The Matrix-Tree Theorem, Cont.

A spanning tree of the graph $G=(V_G, E_G)$ is a tree $T=(V_T, E_T)$ with $V_T = V_G$ and $E_T \subseteq E_G$.

We want to count these.

Def: Let G be a directed graph without loops. Let

$$V_G = \{v_1, \ldots, v_n\}$$

and

$$E_G = \{e_1, \ldots, e_m\}.$$

The incidence matrix of G is the $n \times m$ matrix A defined by

- $A_{i,j} = 1$ if e_j ends at v_i
- $A_{i,j} = -1$ if e_j begins at v_i
- $A_{i,j} = 0$ otherwise.

Theorem 10.20 Let G be a directed graph without loops, and let A be the incidence matrix of G. Remove any row (corresponds to a vertex) from A to obtain the matrix A_0. The number of spanning subtrees of G is $\det A_0 A_0^T$.

But what good is this for counting spanning trees of K_n?

The Matrix-Tree Theorem 10.21:

Let U be a simple, undirected graph on the vertices $[n]$. Define the $(n-1)\times(n-1)$ matrix L by

- $L_{i,i} = \deg i$
- $L_{i,j} = -1$ if $i \neq j$ and $i \sim j$.
- $L_{i,j} = 0$ otherwise.

Then U has precisely $\det L$ spanning trees.

Proof: We convert U into a directed graph and then apply Theorem 10.20.

Construct a directed graph G by replacing each edge of U by a pair of directed edges, one in each direction.

Let A denote the incidence matrix of G, and remove the last row of A to form A_0.

We claim that $A_0 A_0^T = 2L$.

The entry in cell (i,j) of $A_0 A_0^T$ is the product of the i^{th} and j^{th} rows of A_0.

If $i=j$, then every edge of G which begins or ends at i contributes 1, so the (i,i) entry is $2\deg_{\mathcal{U}} i$, as desired.

If $i \neq j$, then every edge from i to j (or vice versa) contributes -1. Since $\mathcal{U}$ was simple, G has either 0 or 2 edges between i and j, so the (i,j) entry is -2 if $i \sim j$ in $\mathcal{U}$, and 0 otherwise.

So, indeed, $A_0 A_0^T = 2L$, and thus

$$\det A_0 A_0^T = 2^{n-1} \det L.$$

Every spanning tree of $\mathcal{U}$ corresponds to 2^{n-1} spanning trees of G, so the theorem follows from Theorem 10.20. ■

Cayley's Theorem

The Matrix-Tree Theorem says that the # of spanning trees of K_n is

$$\det \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Add all rows to the first:

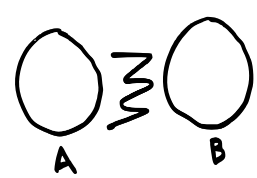
$$= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Add the first to all others:

$$= \det \begin{pmatrix} 1 & 1 & \cdots & 0 \\ 0 & n & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix} = n^{n-2}.$$

Complete Bipartite Graphs

Def: The complete bipartite graph $K_{m,n}$ has $m+n$ vertices partitioned into sets A and B with $|A|=m$ and $|B|=n$, and all edges between A and B.



How many spanning trees?

$$= \det \begin{pmatrix} nI_{m\times m} & J \\ J & mI_{n-1\times n-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & n & \cdots & 0 & -1 & \cdots & -1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n & -1 & \cdots & -1 \\ & & & & m & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ & & & & 0 & \cdots & m \end{pmatrix}$$

$$= n^{m-1} m^{n-1}.$$

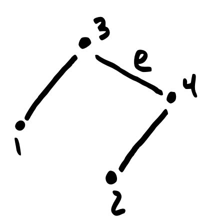
Deletion/Contraction

Here we consider a less efficient method for computing the number of spanning trees of G, which we denote $\tau(G)$.

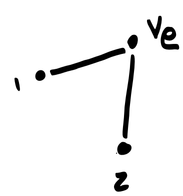
While this method isn't good for this problem, it is good for others.

Def: Let $e = xy$ be an edge of the graph $G = (V, E)$. The graph G/e is constructed by contracting the edge e to form a new vertex v_e adjacent to all neighbors of x and y.

Ex:

$G =$ 

$\downarrow$

$G/e =$ 

For any graph G and edge e, there are two kinds of spanning trees:

1. those that don't use e
   $\|$
   spanning trees of $G - e$
2. those that do use e
   $\updownarrow$
   spanning trees of G/e.

Therefore: $\tau(G) = \tau(G-e) + \tau(G/e)$.