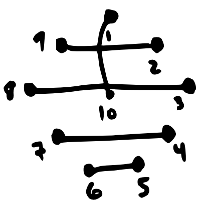
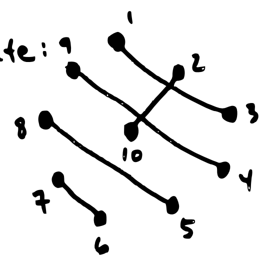
Scheduling games

10 football teams
9 Saturdays
each team must play every other.

Solution:



Rotate:



and repeat...

Terminology

A perfect matching for the graph G is a collection of edges so that every vertex is in precisely one edge.

These are also called 1-factors.

A 1-factorization of G is a partition of its edges into 1-factors.

Aside: How many 1-factorizations does K_{14} have?

98, 758, 655, 816, 833, 727, 741, 338, 583, 040.

(1 million times #stars in universe)

Our construction generalizes to show that K_{2n} has a 1-factorization for all n.

What about generalizing this?

K_{2n} "=" $\{$all 2-subsets of $[2n]\}$.

Suppose $k|n$. Can we partition the k-subsets of $[n]$ nicely?

A parallel class is a set of n/k disjoint elements of $\binom{[n]}{k}$, the set of all k-subsets of $[n]$.

If $k|n$, can we partition $\binom{[n]}{k}$ into parallel classes?

If we could, how many parallel classes would we need?

Answer: $\dfrac{\binom{n}{k}}{\frac{n}{k}} = \binom{n-1}{k-1}$.

Baranyai's Theorem (1973): If $k \mid n$, then $\binom{[n]}{k}$ can be partitioned into $\binom{n-1}{k-1}$ parallel classes.

Proof: We give the proof due to Brouwer and Schrijver.

This proof is inductive, but we need a "catalytic" variable to make the induction work.

Def: An ***m*-partition** of the set X is a multiset $\mathcal{A}$ of m pairwise disjoint subsets of X (some of which may be empty) whose union is X.

Inductive claim: Suppose $k \mid n$. For any $0 \le \ell \le n$, there exists a collection

$$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{\binom{n-1}{k-1}}$$

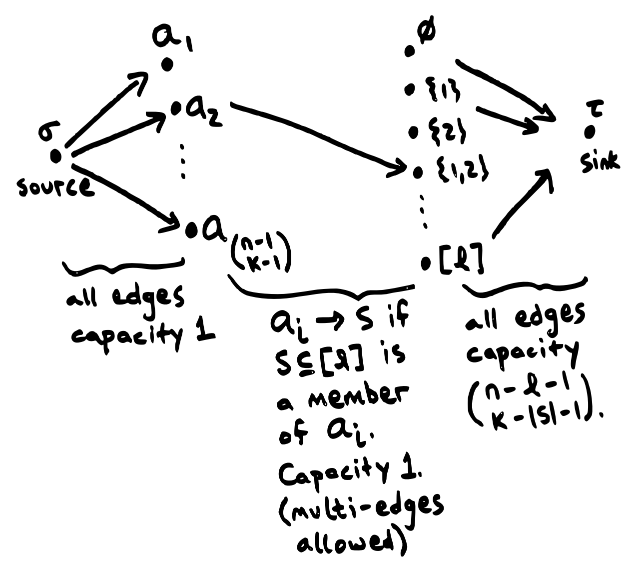
of $\frac{n}{k}$-partitions of $[\ell]$ with the property that each subset $S \subseteq [\ell]$ occurs in precisely

$$\binom{n-\ell}{k-|S|}$$

of the $\frac{n}{k}$-partions $\mathcal{A}_i$.

Note: $\ell = 0$ is trivial... $\mathcal{A}_i = \{\emptyset^{n/k}\}$.
$\ell = n$ implies the theorem.

Suppose the claim holds for $\ell \ge 0$. We construct a network.



A flow in this network:

all edges $\sigma \to a_i$ flow 1

all edges $a_i \to S$ flow $\frac{k-|S|}{n-\ell}$

all edges $S \to \tau$ flow $\binom{n-\ell-1}{k-|S|-1}$.

Out-flow at a_i:

$$\sum_{S \in a_i} \frac{k-|S|}{n-\ell} = \frac{1}{n-\ell}\left(mk - \sum_{S\in a_i} |S|\right)$$
$$= \frac{1}{n-\ell}(mk-\ell)$$
$$= 1$$

(where $m = n/k$.)

In-flow at S:

$$\sum_{i: S \in a_i} \frac{k-|S|}{n-\ell} = \frac{k-|S|}{n-\ell}\binom{n-\ell}{k-|S|}$$
$$= \binom{n-\ell-1}{k-|S|-1}.$$

This verifies that this is a flow.

Since all edges leaving σ are at capacity, this is a maximum flow.

Also, the edges entering τ are saturated. Hence they must be saturated in <u>any</u> maximum flow.

Our proof of the Max-Flow Min-Cut theorem actually proved more — if all capacities are integers, then there is an integral maximum flow.

So this network has a maximum integral flow. What does this flow look like? For each i, it assigns a flow of 1 to the edge $a_i \to S$ for some member S of a_i.

Call this member "special."

Now the flow assigns $\binom{n-\ell-1}{k-|S|-1}$ to each edge $S \to \tau$, so each $S \subseteq [\ell]$ is special for <u>precisely</u> this many n/k-partitions a_i.

It is time to construct a set

$$\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{\binom{n-1}{k-1}}$$

of n/k-partitions of $[\ell+1]$.

Do this by replacing the special member $S \in a_i$ by $S \cup \{\ell+1\}$ to form $\mathcal{B}_i$.

Each subset $T \subseteq [\ell+1]$ occurs

$$\binom{n-(\ell+1)}{k-|T|}$$

times in $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_{\binom{n-1}{k-1}}$,
completing the inductive step. ∎