From last time: The generating function for the sequence $a_0, a_1, a_2, \dots$ is $\sum_{n\geq 0} a_n x^n$.

Examples:

sequence	g.f.
$a_n = 1$	$\frac{1}{1-x}$
$a_n = n$	$\frac{x}{(1-x)^2}$
$a_n = n^2$	$\frac{x(1+x)}{(1-x)^3}$
$a_{n+1} = 4a_n - 100$, $a_0 = 50$	$\frac{50}{1-4x} - \frac{100x}{(1-x)(1-4x)}$
Fibonacci Numbers	$\frac{1}{1-x-x^2}$

These examples are all of a special form.

Def: The generating function $F(x)$ is rational if
$$F(x) = \frac{P(x)}{Q(x)}$$
for polynomials $P(x)$ and $Q(x)$.

Notes:

① We may assume that $\deg P(x) < \deg Q(x)$.

② We may assume that $Q(0) = 1$.

Rational Generating Functions Thm 1:

The sequence $\{a_n\}$ satifies the linear recurrence relation
$$a_{n+d} + c_1 a_{n+d-1} + c_2 a_{n+d-2} + \cdots + c_d a_n = 0$$
for all $n \geq 0$ if and only if its generating function satisfies
$$\sum_{n\geq 0} a_n x^n = \frac{P(x)}{Q(x)}$$
where
$$Q(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_d x^d$$
and $P(x)$ is a polynomial of degree less than d.

Proof: Suppose that $\{a_n\}$ satisfies the linear recurrence given. Then:
$$\left(1 + c_1 x + c_2 x^2 + \cdots + c_d x^d\right)\left(\sum_{n\geq 0} a_n x^n\right)$$
$$\begin{aligned} = a_0 + a_1 x + a_2 x^2 + \cdots &+ a_d x^d + \cdots \\ + c_1 a_0 x + c_1 a_1 x^2 + \cdots &+ c_1 a_{d-1} x^d + \cdots \\ + c_2 a_0 x^2 + \cdots &+ c_2 a_{d-2} x^d + \cdots \\ &\vdots \\ &+ c_d a_0 x^d + \cdots \end{aligned}$$
If $\{a_n\}$ satisfies the linear recurrence, then this shows that
$$Q(x) \sum_{n\geq 0} a_n x^n = \text{a poly of deg} < d.$$
Conversely, if
$$Q(x) \sum_{n\geq 0} a_n x^n$$
is a polynomial of degree $< d$, then $\{a_n\}$ must satisfy the recurrence. ∎

Rational Generating Functions Thm 2:

The sequence $\{a_n\}$ has a rational generating function if and only if there are constants

$$r_1, r_2, \ldots, r_k \in \mathbb{C}$$

and polynomials

$$P_1(n), P_2(n), \ldots, P_k(n)$$

so that

$$a_n = P_1(n) r_1^n + P_2(n) r_2^n + \cdots + P_k(n) r_k^n$$

for all $n \geq 0$.

For example, Binet's formula:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

In order to prove this, we need

Newton's Binomial Theorem: For all real numbers d,

$$(1+x)^d = \sum_{n \geq 0} \binom{d}{n} x^n,$$

where

$$\binom{d}{n} = \frac{d}{n} \cdot \frac{d-1}{n-1} \cdot \frac{d-2}{n-2} \cdot \cdots \cdot \frac{d-n+1}{1}.$$

Proof: Take derivatives...

Example:

$$\frac{x^j}{(1-rx)^d} = x^j \sum_{n \geq 0} (-r)^n \binom{-d}{n} x^n.$$

Note that

$$\binom{-d}{n} = \frac{-d}{n} \cdot \frac{-(d+1)}{n-1} \cdot \cdots \cdot \frac{-(d+n-1)}{1}$$

$$= (-1)^n \binom{d+n-1}{n},$$

so we have

$$\frac{x^j}{(1-rx)^d} = \sum_{n \geq 0} r^n \binom{d+n-1}{n} x^{n+j}$$

$$= \sum_{n \geq 0} r^n \binom{d+n-1}{d-1} x^{n+j}$$

$$= \sum_{n \geq 0} \underbrace{\left[r^{-j} \binom{d+n-j-1}{d-1} \right]}_{\substack{\text{polynomial} \\ \text{of degree } d-1}} r^n x^n$$

Proof: Suppose

$$\sum_{n \geq 0} a_n x^n = \frac{P(x)}{Q(x)}$$

where $Q(0) = 1$ and $\deg P(x) < \deg Q(x)$.

Then

$$Q(x) = \prod_{i=1}^{k} (1 - r_i x)^{d_i}$$

for some set $r_1, \ldots, r_k \in \mathbb{C}$.

Therefore, $\frac{P(x)}{Q(x)}$ is a linear combination of functions of the form

$$\frac{x^j}{(1-r_i x)^{d_i}}$$

with $j < d_i$. This direction now follows from our previous computations. The other direction is HW (not hard). ■

Example 8.9: Let $p^{\le k}(n)$ denote the number of partitions of n into parts at most k. Then

$$\sum_{n\geq 0} p^{\le k}(n)\, x^n = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\cdots\left(\frac{1}{1-x^k}\right)$$

$$= (1+x+x^2+\dots)(1+x^2+x^4+\dots)\cdots(1+x^k+x^{2k}+\dots).$$

Proof: We get an x^n term in this product by choosing a member from each geometric series. So the coefficient of x^n is the number of ways to write

$$n = 1\cdot j_1 + 2\cdot j_2 + 3j_3 + \cdots + k\, j_k.$$

This is the same as the number of ways to write

$$n = \underbrace{k+k+\dots+k}_{j_k} + \dots + \underbrace{2+\dots+2}_{j_2} + \underbrace{1+\dots+1}_{j_1},$$

verifying the g.f. ■

Corollary: Let $P_{\le k}(n)$ denote the number of partitions of n into at most k parts. Then

$$\sum_{n\geq 0} P_{\le k}(n)\, x^n = \prod_{i=1}^{k} \frac{1}{1-x^i}.$$

Proof: Conjugate the previous partitions. ■

Corollary: Let $P_k(n)$ denote the number of partitions of n into precisely k parts. Then

$$\sum_{n\geq 0} P_k(n)\, x^n = \left(\prod_{i=1}^{k-1}\frac{1}{1-x^i}\right)\left(\frac{x^k}{1-x^k}\right).$$

Proof: Subtraction. ■

Example 8.10: The g.f. for $p(n)$ is

$$\prod_{i=1}^{\infty}\frac{1}{1-x^i}$$

Proof: The coefficient of x^n here is the number of ways to write

$$n = 1\cdot j_1 + 2\cdot j_2 + \cdots,$$

which is equal to the number of partitions of n. ■

Example 8.11: The number, $P_{odd}(n)$, of n into odd parts is equal to the number, $P_d(n)$, of n into distinct parts.

Proof: $\sum P_{odd}(n)\, x^n = \prod_{i \text{ odd}} \frac{1}{1-x^i}$

$$\sum P_d(n)\, x^n = \prod_{i\geq 1}(1+x^i)$$

$$= \prod_{i\geq 1}\frac{1-x^{2i}}{1-x^i}$$

$$= \cdots$$