Enumeration under group action (2/2)

Recall the general set-up from last time.

We have a set X of objects, with a group G of symmetries.

The group G "acts" on X.

We identify two objects of X if there is a group element that maps one to the other.

This defines an equivalence relation (why?) on X by

$$y \sim x \quad \text{if} \quad y = gx \text{ for some } g \in G.$$

The equivalence classes are the <u>orbits</u>,

$$\operatorname{orb}(x) = \{ gx : g \in G \}.$$

We want to count inequivalent objects, or in other words, orbits.

<u>Orbit-Counting Lemma</u>: Suppose the group G acts on the set X. Then

$$\#\text{orbits} = \text{average } \# \text{ fixed points} = \frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|,$$

where $\operatorname{fix}(g) = \{ x \in X : gx = x \}$.

Suppose we have a group G acting on a set of objects X. Without loss, we may assume G is a permutation group.

<u>Def</u>: The <u>cycle index</u> monomial of a permutation π is

$$x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k} \cdots$$

where c_k denotes the number of cycles in π of length k.
Note that as long as X is finite, these will rea be monomials.

<u>Def</u>: The <u>cycle index</u> of the group G is the average of the cycle indices of all its elements. We denote this by $Z(G)$.

<u>Example</u>: $G = D_4$

$g \in G$	$g \cdot \begin{smallmatrix}1&2\\4&3\end{smallmatrix}$	cycle index
e	$\begin{smallmatrix}1&2\\4&3\end{smallmatrix}$	x_1^4
σ	$\begin{smallmatrix}4&1\\3&2\end{smallmatrix}$	x_4
σ^2	$\begin{smallmatrix}3&4\\2&1\end{smallmatrix}$	x_2^2
σ^3	$\begin{smallmatrix}2&3\\1&4\end{smallmatrix}$	x_4
τ	$\begin{smallmatrix}1&4\\2&3\end{smallmatrix}$	$x_1^2 x_2$
$\tau\sigma$	$\tau \begin{smallmatrix}4&1\\3&2\end{smallmatrix} = \begin{smallmatrix}4&3\\1&2\end{smallmatrix}$	x_2^2
$\tau\sigma^2$	$\tau \begin{smallmatrix}3&4\\2&1\end{smallmatrix} = \begin{smallmatrix}3&2\\4&1\end{smallmatrix}$	$x_1^2 x_2$
$\tau\sigma^3$	$\tau \begin{smallmatrix}2&3\\1&4\end{smallmatrix} = \begin{smallmatrix}2&1\\3&4\end{smallmatrix}$	x_2^2

So, the cycle index of D_4 is:

$$\frac{1}{8}\left(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4 \right).$$

How can we get the number of 2-colored squares out of $Z(D_4)$?

orbits = average # fixed points.

An object is fixed by $g \in G$ if its colors are constant across cycles.

Therefore:

2-colored squares

= # orbits

= average # fixed points

$= Z(D_4)\big|_{x_1=x_2=x_3=x_4=2}$

$= \frac{1}{8}\left(2^4 + 3\cdot 2^2 + 2\cdot 2^2\cdot 2 + 2\cdot 2\right)$

$= \frac{1}{8}\left(16 + 12 + 16 + 4\right)$

$= 6.$

But we can do more.

If g has a cycle of length k, then (in a fixed object) we can have all k points (b)lue or all k points (r)ed.

So if we subsfitute

$x_1 = r + b$

$x_2 = r^2 + b^2$

$x_3 = r^3 + b^3$

$x_4 = r^4 + b^4$

into $Z(D_4)$, we get

$$\frac{1}{8}\left((r+b)^4 + 3(r^2+b^2)^2 + 2(r^2+b^2)(r+b)^2 + 2(r^4+b^4)\right)$$

$$= r^4 + r^3 b + 2r^2 b^2 + r b^3 + b^4.$$

This is the generating function for 2-colored squares "by color."

Counting graphs

Graphs on labelled vertices $[n]$:

$2^{\binom{n}{2}}$ – trivial

Graphs on labelled vertices $[n]$ that are connected:

Complicated summation via Möbius inversion in the set partition lattice.

Graphs on unlabelled vertices?

Pólya counting...

Unlabeled graphs on 4 vertices

The group of symmetries here is S_4, but unlike previous examples, we want to know about the action of this group on edges.

$\pi \in S_4$	action on edges	#	cycle index
identity	identity	1	x_1^6
2-cycle	2 2-cycles	6	$x_1^2 x_2^2$
2 2-cycles	2 2-cycles	3	$x_1^2 x_2^2$
3-cycle	2 3-cycles	8	x_3^2
4-cycle	2-cycle & 4-cycle	6	$x_2 x_4$

The cycle index is therefore

$$\frac{1}{24}\left(x_1^6 + 9x_1^2 x_2^2 + 8x_3^2 + 6x_2 x_4\right).$$

To find the generating function for unlabeled graphs on 4 vertices by the # of edges, we substitute:

$$x_1 = 1 + t$$
$$x_2 = 1 + t^2$$
$$x_3 = 1 + t^3$$
$$x_4 = 1 + t^4,$$

and we get

$$1 + t + 2t^2 + 3t^3 + 2t^4 + t^5 + t^6.$$

Note that this polynomial is:

1. symmetric, and
2. unimodal.