Math 75, Take home test, due April 21, 2014, 11:15 AM

Instructions. Give careful explanations of your work! The first 4 problems are each worth 20 points. The last 3 problems are each worth 10 points. There are 10 bonus points. No collaboration is allowed. You may read Math 75 course notes from either s08 or s14, or our text, but no other sources. A solution should be in your own words.

1. Let F be a field. Prove using the division algorithm in $F[x]$, that for $f(x) \in F[x]$ and $a \in F$, then $f(a) = 0$ if and only if $x - a \mid f(x)$.

2. Let $A = F[x]/(x^2 + 4x + 1)$, where $F = \mathbb{Z}/(5)$. How many elements are in A? Is A a field? What is the multiplicative inverse of the element represented by x? What is the multiplicative order of x?

3. Give the complete factorization into monic irreducibles of $x^{15} + 1$ in $(\mathbb{Z}/(2))[x]$.

4. Use the Euclidean algorithm to compute the monic gcd of $f(x) = x^2 + x + 2$ and $g(x) = x^4 + 2x^2 + 1$ in $(\mathbb{Z}/(3))[x]$. Use your calculations to express the gcd in the form $a(x)f(x) + b(x)g(x)$, where $a(x), b(x) \in (\mathbb{Z}/(3))[x]$.

5. Let F be a field with p^6 elements, where p is prime. Let K denote the subset of elements from F which are a root of $x^{p^2} - x$. Show that K is a subfield of F with p^2 elements.

6. If F is a finite field of characteristic p, show that each element of F is a pth power of an element of F. That is, for each $a \in F$ there is some $b \in F$ with $a = b^p$. Find a formula for b that shows that it is fairly easy to compute.

7. Show that if p is any prime number, then there is an irreducible polynomial in $(\mathbb{Z}/(p))[x]$ of the form $x^3 - x + a$. For example, $a = 1$ works for $p = 3$, and $a = 2$ works for $p = 5$. (Hint: show that $x^3 - x$, considered as a function on $\mathbb{Z}/(p)$ to itself, is not onto.)