Clearly $\mu(x,x) = 1$ because $\frac{x}{x}$ is the product of 0 distinct primes. Also, $\mu(x,y) = 0$ if $x \nmid y$. Therefore it suffices to check that

$$\sum_{x|z|y} \mu(x,z) = 0$$

when $x \mid y$.

But we have $\mu(x,z) = 0$ whenever a square divides $\frac{z}{x}$, so

$$\sum_{x|z|y} \mu(x,z) = \sum_{\substack{x|z|y \\ \frac{z}{x} \text{ is square-free}}} \mu(x,z)$$

Now let $p_1, \ldots, p_t$ denote the set of distinct primes that divide $\frac{y}{x}$. There are $\binom{t}{i}$ integers z such that $x|z|y$ and $\frac{z}{x}$ is the product of i distinct primes, so this sum reduces to

$$\sum_{x|z|y} \mu(x,z) = \sum_{i=0}^{t} (-1)^i \binom{t}{i},$$

which is 0 by the Binomial Theorem. ∎

Interlude: A million dollar question.

The Mertens function is

$$M(n) = \sum_{1 \le i \le n} \mu(1,i).$$

If for every $\varepsilon > 0$ there is a constant C so that

$$M(n) < C n^{1/2+\varepsilon}$$

then the Riemann hypothesis is true.

A common generalization of these two: submultisets of a multiset.

Let M be a multiset. Then the Mobius function for all submultisets of M is

$$\mu(S,T) = \begin{cases} (-1)^{|T-S|} & \text{if } S \subseteq T \text{ and } T-S \text{ has no repeated elements} \\ 0 & \text{otherwise.} \end{cases}$$

The proof is similar to the divisor poset proof.

Products of Posets

Let P_1 and P_2 be two posets. Their product, $P_1 \times P_2$, is the poset defined on ordered pairs (x_1, x_2) with $x_1 \in P_1$ and $x_2 \in P_2$ in which $(x_1,x_2) \le (y_1,y_2)$ if and only if $x_1 \le y_1$ (in P_1) and $x_2 \le y_2$ (in P_2).

Isomorphism

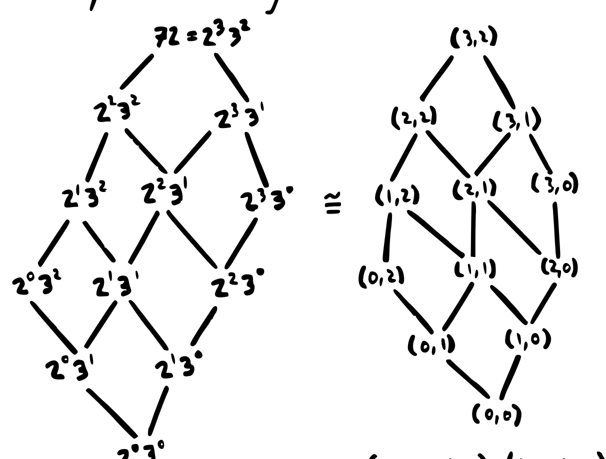
Two posets P_1 and P_2 are isomorphic if there is a bijection

$$\varphi : P_1 \to P_2$$

such that $x \le y$ in P_1 if and only if $\varphi(x) \le \varphi(y)$ in P_2.

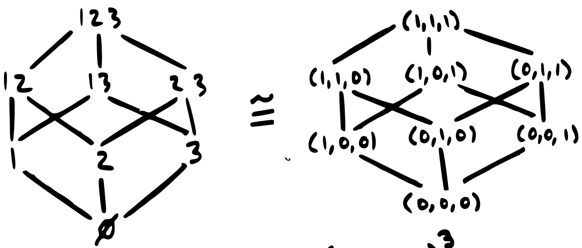
Claim: Our previous examples are isomorphic to products.

Ex: Divisors of $72 = 2^3 3^2$ ordered by divisibility:



$(\{0,1,2,3\},\le) \times (\{0,1,2\},\le)$

Ex: Subsets of $\{1,2,3\}$ ordered by $\subseteq$:



$(\{0,1\},\leq)^3$

So, it would be nice to figure out the Mobiüs functions of products...

Product Formula 16.24

Let $P = P_1 \times P_2$. The Mobiüs function of P is given by

$$\mu_P((x_1,x_2),(y_1,y_2)) = \mu_{P_1}(x_1,y_1)\,\mu_{P_2}(x_2,y_2).$$

Proof: Clearly this function satisfies

$$\mu_P((x_1,x_2),(x_1,x_2)) = 1$$

and $\mu_P((x_1,x_2),(y_1,y_2)) = 0$ if $(x_1,x_2) \not\leq (y_1,y_2)$. So it suffices to prove that for $(x_1,x_2) < (y_1,y_2)$,

$$\sum_{\substack{(x_1,x_2) \\ \leq (z_1,z_2) \\ \leq (y_1,y_2)}} \mu_P((x_1,x_2),(z_1,z_2))$$

$$= \sum_{\substack{x_1 \leq z_1 \leq y_1 \\ x_2 \leq z_2 \leq y_2}} \mu_{P_1}(x_1,z_1)\,\mu_{P_2}(x_2,z_2) \boxed{= 0}$$

But this sum can be rewritten:

$$= \left(\sum_{x_1 \leq z_1 \leq y_1} \mu(x_1,z_1)\right)\left(\sum_{x_2 \leq z_2 \leq y_2} \mu(x_2,z_2)\right),$$

so since either $x_1 < y_1$ or $x_2 < y_2$ (or both), we are done. ∎