

Problems

Math 68, Algebraic Combinatorics, Fall 1999

1. We draw six geometric figures in the plane. In how many ways can we color them (each figure getting just one color) with four colors? With 8 colors?
2. In how many ways can we color the figures in Problem 1 if each one must get a different color?
3. In how many ways can we color the figures in Problem 1 if each color must be used?
4. How many functions are there from a six element set to a four element set? An eight element set? How many of these functions are one-to-one? How many are onto? What is the point of this question?
5. What general principles about counting can you draw from the techniques you used in doing the counting problems above?
6. In a group of 30 students, 12 are studying mathematics, 18 are studying English, 8 are studying science, 7 are studying mathematics and English, 6 are studying English and science, 5 are studying mathematics and science, and 4 are studying all three subjects. How many are studying only English? How many are not studying any of these subjects?
7. Let A_1 , A_2 and A_3 be finite sets. Write a formula for $|A_1 \cup A_2|$ in terms of $|A_1|$, $|A_2|$ and $|A_1 \cap A_2|$. Find a similar formula for $|A_1 \cup A_2 \cup A_3|$.
8. Find a formula for the number of onto functions from an m -element set to the n -element set $\{1, 2, \dots, n\}$ when $n = 1, 2, 3$ and 4.
9. Using Problem 8, guess a formula for the number of onto functions from an m -element set to an n -element set.
10. Let A be a set, P be a set of properties that each element of A might or might not have, and for each subset S of P , let $N_a(S)$ be the number of elements in A that have at least all of the properties in S (they may have more) and let $N_e(S)$ be the number of elements in A that have exactly those properties in S (and no more).

For example, if $A = A_1 \cup A_2 \cup A_3$ in Problem 7, we might say $a \in A$ has the property p_i if $a \in A_i$. Then $N_a(\{p_i\}) = |A_i|$, $N_a(\{p_i, p_j\}) = |A_i \cap A_j|$, etc.

Rewrite the formulas you found in Problem 7 in terms of these $N_a(S)$.
11. Let A be the set of functions from an m -element set to the n -element set $\{1, 2, \dots, n\}$. Let $P = \{p_1, p_2, \dots, p_n\}$ be the set of properties where $f \in A$ has the property p_i if $\{i\}$ is not in the image of f . Rewrite the formula you found in Problem 9 in terms of the $N_a(S)$ for subsets S of P .
12. What general principle can you determine for such counting problems?
13. After recess, n children throw their winter gloves into a pile near the radiator to dry. When the school day ends, each child takes a pair of gloves from the pile. In how many ways can each child leave without their own gloves if they each leave with a matching pair? What if the pairs do not necessarily match?
14. Find at least two ways to compute the number of ways to seat 6 people around a round table.

15. Find at least two ways to compute the number of one-to-one functions from a k -element set into an n -element set.
16. Find at least two ways to compute the number of k -element subsets of an n -element set.
17. A child has a dozen distinct “Popit” beads. Find at least two ways to compute the number of necklaces she can make by popping all the beads together in a circle. How is this different from Problem 14?
18. Some of the derivations of the formulas in Problems 14-17 can be explained by talking about counting equivalence classes of an equivalence relation. For each of these problems, on what set is the equivalence relation defined? How big is this set? Give a description of the equivalence relation. How big is an equivalence class? How many equivalence classes are there? What general principle are we using when we use this counting technique? State this principle as a theorem and prove it.
19. Suppose the six geometric figures in Problem 1 were six identical circles drawn in the corners of a (cardboard) regular hexagon, with all circles drawn on the same side of the hexagon. Can we answer any of Problems 1-3? If so, answer them.
20. Suppose the six geometric figures of Problem 1 were the six squares that are the faces of a cube. Can we answer any of Problems 1-3? If so, answer them.
21. Suppose we draw circles at the corners of a hexagon as in Problem 19, and we wish to color the circles with two colors. In how many ways may we do this?
22. Express the results of the previous problem in terms of describing the equivalence classes of a certain equivalence relation defined on a set of functions. Find a good notation for describing these functions compactly. Write down the equivalence classes of this equivalence relation.
23. The equivalence classes in the previous problem have either 1, 2, 3, or 6 elements. Explain how this observation is related to the fact that there are six possible ways we can rotate the hexagon around its center and have it end up in a position indistinguishable from its previous position.

We say a group G acts on a set S if for each element g of G and each element s of S , there is an element gs of S , with the elements gs defined in such a way that

1. $g_1(g_2s) = (g_1g_2)s$ and
2. $1_Gs = s$, where 1_G denotes the identity of G .

We say that the set $\{gs|g \in G\}$ is the “orbit of s under the action of G .”

24. The equivalence classes of Problems 14-17 and Problem 21 may all be described as orbits of the action of some group on some set. Figure out a group and set that makes this statement true in each of these cases. Give an informal description of the action of the group on the set.
25. We need to think carefully to get our notation to work smoothly. In Problem 21, we can think of each coloring of the circles as a function from the six vertices of the hexagon to the set $\{R, B\}$ of colors. Now imagine your hexagon on a piece of paper and next to the vertices (on the paper) write the numbers 1 through 6. Then $f(1) = R, f(2) = R, f(3) = B, f(4) = B, f(5) = R, f(6) = B$ is one such coloring. Let’s use ρ to stand for the rotation of the hexagon through one sixth of a full turn. We can think of ρ as a function defined on the vertices of the hexagon by $\rho(1) = 2, \rho(2) = 3, \rho(3) = 4, \rho(4) = 5, \rho(5) = 6, \rho(6) = 1$. After we rotate the hexagon through one turn, the color that is now in place 2 is the color that used to be in place 1. In terms of ρ and f , what is the color that is in place 2 after this rotation?

26. When a group acts on a set, each group element can be thought of as a permutation of S . Why? Show that if a group G acts on a set S , then G acts on the functions from S to another set T by the rule

$$gf = f \circ g^{-1},$$

where \circ denotes function composition. What if we were to use the rule $gf = f \circ g$?

27. When a group G acts on a set S , we say two elements x and y of S are equivalent if there is a group element g such that $gx = y$. Show that this is an equivalence relation. How are the equivalence classes related to the orbits of G acting on S ?

The *Equivalence Principle* tells us that if we have an equivalence relation on a set of size n , and each equivalence class has size m , then the number of equivalence classes is n/m . In many of the cases where a group acts on a set, the equivalence classes don't have the same size, so we can't apply the equivalence principle. By thinking of the equivalence classes as objects called "multisets," we can fix this problem. Informally a *multiset* is a set which may have repeated elements; as an example, $\{f, o, o, r\}$ is the multiset of letters of the word roof. Formally, a multiset M chosen from a set S is given by a *multiplicity function* m (or m_M if there is a chance of confusion) from S to the nonnegative integers. We call $m(x)$ the multiplicity of the element x of the multiset M . The *size* of M is $\sum_{x \in S} m(x)$. If infinitely many values of m are nonzero, then the multiset has infinite size. All the usual ideas about cardinalities of sets can be made to apply to multisets. We define the *union* of two multisets M_1 and M_2 given by multiplicity functions m_1 and m_2 to be the multiset whose multiplicity is $\max(m_1, m_2)$ and the intersection to be the multiset whose multiplicity is $\min(m_1, m_2)$. We say two multisets are disjoint if the multiplicity function of their intersection is the zero function. (There is a bit of sorting out you need to do with these definitions if M_1 is a multiset chosen from one set S and m_2 is a multiset chosen from some other set T , but you can make everything work out smoothly.) [By the way, a comment like the last one in parentheses is actually a suggestion that you do the sorting out or whatever.] When two multisets are disjoint, the multiplicity function of their union is the sum of their multiplicity functions. What does all this have to do with groups acting on sets? Well if a group G acts on a set S , then there is a very natural way to think about the orbits of G on S as multisets of size $|G|$ rather than as sets. In particular, the multiplicity of the element y in the multiset $\{gx \mid g \in G\}$ (note the use of double braces for multisets) is the number of group elements g such that $gx = y$. When we want you to think of an orbit as a multiset, we will try to remember to call it a *multiorbit*.

28. If $y \in Gx = \{gx \mid g \in G\}$, then is $x \in Gy$? Under what circumstances can we have $Gy \subseteq Gx$? Resolve the following apparent conflict: Suppose $Gx = Gy$. Then the multiplicity of y is both the number of elements g such that $gx = y$ and the number of elements g such that $gy = y$, the number of elements of G that fix y .

What is the multiplicity of x in any multiorbit it lies in? How many multiorbits does x lie in?

29. In this problem we want to lead you to develop a formula that, despite its simplicity, is very helpful for counting orbits (multiorbits) under the action of a group. First, in terms of $|G|$ and the number of (multi)orbits, what is the size of the union of the multiorbits of the action of G on a set S ? Second, how does this relate to the number of ordered pairs (g, x) such that $gx = x$? Third, note that this number of ordered pairs is the sum

$$\sum_{g \in G} \chi(g),$$

in which $\chi(g)$ denotes the number of elements of S left fixed by g . (χ is called a *permutation character* of the group.) Explain why we have proved the following:

Theorem 1 *Suppose a group G acts on a set S . Then the number of orbits of G acting on S is*

$$\frac{1}{|G|} \sum_{g \in G} \chi(g).$$

This theorem is often called Burnside's Lemma, which is unfortunate, since it was discovered for permutation groups by Cauchy and for groups in general by Frobenius. Why don't we call it the Cauchy-Frobenius-Burnside Theorem or the CFB theorem for short?

30. Now return to Problem 19. By now you know the relevant group is the group of all rotations of the hexagon. Let ρ stand for the rotation through sixty degrees. Then ρ generates the rotation group, so to find the number of colorings with four or eight colors, all we have to do is to figure out for each of these six rotations, how many coloring functions it leaves fixed. Find the number of ways to color the circles on the hexagon using four colors. Is eight colors any harder? We imagine you have already answered the question in Problem 19 for colorings where every color is different. Find the number of ways to color the circles using four colors if every color must be used.
31. To what extent can you extend the results of the previous problem to an n -gon and m colors?
32. Now do Problem 20.
33. Now the child in Problem 17 has identically shaped "Popit" beads of four different colors; she has at least 12 of each color. In how many ways may she make a necklace of 12 beads?
34. When a group acts on a set, what can you say about the multiplicities of the elements in a multiorbit? How does this let you generalize Problem 23 and find a nice description of the answer to the generalized question? State and prove the relevant theorem. This problem, Problem 23, and the apparent conflict in Problem 28 all have very nice explanations in terms of a subgroup of the group in question and the cosets of that subgroup. Find and explain this connection.
35. Suppose $2n$ people go to a restaurant for dinner, and everyone arrives in a group of two. In how many ways may these people seat themselves around a round table so that nobody is adjacent to the person he or she came with? Now suppose the groups of two are each two people of the opposite sex. In how many ways may these people seat themselves around a round table so that the seating alternates gender and nobody sits next to someone he or she came with?

Now we are going to change our focus for a while and study a topic known as *generating functions*. We will begin our study by introducing George Polya's idea of picture writing.

Suppose we are planning a snack and we decide we'll have some pears. We can symbolize the statement "We will have some pears" by

$$\delta + \delta\delta + \delta\delta\delta + \dots$$

Similarly, we can say "We will take some apples" by writing

$$\acute{\delta} + \acute{\delta}\acute{\delta} + \acute{\delta}\acute{\delta}\acute{\delta} + \dots$$

In both cases you can think of the plus sign as the "exclusive or", so in the first case we are saying "We will take one pear, or we will take two pears, or we will take 3 pears,..." and so on. What if we want to take both some pears and some apples for our snack? We could, for example, say we are going to have two pears and three apples by writing $\delta\delta\acute{\delta}\acute{\delta}\acute{\delta}$. However, it would be nice to be able to combine the statement "We will take some pears" and the statement "We will take some apples". By slightly extending the notation we have used to say we are taking two pears and three apples, we can say we will take some pears and some apples by writing

$$(\delta + \delta\delta + \delta\delta\delta + \dots)(\acute{\delta} + \acute{\delta}\acute{\delta} + \acute{\delta}\acute{\delta}\acute{\delta} + \dots).$$

Because “and” distributes over “exclusive or” in symbolic statements, this is the same as the sum of every possible combination of apples and pears with at least one of each.

A somewhat simpler way to write the expression

$$\Delta + \Delta\Delta + \Delta\Delta\Delta + \dots$$

is

$$\Delta + \Delta^2 + \Delta^3 + \dots,$$

which suggests that we can include the possibility of having no pears by

$$\Delta^0 + \Delta + \Delta^2 + \Delta^3 + \dots$$

So now if we wish to choose a snack, choosing from pears, apples, and bananas, we can picture our outcomes as

$$(\Delta^0 + \Delta + \Delta^2 + \Delta^3 + \dots)(\acute{\Delta}^0 + \acute{\Delta} + \acute{\Delta}^2 + \acute{\Delta}^3 + \dots)(\mathcal{L}^0 + \mathcal{L} + \mathcal{L}^2 + \mathcal{L}^3 + \dots).$$

In a more convenient notation to write or type, we could write our outcomes as

$$(P^0 + P^1 + P^2 + P^3 \dots)(A^0 + A^1 + A^2 + A^3 \dots)(B^0 + B^1 + B^2 + B^3 \dots)$$

We will refer to infinite series of this type informally as picture series.

If we multiply this out (we will also call the result a picture series), the display will represent all possible combinations of fruit we can choose. Now what if we would like to know the number of snacks we can choose that will have a total of six pieces of fruit?

36. What is counted by the coefficient of x^6 in the infinite series we get by substituting x for P , A , and B in all the monomials that appear after we write out the product

$$(P^0 + P^1 + P^2 + P^3 \dots)(A^0 + A^1 + A^2 + A^3 \dots)(B^0 + B^1 + B^2 + B^3 \dots)?$$

What is counted by the coefficient of x^6 we get by substituting x for P in the first series, x for A in second series, and x for B in third series and then multiplying out the three series? This illustrates a rather subtle issue. We were initially thinking of our series as standing for logical statements, with the plus sign meaning “exclusive or.” Now we are thinking of them as arithmetic objects, and this is a big difference, because if $+$ means “logical or,” then $P + P = 0$, where 0 is a logically false statement, but if $+$ means an arithmetical sum over the integers, then $P + P = 2P$. So why doesn’t it matter whether we make our change from using the plus sign logically to using it arithmetically before or after we expand out the products?

37. Suppose pears cost 40 cents each, bananas cost 30 cents each, apples cost 25 cents each and tangerines cost 20 cents each. Write down an appropriate picture series for describing the possible selections of fruit, and determine the substitutions we need to make in order to get an infinite series in x in which the coefficient of x^n is the number of snacks we can choose that cost n cents.
38. In Problem 36, suppose we have four each of the three kinds of fruit. Write down the picture series we get for possible fruit selections. The result that you get after substituting x in (so that the coefficient of x^i is the number of ways to select i pieces of fruit) can be written as the quotient of two powers of binomials. Do so.

We are using picture writing as a way of introducing generating functions. Given a sequence a_n of numbers, the (*ordinary*) *generating function* for the sequence is the “formal power series” $\sum_{i=0}^{\infty} a_i x^i$. Why do we call it a formal power series? In calculus we study something called power series, and make a big deal about convergence. In combinatorics we frequently use power series in blind disregard of whether they converge to a function or not. So what do we mean by a power series? We’ll turn the question around and ask you “What do we mean by a polynomial?” You’ll probably tell us that a polynomial over the ring R is a member of the ring $R[x]$. So what is that ring? As a set, it consists of all symbol strings of the form $\sum_{i=0}^n a_i x^i$ where n can be any nonnegative integer and the elements a_i can be any members of R . You tell us to add these symbol strings in the obvious way and to multiply them by the rule

$$\sum_{i=0}^m a_i x^i \cdot \sum_{i=0}^n b_i x^i = \sum_{i=0}^{m+n} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i$$

(If that isn’t what you’d tell us, it’s what we were hoping you’d tell us, and so we’re going to pretend you did.) In analogy, the ring $R[[x]]$ consists of all symbol strings of the form $\sum_{i=0}^{\infty} a_i x^i$, where the elements a_i can be any elements of the ring R . The elements of this ring are called (*formal*) *power series*. You add two formal power series in the obvious way, and you multiply them by the rule

$$\sum_{i=0}^{\infty} a_i x^i \cdot \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i$$

There shouldn’t be any real difference between proving that $R[x]$ is a ring and proving that $R[[x]]$ is a ring, and so we’re not going to ask you to do so. The point is that now that we have this ring, the answer to the question “What is a formal power series?” is “It is a member of the ring $R[[x]]$ ”. So now we know what the phrase “generating function,” or at least the phrase “ordinary generating function” means.

39. Expand $\frac{1}{1-x}$ as a power series.
40. In Problem 36, a selection of six pieces of fruit is a six element multiset of the set $\{A, P, B\}$. We use the notation $\binom{n}{k}$ to stand for the number of k -element multisets of an n -element set. Problems 36 and 39 suggest a way to expand $(1-x)^{-n}$ as a power series that is very similar to the binomial theorem. Figure out what this expansion is; explain your reasoning. Because we’ve come at this from a picture writing point of view, it might seem that all we can use to justify this is intuition. However, there is a combinatorial proof (similar to a standard proof of the binomial theorem) that relies only on Problem 39. Clearly state a theorem about expanding $(1-x)^{-n}$ and prove it.
41. The theorem we developed in the previous problem would be a lot more exciting if we had a formula for $\binom{n}{k}$. There is an approach to finding a formula which is similar to the following. The number of 1-1 functions from a six element set to an eight element set is $\binom{8}{6} \cdot 6!$. Counting it another way, it is $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$. Setting these two equal and dividing by $6!$ gives a formula for $\binom{8}{6}$.
 - (a) In terms of the number $\binom{n}{k}$, in how many ways can you arrange k distinct books on n shelves, assuming all the books could fit onto one of the shelves? (An arrangement of books on a shelf tells us what books go on the shelf in which order.)
 - (b) Find a formula not involving $\binom{n}{k}$ for the number of ways to arrange k distinct books on n shelves, assuming all the books could fit onto one of the shelves.
 - (c) Find a formula for $\binom{n}{k}$.
 - (d) With luck, the formula you found in part (c) is a binomial coefficient. This means there must be another explanation of the formula in terms of choosing something from something else. Try to figure one out.

42. The Vermont Maple Candy company makes 5 kinds of candy that weigh 1 ounce, 3 kinds of candy that weigh two ounces and one kind of candy that weighs 4 ounces. Write down what you consider to be the “nicest” form of the generating function for the number of ways to choose an n -ounce assortment of their candy.
43. How many 8 ounce assortments of candy can we make in Problem 42

In Problems 36-38 and 42, we have seen several examples of multiplying generating functions together in order to count something interesting. In each of these examples, we have some sets of interest. For example in Problem 38, we have the set

$$\{\text{no pears}, \Delta, \Delta\Delta, \Delta\Delta\Delta, \Delta\Delta\Delta\Delta\}$$

which we can write more conveniently as

$$\{P^0, P^1, P^2, P^3, P^4\}.$$

We have similar sets for apples and bananas. We have the more interesting set

$$\{P^0, P^1, P^2, P^3, P^4, A^0, \dots, A^4, B^0 \dots B^4, PA, P^2A, \dots, P^4A^4, \dots, P^4A^4B^4\},$$

which we can think of as the set of ordered triples of elements chosen from the sets of pear selections, apple selections, and banana selections. Each element of each one of these sets has associated with it a value. In Problem 38, the value of P^2 is 2 and the value of P^2AB^3 is 6. In other words, the value of any of the monomials we are examining is the number of fruits it represents. In Problem 37 the value of a monomial of fruit is how much it costs to buy the fruits it represents, and in Problem 42, the value of a monomial of candy is weight of the candy it represents. Our generating functions then have a special interpretation; in $\sum_{i=0}^{\infty} a_i x^i$, a_i represents the number of elements in one of our sets that have value i . Loosely speaking, values appear as exponents and numbers of elements with a given value appear as coefficients. We will call this kind of interpretation of an ordinary generating function a *counting interpretation*. In this context, there is a product principle that explains a counting interpretation of the product of two (or more) generating functions in terms of counting interpretations of the generating functions being multiplied together.

44. State and prove a product principle for (ordinary) generating functions that have counting interpretations.
45. A partition of a positive integer n is a multiset of positive integers that adds to n . What is the generating function for the number of partitions of an integer into parts of size no more than 10? On the basis of this, what would you expect the generating function for partitions of an integer to be? Why didn't we word this second part more directly, as in “What is the generating function ...?” Can you resolve this problem?
46. Show (using generating functions) that the number of partitions of an integer n into parts each used an even number of times is the same as the number of partitions of n into parts which are even.
47. The generating function in Problem 38 is actually a product of two generating functions. Since the coefficients in one of these two generating functions are not always nonnegative, however, the product principle for generating functions does not apply. Express the product as a single power series (which may have rather complicated coefficients). Find a formula for the number of k -element multisets of an n -element set such that no element has multiplicity p or more. (In other words, in how many ways may k identical pieces of candy be passed out to n children so that no child gets p or more?) Prove the formula in two different ways.
48. We have been making a lot of use of the fact that $1-x$ is the inverse of the formal power series $\sum_{i=0}^{\infty} x^i$. What power series have multiplicative inverses? Why is every ideal of $F[[x]]$ (except for the whole ring itself) a subset of the ideal (x) generated by x ?

49. Assume that $(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i$ is valid for all positive and negative integers n . How does this tell you to define $\binom{n}{k}$ when n is negative?

50. Give as many proofs as you can of the following statements about binomial coefficients.

(a) $\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$

(b) $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$

(c) $\sum_{i=0}^n \binom{k+i}{i} = \binom{k+n+1}{n}$

(d) $\sum_{i=0}^n i \binom{n}{i} = n2^{n-1}$.

51. Make sense of the formula

$$(1-4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

What is the generating function for $\binom{2n}{n}$?

Prove the formula

$$\sum_{i=0}^n \binom{2i}{i} \binom{2(n-i)}{n-i} = 4^n.$$

There is a combinatorial proof of this formula, i.e. a proof that shows both the left and right side count the same thing. For extra credit, try to find such a proof.

We are now going to go back to the idea of a group acting on a set S where we will construct generating functions for the orbits of the action of G on the functions from S to another set T . After you work through these problems (or while you work on them, if you choose), we suggest you read the book by Polya and Read to get an idea of how Polya found and understood these generating functions.

Our language will be more intuitive if we think of T as a set of “colors.” To illustrate that using this language is not restrictive, the set S might be the positions in a hydrocarbon molecule which are occupied by hydrogen, and the group could be the group of spatial symmetries of the molecule. The colors could then be radicals (including hydrogen itself) that we could substitute for each hydrogen position in the molecule. Then the number of orbits of colorings is the number of chemically different compounds we could create by using these substitutions.

So think intuitively about some “figure” that has places to be colored. (Think of the sides of a square, the beads on a necklace, circles at the vertices of an n -gon, etc.) How can we picture the coloring? If we number the places to be colored, say 1 to n , then a function from $[n]$ to the colors is exactly our coloring; if our colors are blue, green and red, then $BGRRGBG$ describes a typical coloring of 8 such places. Unless the places are somehow “naturally” numbered, this idea of a coloring imposes structure that is not really there. Even if the structure is there, visualizing our colorings in this way doesn’t “pull together” any common features of different colorings; we are simply visualizing all possible functions. We have a group (think of it as symmetries of the figure you are imagining) that acts on the places. That group then acts in a natural way on the colorings of the places and we are interested in orbits of the colorings. Thus we want a picture that pulls together the common features of the colorings in an orbit. One way to pull together similarities of colorings would be to let the letters we are using as pictures of colors commute; then our picture $BGRRGBG$ becomes $B^3G^3R^2$, so our picture now records simply how many times we use each color. If you think about how we defined the action of a group on a set of functions, a group element won’t change how many times each color is used; it simply moves colors to different places. Thus the picture we now have of a given coloring is an equally appropriate picture for each coloring in an orbit. One natural question for us to ask is “How many orbits have a given picture?” We can think of a multivariable generating

function in which the letters we use to picture individual colors are the variables, and the coefficient of a picture is the number of orbits with that picture. Such a generating function is an answer to our natural question, and so it is this sort of generating function we will seek. Since the CFB theorem was our primary tool for saying how many orbits we have, it makes sense to think about whether the CFB theorem has an analog in terms of pictures of orbits.

So what do we really mean by a picture of an element of a set? Abstractly a picture is a symbol, and we want to be able to multiply picture symbols together, add them and multiply them by integers. If these symbols are all members of a common ring, then we can do the desired operations. We have seen quite a few examples where we want to add infinitely many pictures together, and that is not something that is possible in every ring. (For example, it is not possible to add infinitely many integers together, unless all but a finite number of them are zero.) Power series rings in one or several variables (or, more generally, complete local rings) let us add together infinitely many elements in many cases.

52. You've shown that in $F[[x]]$, the ideal (x) is a unique maximal proper ideal. Thus it is a maximum proper ideal. For reasons that we don't quite understand, algebraists seem to like the phrase "unique maximal ideal" better than the phrase "maximum ideal" in this context, so we will sometimes use the preferred phrase to help you get used to it. (If you haven't pondered the difference between maximal and maximum before, note that (x) is a maximal proper ideal but not a maximum proper ideal in $F[x]$.) State and prove the corresponding result (to the result in the first sentence of this problem) about ideals of a power series ring in an arbitrary number of variables.
53. We will say a function P from a set S to the ring $F[[x_1, x_2, \dots, x_n]]$ (or to a complete local ring) is a *picture function* if, for any integer k , only finitely many values of P do not lie in M^k , the k th power of the unique maximal ideal of $F[[x_1, x_2, \dots, x_n]]$. Explain why it makes sense to talk about $\sum_{x:x \in S} P(x)$ even if S is infinite. We call $\sum_{x:x \in S} P(x)$ the *picture enumerator* $E(P)$ of P .
54. Suppose that P_1 and P_2 are picture functions on sets S_1 and S_2 . Define P on $S_1 \times S_2$ by $P(x_1, x_2) = P_1(x_1)P_2(x_2)$. How are $E(P)$, $E(P_1)$, and $E(P_2)$ related?
55. Suppose P is a picture function on a set T . Suppose that we define the picture of a function from some other set S to the set T to be the product of the pictures of the values of f , i.e.

$$\hat{P}(f) = \prod_{x:x \in S} P(f(x)).$$

What is the picture enumerator $E(\hat{P})$ of the set T^S of all functions from S to T ? (You may assume that both S and T are finite.)

56. Suppose now we have a group G acting on a set and we have a picture function on that set with the additional feature that for each orbit of the group, all its elements have the same picture. In this circumstance we define the picture of an orbit to be the picture of any one of its members. The *orbit enumerator* $Orb(G, S)$ is the sum of all the pictures of all the orbits. The *fixed point enumerator* $Fix(G, S)$ is the sum of all the pictures of all the fixed points of all the elements of G . We are going to construct a generating function analog of the CFB theorem. The main idea of the proof of the CFB theorem was to try to compute in two different ways the number of elements (i.e. the sum of all the multiplicities of the elements) in the union of all the multiorbits of a group acting on a set. Suppose instead we try to compute the sum of all the pictures of all the elements in the union of the multiorbits of a group acting on a set. By thinking about how this sum relates to $Orb(G, S)$ and $Fix(G, S)$, find an analog of the CFB theorem that relates these two enumerators. State and prove this theorem.
57. Apply the "Orbit-Fixed Point" theorem of the previous exercise to determine the Orbit Enumerator for the colorings, with two colors, of six circles placed at the vertices of a hexagon as in Problem 21. Compare the coefficients of the resulting polynomial with the various equivalence classes you found in Problem 22.

58. Find the generating function (in variables R, B) for colorings of the faces of a cube with two colors (red and blue).

Polya's famed enumeration theorem deals with situations such as those in Problems 57 and 58 in which we want a generating function for the set of all functions from a set S to a set T on which a picture function is defined, and the picture of a function is the product of the pictures of its multiset of values. The point of the next series of problems is to analyze the solution to Problems 57 and 58 in order to see what Polya saw.

59. In Problem 57 we have four kinds of group elements: the identity (which fixes every coloring), the rotations through 60 or 300 degrees, the rotations through 120 and 240 degrees, and the rotation through 180 degrees. The fixed point enumerator for all coloring functions is the sum of the fixed point enumerators of colorings fixed by the identity, of colorings fixed by 60 or 300 degree rotations, of colorings fixed by 120 or 240 degree rotations, and of colorings fixed by the 180 degree rotation. Why? Write down each of these enumerators individually and factor each one as completely as you can.
60. In Problem 58 we have five different kinds of group elements, and the fixed point enumerator is the sum of the fixed point enumerators of each of these kinds of group elements. For each kind of element, write down the fixed point enumerator for the elements of that kind. Factor the enumerators as completely as you can.
61. As in the Problems 58 and 60, find the generating function for colorings of the faces of a cube with four colors, all of which are used in each coloring.
62. Show that an action of a group G on a set X gives a homomorphism from G into the symmetric group of all permutations of X , and a homomorphism φ from G into the symmetric group of all permutations of X gives a permutation representation of G on X by $gx = \varphi(g)(x)$. Note that among other things, this tells us that when we have a permutation representation of a group G , each element of G corresponds to a unique permutation of X (but a permutation of X could correspond to 0, 1, or many group elements.)
63. In Problems 59 and 60, each "kind" of group element has a "kind" of cycle structure (namely the cycle structure of the associated permutation). Discuss the relationship between the cycle structure and the factored enumerator.
64. The usual way of describing Polya's enumeration theorem involves the "cycle indicator" or "cycle index" of a group acting on a set. Suppose we have a group G acting on a finite set S . Since each group element can be thought of as a permutation of S , as such it has a decomposition into disjoint cycles. Suppose g has c_1 cycles of size 1, c_2 cycles of size 2, ..., c_n cycles of size n . Then the *cycle monomial* of g is

$$z(g) = z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}.$$

The *cycle indicator* or *cycle index* of G acting on S is

$$Z(G, S) = \frac{1}{|G|} \sum_{g: g \in G} z(g).$$

How can you compute the Orbit Enumerator of G acting on functions from S to a finite set T from the cycle index of G acting on S ? (You will need to choose a notation for the pictures of elements of T .) State and prove the relevant theorem! This is Polya's famous enumeration theorem.

65. Suppose we make a necklace by stringing 12 pieces of brightly colored plastic tubing onto a string and fastening the ends of the string together. We have ample supplies blue, green, red, and yellow tubing available. Give a generating function in which the coefficient of $B^i G^j R^k Y^h$ is the number of necklaces we can make with i blues, j greens, k reds, and h yellows. How many of these necklaces have 3 blues, 3 greens, 2 reds, and 4 yellows?

66. Show how to find numbers a , b and c such that

$$\frac{1}{(1-x)(1-2x)(1-3x)} = \frac{a}{1-x} + \frac{b}{1-2x} + \frac{c}{1-3x}$$

67. The tower of Hanoi puzzle consists of three equal sized pegs attached to a base. Around one peg there are n rings of different diameters, each of which is above only rings with larger diameter.



As the picture shows, the goal of the puzzle is to move all the rings to another peg. You are allowed to move only one ring at a time, and you can only move it to another peg. You are not allowed to put a ring on top of one with smaller diameter. Let a_n be the minimum number of moves needed to move all the rings from one peg to another peg. Explain why $a_n = 2a_{n-1} + 1$. What are a_0 and a_1 ? Are they properly related by the equation for a_n ? Guess a solution to this recurrence relation and prove your guess is correct. (We want you to write your proof especially nicely please.) Multiply both sides of this equation by x^n and sum over all values of n that make sense. With some algebraic manipulation, you can write the result as

$$p(x) \sum_{n=0}^{\infty} a_n x^n = q(x)$$

for appropriately chosen polynomials or power series $p(x)$ and $q(x)$. Do so, and by using your known value for a_0 , find a formula for a_n .

An equation like the one for a_n in the previous problem is called a recurrence relation. More precisely, a *recurrence relation* for a sequence a_n is an equation that tells how to compute a_n from some (or all) preceding value of a_i , that is from values of a_i with $i < n$. Recurrence relations abound in combinatorics, and not just for functions of one variable. The classic Pascal recurrence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ is an example of a two-variable recurrence. Generating functions provide useful tools for solving recurrence relations.

68. The “Fibonacci Numbers” involve reproduction in imaginary families of rabbits. These rabbits live in pairs, and require one month to mature before they can reproduce. They do not reproduce in the month after they are born, but each pair of rabbits produces new offspring during their second month of life. Suppose we start a population (at time 0) with one pair of immature rabbits and let a_n be the number of pairs present after n months. Then a_0 and a_1 are both 1, but a_2 is 1 plus the number of rabbits born during month 1. (Note that our rabbits become mature after the end of the first month and produce new offspring after the end of the first month and before the end of the second month.) The rabbits reproduce monthly after they mature, and we don’t observe the population long enough to see them die.

- (a) If each pair of mature rabbits produces two pairs of rabbits each time they reproduce, what recurrence relation can you write down for a_n ? Solve this recurrence relation.
- (b) In Fibonacci’s original problem the rabbits produce one new pair of rabbits each time they reproduce. The numbers a_n that result are known as Fibonacci numbers. Find a formula for Fibonacci numbers and explain (in as many ways as you can) why your formula gives integers.

69. A second order linear constant coefficient recurrence relation has the form

$$a_n + ba_{n-1} + ca_{n-2} = d_n,$$

where d_n is a known sequence. The recurrence relation is called homogeneous if $d_n = 0$. A general solution for a recurrence relation is a formula involving a_0 and a_1 such that any solution can be obtained by substituting appropriate numerical values of a_0 and a_1 .

- (a) Describe the general solution to a second order linear constant coefficient homogeneous recurrence relation. The form of the solution might depend on the roots of a certain quadratic equation.
- (b) Describe the general solution to a second order linear constant coefficient recurrence relation.
70. When we evaluate a product of n matrices, we multiply together two of the matrices first, replace them by their product and continue to work on the product of $n - 1$ matrices that results. To show which matrices we multiply together, we typically use parentheses. Thus we might evaluate $M_1M_2M_3M_4$ as $(M_1((M_2M_3)M_4))$. If we think of each set of parentheses as an injunction to multiply, then the outside set of parentheses we used makes sense; ordinarily it would be superfluous. Including the outside parentheses makes this problem a bit easier; that is the real reason we have used it! Let a_n stand for the number of ways to parenthesize a product of n matrices to specify the multiplications to make in order to evaluate the product.

When we remove the matrices from a parenthesized product of matrices we get a “balanced list of parentheses.” A *balanced list of n parentheses* is a list of n left and n right parentheses such that as we read the list from left to right, the total number of left parentheses we have seen is never less than the total number of right parentheses we have seen. Notice that $(M_1((M_2M_3)M_4))$ and $(M_1(M_2(M_3M_4)))$ both give the same balanced list of parentheses when we remove the matrices. Let b_n be the number of balanced lists of parentheses we can make with n left and n right parentheses.

Let c_n be the number of ways to arrange n ones and $n - 1$ minus ones (or n red beads and $n - 1$ green ones) in a circle.

- (a) What relationships can you find among a_n , b_n and c_n ? (You may want to revisit this question in part (d).)
- (b) There is a (rather wild) recurrence relation for a_n that you can find by examining the two terms that the outer parentheses in a parenthesized matrix product tell you to multiply together. Find it. Solve it! (Easier said than done.)
- (c) There is a formula for c_n that arises by examining ways to make choices and rotational symmetry of a circle. Find it.
- (d) Comment on part (a) in light of (b) and (c). The numbers a_n are called Catalan numbers. There are lots of things they count.
71. The derangement (or hatcheck) numbers d_n are the number of permutations of $[n]$ with no fixed points. Show that

$$d_n = (n - 1)d_{n-1} + (n - 1)d_{n-2}.$$

Use this to show that $d_n = nd_{n-1} + (-1)^n$.

72. The number b_n of bijections from an n element set to itself satisfy the recurrence $b_n = nb_{n-1}$. As with the derangement numbers, this is a linear recurrence relation, but not constant coefficient. The technique we’ve introduced for solving recurrence relations with generating functions don’t work quite so smoothly for this kind of recurrence relation. (Try it if you’d like.) The problem seems to be that when we multiply both sides of the equation by x^n and sum, we can factor an x out of a series with no problem, but we can’t factor an n out because it is the summation index. If you are clever with derivatives of power series and differential equations, you can sometimes push the method into finding the sequence for you. Another approach is to multiply both sides of the recurrence by $x^n/n!$, sum over all appropriate values of n , and solve for the “exponential generating function” $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$. Try this for b_n . You get that

$$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n,$$

and if you stare at this for a few minutes you realize that that means $b_n = n!$, as it should. Try the same process on d_n .

The previous problem introduces us through the back door to the *exponential generating function* for the sequence a_n , namely the infinite series $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$. What is the basis for wanting to use an exponential generating function rather than an ordinary one? The previous problem suggests that maybe when order matters, exponential generating functions will help us solve problems that ordinary generating functions can't help us with. Our take on this is that ordinary generating functions help us in cases such as passing out apples to children, when the number of objects (apples each child gets) is the subject of interest. If we are arranging books on bookshelves (and the books are distinct), then we care what set of books goes on each shelf, so we have a value function which is set-valued rather than numerically valued. So suppose we have two bookshelves, one which can hold less than 10 books and one which can hold less than 20 books. Once we select a set of i books to put on shelf j , we can arrange it in $i!$ ways. Once we know which books are going on the first shelf, the remaining books go to the second shelf. The number of ways to arrange the books onto their shelves is the product of the number of ways to arrange the assigned books on each shelf. In the case we are discussing, if i books are assigned to shelf 1, then the number of ways to arrange them is

$$a_i = \begin{cases} i!, & \text{if } 0 \leq i < 10 \\ 0, & \text{otherwise} \end{cases},$$

Similarly, if j books are assigned to shelf 2, then the number of ways to arrange them is

$$b_j = \begin{cases} j!, & \text{if } 0 \leq j < 20 \\ 0, & \text{otherwise} \end{cases}.$$

Now, what is the coefficient c_k in the product

$$\sum_{k=0}^{\infty} c_k \frac{x^k}{k!} = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!} \sum_{j=0}^{\infty} b_j \frac{x^j}{j!}?$$

In words, $\frac{c_k}{k!}$ is the sum of terms of the form $\frac{a_i}{i!} \frac{b_{k-i}}{(k-i)!}$ where i ranges from 0 to k , giving us

$$c_k = \sum_{i=0}^k \frac{k!}{i!(k-i)!} a_i b_{k-i}.$$

Thus we may interpret c_k as the number of ways to choose i books out of a set of k books for shelf 1 times the number of ways to arrange these books on shelf 1 times the number of ways to arrange the remaining books on shelf 2. Suppose we agree that the *value* of an arrangement of a set B of books on either shelf or on the two shelves is the set B itself. Thus a_i is actually the number of arrangements of books on shelf 1 of value I , where I is a set of books of size i , and b_j is the number of arrangements of books on shelf 2 of value J , where J is a set of j books. Note that c_k is the number of ways to divide a set K into two sets, I of size i , and J of size $j = k - i$ and then to arrange the books of I on shelf 1 and the books of J on shelf 2. Thus it is the number of arrangements on the two shelves of a specific set K of k books. This suggests (at least to us) the following theorem.

Theorem 2 *Let S_1 and S_2 be sets and V_1 and V_2 be functions from S_1 and S_2 , respectively, to the subsets of a set T . Suppose that, for each i , the number of x in S_i with $V_i(x) = J$ is a function of i and the size of J alone. If a_i is the number of x in S_1 with $V_1(x) = I$ for any given subset I of T of size i and b_j is the number of y in S_2 with value J for any given subset J of T of size j , and if*

$$\sum_{k=0}^{\infty} c_k \frac{x^k}{k!} = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!} \sum_{j=0}^{\infty} b_j \frac{x^j}{j!},$$

then c_k is the number of ordered pairs (x, y) in $S_1 \times S_2$ such that $V_1(x)$ and $V_2(y)$ are disjoint sets whose union is any given subset K of T of size k .

In essence, we proved this theorem above. Let us see how it (and the natural extension by induction to a product of any number of generating functions) applies to our bookshelf problems. The exponential generating function for shelf 1 is $1 + x + \cdots + x^9$, and the exponential generating function for shelf 2 is $1 + x + \cdots + x^{19}$, so the generating function for assigning books to both shelf 1 and shelf 2 is

$$\frac{1-x^{10}}{1-x} \frac{1-x^{20}}{1-x} = (1-x^{10}-x^{20}+x^{30}) \sum_{i=0}^{\infty} \binom{1+i}{i} x^i = 1-x^{10}-x^{20}+x^{30} \sum_{i=0}^{\infty} (1+i)x^i.$$

The coefficient of $\frac{x^k}{k!}$ is thus

$$c_k = \begin{cases} 0, & \text{if } k \geq 30; \\ (29-k)k!, & \text{if } 20 \leq k < 30 \\ 10k!, & \text{if } 10 \leq k < 20 \\ (k+1)k! & \text{if } 0 \leq k < 10 \end{cases}$$

If we have one bookcase with three shelves, each of which can hold fewer than 10 books and a second bookcase with 4 shelves each of which can hold fewer than 20 books, then our exponential generating function for the number of ways to place k books into the bookcases is

$$\frac{(1-x^{10})^3(1-x^{20})^4}{(1-x)^7}$$

73. In as many ways as you can, compute the number of ways to place n books onto four shelves in a bookcase, assuming that each shelf gets at least two books and all the books could fit onto one shelf.
74. In as many ways as you can, compute the number of ways to place n books onto the four shelves in a bookcase, assuming that each shelf gets fewer than 10 books.
75. In as many ways as you can, compute the number of functions from an n -element set onto the set $\{1, 2\}$ and extend the computations to the number of functions from an n -element set onto a k -element set.
76. The number $\frac{n!}{i_1!i_2!\cdots i_k!}$ is called a *multinomial coefficient* and denoted by $\binom{n}{i_1, i_2, \dots, i_k}$ if $\sum_{j=1}^k i_j = n$.
 - (a) Show that the number of functions from $[n]$ to $[k]$ such that the inverse image of j has size i_j is a multinomial coefficient. Relate this to the number of ways of choosing k_1 elements to label with the label 1, k_2 elements to label with the label 2, and so on.
 - (b) What is the coefficient of $x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$?
77. State and outline the proof of the product theorem for a product of n exponential generating functions.
78. What is the exponential generating function for the number of functions from an n -element set to a 1-element set? Now fix an integer k . What is the exponential generating function for the number of functions from an n -element set onto a k -element set? Use your generating function to give a formula for the number of functions from an n -element set onto a k -element set. Compare the result to our inclusion-exclusion result. A function $f : N \rightarrow K$ is doubly onto the set K if each for each y in K , there are at least two elements x of N with $f(x) = y$. Find a formula for the number of doubly onto functions form an n -element set to a k -element set.
79. So far the recurrence relations we have worked with have two variables. There are quite a few two-variable recurrence relations of importance in combinatorics. The Pascal relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ is one we all know, and one of its proofs is a paradigm for the study of many other recurrence relations. That proof starts out with something like “A k -element subset of $\{1, 2, \dots, n\}$ either contains n or it doesn't.” You probably already know this proof, but we'd like you to write it down briefly for practice. If you've done this before, feel free to be very brief (but also very accurate).

80. The number $S(n, k)$ of partitions of an n -element set into k (nonempty) subsets is called a “Stirling number of the second kind.” We agree that $S(0, 0) = 1$ (the empty set of subsets of the empty set is a partition of the empty set into 0 parts, if you like), but $S(0, k) = 0$ for $k \neq 0$.

- (a) Find and prove a recurrence like the Pascal recurrence for $S(n, k)$. Just to get a feeling for the numbers, give a “Stirling’s triangle” for n up to 7.
- (b) How is $S(n, k)$ related to the number of functions from an n -element set onto a k -element set, and what formula does this give you for $S(n, k)$?
- (c) Every function from an n -element set S to an m -element set T maps onto some k -element subset K of T . Use this to explain the formula

$$\sum_{i=0}^n S(n, i)(x)_i = x^n,$$

where

$$(x)_i = \prod_{j=0}^{i-1} (x - j).$$

As with almost any empty product, we let $(x)_0 = 1$. The function $(x)_i$, is called the “ i th falling factorial power of x .” Knuth suggests we use the notation $x^{\underline{i}}$ for the function as well, and though the notation is not yet standard among mathematicians, you are welcome to use it instead. For this reason we say the x^n is the generating function for the Stirling numbers of the second kind relative to the basis of the falling factorial powers.

81. The number $s(n, k)$ of permutations with k cycles is called the Stirling number of the first kind. We agree that $s(0, 0) = 1$, and $s(0, k) = 0$ for $k \neq 0$.

- (a) Find and prove a recurrence like the Pascal relation for the numbers $s(n, k)$.
- (b) The “signed Stirling numbers” of the first kind, $s^{\pm}(n, k)$ are defined by the equation

$$(x)_n = \sum_{i=0}^n s^{\pm}(i)x^i.$$

Thus the ordinary generating function for the signed Stirling numbers of the first kind is simply $(x)_n$. Find a recurrence relation for $s^{\pm}(n, k)$ similar to your recurrence for $s(n, k)$.

- (c) Make small tables of the values in the Stirling and signed Stirling triangles of the first kind and conjecture and prove a theorem about the Stirling numbers and signed Stirling numbers of the first kind.
82. The powers of x and the falling factorial powers of x are both bases for the vector space of polynomials of degree less than or equal to n . What does this say about $n + 1$ by $n + 1$ matrices whose (i, j) entries are $S(i, j)$ and $s^{\pm}(i, j)$ respectively?
83. On our exam we learned about the Lah numbers, $L(n, k)$, the number of ways to make k nonempty lists of distinct elements of an n -element set S so that each element is in one and only one list. They turn out to be related to the rising factorial powers $(x)^{\overline{i}} = \prod_{j=0}^{i-1} x + j$, also denoted by Knuth as $x^{\overline{i}}$. Discover this relationship and prove you are right.

Here is a mathematical quandry for all of us. While we’ve derived formulas for the Lah numbers and the Stirling numbers of the first kind, we have no formula (and so far as we know, nobody else has a formula) for the Stirling numbers of the first kind. Having noted this quandry, w now change subjects!

84. Here are several examples of relations:

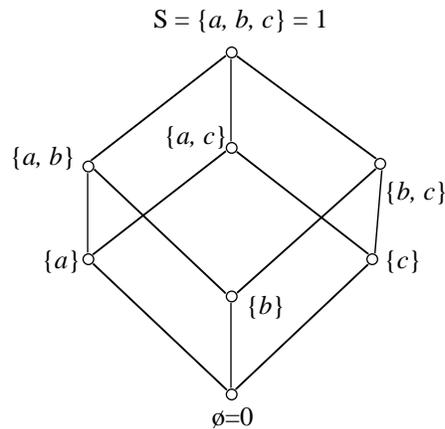
- The less than or equal to relation on the set of positive integers.
 - The subset or equal to relation on the set of subsets of a set.
 - The “divides” relation (given by m divides n if m is a factor of n , and denoted by $m|n$) on the positive integers.
 - The “refinement” relation of the set of partitions of a set S , given by $\mathbf{P} = \{P_1, P_2, \dots, P_n\}$ is a *refinement* of $\mathbf{Q} = \{Q_1, Q_2, \dots, Q_m\}$ (where $\mathbf{P} = \{P_1, P_2, \dots, P_n\}$ is a family of disjoint sets whose union is S , and similarly for \mathbf{Q}) if every set P_i is a subset of some set Q_j . We say \mathbf{Q} is *coarser* than \mathbf{P} if \mathbf{P} is a refinement of \mathbf{Q} .
- (a) A relation R is said to be *antisymmetric* if whenever x is related to y (i.e., $(x, y) \in R$) and y is related to x (i.e. $(y, x) \in R$), then $x = y$. Each of the relations above is antisymmetric. Explain why.
- (b) Antisymmetry is an example of a property that all these relations have in common. Find some other properties these relations all have in common.

The examples in Problem 84 are examples of order relations. An *order* or *ordering* on a set X , also known as (aka) a *partial order*, is a relation R on X that is reflexive, antisymmetric, and transitive. An *ordered set*, aka a *partially ordered set*, aka a *poset*, is a pair (X, P) where P is an ordering of X . We say that elements x and y are *comparable* in the ordering P if either $(x, y) \in P$ or $(y, x) \in P$. We say that an ordering of a set S is *linear* if every pair of elements of S is comparable in the ordering. The only linear ordering in Problem 84 is the less than relation on the integers. A *chain* in an ordered set is a subset C that is linearly ordered by the restriction of P to C . For example, $\emptyset, \{a\}, \{a, b\}, \{a, b, c, d\}$ is a chain in the poset of subsets of the set $\{a, b, c, d, e\}$, ordered by set inclusion. We say the ordering P of the set S is an *extension* of the ordering R on S if every ordered pair in R is also in P . In other words, P is an extension of R if, as sets of ordered pairs, $R \subseteq P$. Not surprisingly, if P is a linear order and P is an extension of R , we call P a *linear extension* of R . Because of the similarity between order relations in general and the \leq relationship on integers or the \subseteq relation on subsets of a set, we have a special notation we use with orderings. We use the phrase “ $x \preceq y$ in P ” to mean $(x, y) \in P$, and when we are really only talking about one order P , we simply write “ $x \preceq y$ ” in place of “ $x \preceq y$ in P ”. If a and b are elements of an ordered set, we define the *interval* $[a, b]$ to be the set

$$[a, b] = \{x : a \preceq x \preceq b\}.$$

Thus $[a, b]$ is empty if $a \not\preceq b$, and otherwise it consists of all elements “between” a and b .

85. Is any order in Problem 84 a linear extension of any other one? Think of the subsets of the set $\{a, b, c, d\}$ as “nonsense words” by writing the letters of a subset next to each other in alphabetical order. Thus the subset $\{b, a, c\}$ is the nonsense word abc . The empty subset corresponds to the empty word; let us agree to denote this word by \emptyset , and let us agree that this word comes before any other one in alphabetical order. Is alphabetical order on nonsense words a linear extension of the subset ordering? Now think about the subsets of $\{1, 2, 3, 4\}$ in the same way, except as “nonsense numbers” rather than nonsense words. Thus the set $\{3, 2, 4\}$ corresponds to the number 234. Is numerical order a linear extension of the subset ordering?
86. The following picture is a picture of the subset ordering on the set $\{a, b, c\}$. Note that we draw an upward sloping line between the circles representing two sets if one is a subset of another, always putting the subset lower than the superset in the picture. Further, note we don’t draw any lines that are implied by the transitive law from lines already present in the figure. We say x is *covered by* y or y *covers* x if $x \preceq y$, $x \neq y$, and, for every z , if $x \preceq z \preceq y$, then $z = x$ or $z = y$. Note the lines that we drew in the diagram correspond to the covering relation of the subset ordering.



Draw similar pictures for the divides order on the set of positive factors of 36, for the positive factors of 60 (if you get a really nice picture for 60, try 180 which has a similar nice picture), for the refinement ordering on the partitions of a three element set and for the refinement ordering on the partitions of a four element set. Describe (up to isomorphism) the possible intervals of the refinement ordering on the partitions of a four element set and the possible intervals of the divides order on the positive factors of 60 (or 180 if you drew it).

87. Does every finite ordered set have a linear extension? What do you think about infinite ordered sets? If you know how to deal with infinite sets, try to prove that what you say is true.
88. We count the people in a room, and then we ask “How many people have sisters?” Then we ask “How many people have two or more sisters?” To belabor the obvious, how do we find out the number of people with no sisters? With exactly one sister? Inclusion-exclusion counting is quite similar. Suppose we are trying to compute the number of permutations of $\{1, 2, 3\}$ with no fixed points. We ask how many permutations are there? (Play along and answer the questions as we go.) How many permutations have 1 as a fixed point? How many have 2 as a fixed point? How many have three as a fixed point? How many have 1 and 2 as fixed points? 1 and 3? 2 and 3? How many have 1, 2, and 3 as fixed points? Now we ask how many permutations have 1 and 2 and only 1 and 2 as fixed points? How many have 3 and only 3 as fixed points? How many have no fixed points? In both these examples, we have 2 functions f_a and f_e (standing for “ f -sub **at least**” and “ f -sub **exactly**”) defined on an ordered set (S, \preceq) . The two functions are related by the system of equations

$$f_a(x) = \sum_{y: x \preceq y} f_e(y) \tag{1}$$

In both cases, it turned out that we knew the numbers $f_a(x)$. We had $|S|$ equations in $|S|$ unknowns (in the first case S is $\{0, 1, 2\}$ and in the second case, S is the power set of $\{1, 2, 3\}$) and as luck would have it, we can solve the system of equations for $f_e(x)$ for every element x of S . The main question of this problem is “Was it just luck?” In other words, suppose we have an ordered set (S, \preceq) , and we have two functions related by the system of equations (1). Can we always solve the equations for f_e in terms of f_a ? In each case, take a linear extension of (S, \preceq) , and write down the system of equations so that the equation for $f_a(x)$ precedes the equation for $f_a(y)$ if and only if x precedes y in the linear extension. Think of this system of equations as the system of a matrix equation. What is special about the matrix of coefficients? Was it just luck that we were able to solve the systems for the unknowns $f_e(x)$? What if I have the system of equations defined by (1) for some arbitrary known function f_a ? Is it possible to solve for the values of the function f_e ?

89. Let $f_e(k)$ be the number of generators of a cyclic group of order k . What is

$$f_a(n) = \sum_{k:k|n} f_e(k)?$$

(Think of f_a here as standing for “f-sub **at least**.”) For a fixed m , determine whether the system of equations we get by considering all divisors n of m can be solved to give the unknowns $f_e(k)$ for all the divisors k of m .

Solving a system of equations is, in practice, often different from determining whether such a solution is guaranteed to exist. We are going to analyze the process of solving a system of equations of the form (1). One tool we could use in this analysis is matrix algebra. The subscripts that are involved in dealing with matrices would, however, slow us down, and they obscure patterns involving ordered sets. Thus we will use an approach which parallels matrix algebra, but is more faithful to the structure of our underlying ordered set. It is good for intuition’s sake to translate our results back into the terminology of matrix algebra as we go. For a partially ordered set $\mathbf{P} = (X, \preceq)$ are going to introduce a ring called the Incidence Algebra of \mathbf{P} , and a module over that ring called the Möbius Module of \mathbf{P} . For an n -element poset, the ring is isomorphic to a subalgebra of the algebra of n by n matrices, and the module is isomorphic to the vector space of column vectors of length n . The action of the ring on the module is analogous to the multiplication of column vectors by matrices.

Choose a field F . The Möbius Module of \mathbf{P} over F is simply the set of functions from X to F . (Remember, $\mathbf{P} = (X, \preceq)$.) We add two members of the module in the natural way, namely $(f + g)(x) = f(x) + g(x)$. The incidence algebra $\mathcal{I}(\mathbf{P})$ of \mathbf{P} consists of all functions φ of two variables with the property that $\varphi(x, y)$ is zero unless $x \preceq y$. The product in the ring is defined by

$$\varphi \circ \psi(x, y) = \begin{cases} \sum_{z:x \preceq z \preceq y} \varphi(x, z)\psi(z, y) = \sum_{z=x}^y \varphi(x, z)\psi(z, y) & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

We define the addition in the ring in the natural way (what is it?). Notice that for the product we have defined to make sense, we need to require that every interval $[x, y]$ of our ordered set is finite. (An ordered set is called *locally finite* if its intervals are all finite.) We make the Möbius Module into a (right) module over the incidence algebra by defining $\varphi \circ f$ by

$$(\varphi \circ f)(x) = \sum_{y:x \preceq y} \varphi(x, y)f(y).$$

(This is actually the “upper Möbius module” of (X, P) ; the same set is also a “lower Möbius module” of (X, P) which is a left $\mathcal{I}(\mathbf{P})$ module. If you are having fun with this, figure out what the action is.) Notice that for this to make sense, we need a stronger “finiteness” condition than simply that all intervals are finite. For now, when we speak of the Möbius Module of \mathbf{P} , we will assume \mathbf{P} is finite.

90. Prove that the product we have defined in the incidence algebra is associative. What function δ is the multiplicative identity?

91. When does an element of the incidence algebra have a multiplicative inverse? In particular, does the element ζ (zeta) given by

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

have an inverse?

92. The elements ζ and δ in the incidence algebra have been described above. What does $(\zeta - \delta)^n(x, y)$ count?

93. Define κ by

$$\kappa(x, y) = \begin{cases} 1 & \text{if } y \text{ covers } x \\ 0 & \text{otherwise.} \end{cases}$$

Express ζ in terms of powers of κ , using the convention that $\varphi^0 = \delta$ for any element φ of the incidence algebra. Express κ in terms of ζ .

94. The inverse of the zeta function is called the Möbius function, denoted by μ . What is $\mu(x, x)$, for any x in an ordered set \mathbf{P} ? If y covers x in \mathbf{P} , what is $\mu(x, y)$? If $x \prec y$ (which means $x \preceq y$ and $x \neq y$), what is $\mu \circ \zeta(x, y)$? What is $\zeta \circ \mu(x, y)$? What is $\sum_{z=x}^y \mu(x, z) = \sum_{z:x \preceq z \preceq y} \mu(x, z)$? What is $\sum_{z=x}^y \mu(z, yx)$?

95. What is $\mu(X, Y)$ in the ordered set of subsets of a set S if $X \subseteq Y$? What if $X \not\subseteq Y$?

96. Suppose now that \mathbf{P} is a finite ordered set. Recall that if φ is in the incidence algebra and f is in the Möbius Module, we define $\varphi \circ f(x) = \sum_{y:x \preceq y} \phi(x, y)f(y)$. Let S be a finite set of objects, let R be a set of properties that elements of S may or may not have, and let \mathbf{P} be the ordered set of subsets of R , ordered by set inclusion. Suppose that for each subset X of R , $f_e(X)$ is the number of elements of S with exactly the properties in the set X . What does $\zeta \circ f_e(X)$ count? Suppose that

$$f_a(X) = \sum_{Y:X \subseteq Y} f_e(Y).$$

How can the values of f_e be computed from the values of f_a ?

97. (The Fundamental Theorem of Möbius Inversion.) Let f be a function defined on an ordered set \mathbf{P} , and let $g(x) = \sum_{y:x \preceq y} f(y)$. How can we use μ to compute f from g ?

98. Explain how The Principle of Inclusion and Exclusion is a special case of Möbius Inversion.

99. The product of two ordered sets (X_1, P_1) and (X_2, P_2) is the ordered set on $X \times Y$ given by $(x_1, x_2) \preceq (y_1, y_2)$ if and only if $x_1 \preceq y_1$ in P_1 and $x_2 \preceq y_2$ in P_2 . This ordering, called the product ordering, is denoted by $P_1 \times P_2$. Without being too detailed, tell us how we know the product ordering is an ordering. If ζ_1 and ζ_2 are the zeta functions of (X_1, P_1) and (X_2, P_2) , and ζ is the zeta function of $P_1 \times P_2$, how can $\zeta((x_1, y_1), (x_2, y_2))$ be computed from values of ζ_1 and ζ_2 ?

100. If φ_1 and φ_2 are members of $\mathcal{I}(P_1)$ and $\mathcal{I}(P_2)$, define $\varphi_1 \cdot \varphi_2$ in $\mathcal{I}(P_1 \times P_2)$ by

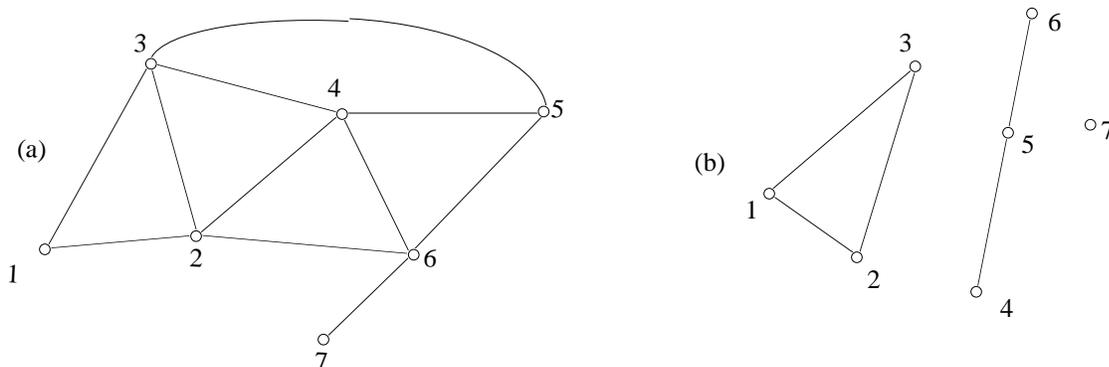
$$\varphi_1 \cdot \varphi_2((x_1, x_2), (y_1, y_2)) = \varphi_1(x_1, y_1)\varphi_2(x_2, y_2)$$

Note that we are using a raised dot rather than a raised circle to denote this “product,” and that the product takes elements of $\mathcal{I}(P_1)$ and $\mathcal{I}(P_2)$ and gives us an element of $\mathcal{I}(P_1 \times P_2)$. By the way, how do we know that this product lies in the incidence algebra of $P_1 \times P_2$? Express your answer about the zeta function of the product in the previous problem in this notation. What is the Möbius function of the product? Prove your answer is correct. What about the Möbius function of the product of a finite number of ordered sets?

101. What is the Möbius function of the ordered set of 3-tuples of zeros and ones with the product ordering? What does this have to do with the Möbius function of the ordered set of subsets of a set? What is the Möbius function of the ordered set of divisors of 180, ordered by divisibility?

102. Solve the system of equations you gave in Problem 89

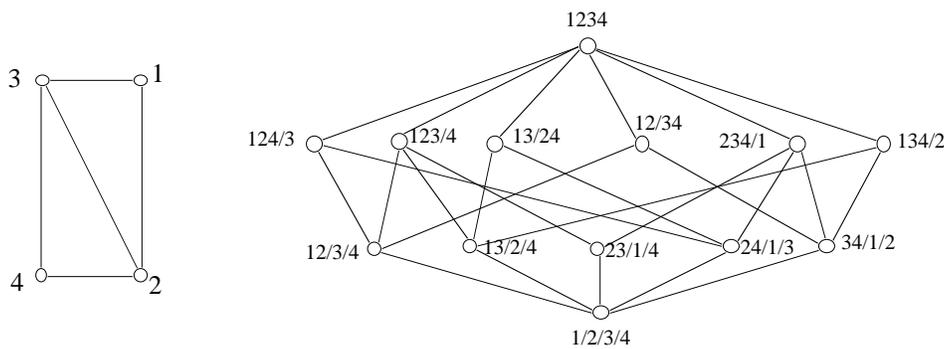
A *graph* consists of a set X , called a vertex set, and a set E , called an edge set, of two-element subsets (called *edges*) of X . We use the notation $G = (X, E)$ to stand for a graph with vertex set X and edge set E . (There are other non-equivalent definitions of graphs; let us ignore other possible definitions for now.) We draw pictures of graphs by drawing dots or circles for the vertices, and drawing a line between two vertices if they form an edge. Two graphs are illustrated in the following figure.



The graph in part (a) of the figure is called a connected graph, while the graph in part (b) is called a disconnected graph. To be more precise, a sequence $x_0e_1x_1e_2\cdots e_nx_n$ of vertices and edges is called a *walk* from x_0 to x_n if each e_i is the edge $\{x_{i-1}, x_i\}$. If no edges or vertices are repeated in the sequence, a walk is called a *path*. We say x is connected to y if there is a walk starting at x and ending at y . It is straightforward to show that the relation of being connected is an equivalence relation. The equivalence classes of this relation are called *connected components*, and a graph is called *connected* if it has one connected component.

103. How many graphs are there with vertex set $\{a, b, c, d, e, f, g\}$? How many graphs have a connected component partition that is a refinement of $\{a, c, g\}$, $\{b, e\}$, $\{d, f\}$? If $f_e(\mathbf{R})$ is the number of graphs whose connected component partition is exactly \mathbf{R} , write down a system of equations that will allow you (at least in principle) to compute $f_e(\mathbf{R})$ for every partition \mathbf{R} of the vertex set. If \mathbf{R} is the partition with one (equivalence) class, what does $f_e(\mathbf{R})$ count? Express $f_e(\mathbf{R})$ in terms of the Möbius function of an appropriate ordered set. Since we do not yet know this Möbius function, this is all we can do for now. We will come back to this problem later.

104. As you may know, “coloring graphs” is a colorful chapter of the history of combinatorial mathematics. A coloring of a graph is an assignment of colors to its vertices, i.e., a function from the vertex set to some set K of colors. (We use K so C can stand for class, as in equivalence class.) A coloring is called proper if no two vertices joined by an edge are given the same color. (The four color theorem asserts that a graph that can be drawn in the plane with no crossings among the edges can be properly colored in 4 colors.) We are interested here in the number of proper colorings of a graph G in k colors. It is probably not particularly surprising that we count proper colorings by thinking first about all colorings, subtract out those that give some two adjacent vertices the same color, add back in However we will attack the problem in a more organized way. A partition of the vertex set of a graph is called a *bond* if, for each class C of the partition, each two vertices in C are connected by a path all of whose vertices are in C . Explain why any bond is a refinement of the connected component partition of the graph. The bonds of a graph are ordered by the refinement ordering for partitions. In the figure that follows, we show a graph and the ordered set of all its bonds. Can you find a partition that is not a bond.



A coloring of a graph gives a bond as follows. We say two vertices are equivalent if there is a walk, all of whose vertices have the same color, joining them. Explain why this is an equivalence relation and why the equivalence class partition of this equivalence relation is a bond. What is the bond of a proper coloring? For a given bond \mathbf{B} of G , how many colorings of G are there whose bond is a coarser than (or equal to) \mathbf{B} ? Use this information to find a formula (which could have a Möbius function in it) for the number of proper colorings of G with k colors. Notice that you have found a polynomial function of k ; this polynomial is called the Chromatic Polynomial of the graph.

105. The complete graph on n vertices is the graph on n vertices which has all two element subsets of the vertex set as edges. What partitions are bonds of this graph, i.e. what is the ordered set of bonds of this graph? How big does k have to be for this graph to have any proper colorings with k colors? Give a formula for the number of proper colorings of this graph with $k \geq 0$ colors. Thinking of k as a variable, what is the coefficient of k in this polynomial?
106. Use \mathbf{b} (for bottom) to stand for the partition of a set X into parts of size one, and use \mathbf{t} (for top) to stand for the partition of a set X into one part. What is $\mu(\mathbf{b}, \mathbf{t})$ in the ordered set of partitions of an n -element X ? What is $\mu(\mathbf{P}, \mathbf{t})$ for an arbitrary partition \mathbf{P} of X into m classes?
107. Return to Problem 103 with the Möbius function of Problem 106, and convert your formula for the number of connected graphs on n vertices into a sum of known quantities over all partitions of the integer n .

There is a special property the four ordered sets in Problem 88 have that is not shared by many other ordered sets, namely each pair of elements has a “greatest lower bound” and a “least upper bound.” To see what this means, start by thinking about the subsets of a set. Given two subsets, their union contains both of them and is contained in any other set that contains both of them. Similarly their intersection is contained in both of them and contains any other set that is contained in both of them. Now think about the integers with their usual ordering. Given any two integers, the larger one is greater than or equal to both, and is the smallest number greater than or equal to both, and similarly for the smaller one. In the case of the positive integers ordered by divisibility, given two positive integers, their least common multiple is the smallest number that has both as a factor, and their greatest common divisor is the largest number that is a factor of both. The situation is similar with partitions of a set, but it requires thought. With these examples in mind, we say that L is a least upper bound of a subset S of an ordered set if

- $x \leq L$ for all x in S , and
- For any y in S such that $y \geq x$ for all x in S , $y \geq L$ as well.

For a two element set $\{x, y\}$, we write $x \vee y$, read as “ x join y ” for the least upper bound of S , if it exists. Greatest lower bounds are defined similarly, and the greatest lower bound of x and y is denoted by $x \wedge y$, read as “ x meet y ” if it exists. We say an ordered set is a *lattice* if each pair of elements has a greatest lower bound and a least upper bound.

108. Show that the ordered set of partitions of a set is a lattice. If we had more time we would now study lattices and especially geometric lattices, of which the partition lattice is an example.