13. Let C be a cyclic group, and let $\varphi: G \to C$ be any surjective homomorphism with $ker\ \varphi \subseteq Z(G)$. Show that G is abelian. *Hint:* Every $g \in G$ can be written as zx^i where $z \in Z(G)$ and $C = \langle \varphi(x) \rangle$.

14. Show that if a group G has a subgroup of finite index, then G has a *normal* subgroup of finite index. What is an upper bound for the index of the normal subgroup in terms of the index of the subgroup? *Hint:* Consider the action of G on G/H, where $H \leq G$ has finite index.

15. Let S be a G-set. If $s \in S$ and $a \in G$, show that the stabilizers of s and $a{\cdot}s$ are related by $G_{as} = aG_s a^{-1}$.

16. If a group G has order pq where p and q are (not necessarily distinct) primes, show that either G is abelian or $Z(G) = \{1\}$. (Hint: Use problem 13 above.) Can both possibilities occur?

17. Which of the class equations in problem 6 on page 229 is the class equation of the dihedral group D_5?

18. Is $D_4 \approx H$, where H is the quaternion group?

19. Find all subgroups of $\mathbf{Z}$ containing $12\mathbf{Z}$ and arrange them in a lattice. Then use the correspondence theorem to exhibit the lattice of all subgroups of $\mathbf{Z}/12\mathbf{Z}$.

20. Show that every group of order p^3, where p is a prime, has normal subgroups of order p and p^2. (Hint: Use the correspondence theorem, as well as other theorems proved in class.)

21. Show that no group of order 300 is simple. (A group G is simple if the only normal subgroups of G are $\{1\}$ and G itself.) Hint: Consider the action of G on the set of p-Sylow subgroups of G, for various primes p, and the kernels of the corresponding homomorphisms.

22. Assume that R and R' are rings and $\varphi: R \to R'$ is a function which preserves addition and multiplication. Show that if $\varphi(1) \neq 1$, then $\varphi(1)$ is a zero divisor of R'. (Zero divisors are defined near the bottom of page 368.)

23. Show that every ideal of the ring $\mathbf{Z}/n\mathbf{Z} = \mathbf{Z}/(n)$ is principle. (Hint: Show every subgroup is cyclic. This can be done quickly using the usual ring map $\pi: \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$, the first test problem and.... Another approach is to use the correspondence theorem.)

24. Show that the characteristic of an integral domain is a prime or zero.

25. Let R be an integral domain. Assume that $r \in R$ has the property that whenever (a) is a principle ideal in R satisfying $(r) \subseteq (a) \subseteq R$, either $(a) = R$ or $(a) = (r)$. Show that r is irreducible. That is, show that if $r = ab$ in R, then either a or b is a unit.

26. Show that irreducible elements in a UFD are prime.