1. (12) Determine whether the following integral is convergent or divergent:

$$\int_1^\infty \frac{4(2+\ln x)^3}{x}\,dx.$$

$$\int \frac{4(2+\ln x)^3}{x}\,dx$$

$u = 2+\ln x$

$du = \frac{1}{x}\,dx$

$$\int 4u^3\,du = u^4 \ (+C) = (2+\ln x)^4 \ (+C)$$

$$\lim_{t\to\infty}\int_1^t \frac{4(2+\ln x)^3}{x}\,dx = \lim_{t\to\infty}\left[(2+\ln t)^4 - (2+\ln 1)^4\right]$$

$$= \infty$$

The integral Diverges

2. (10) Evaluate

$$\int e^{2x} \cos x \, dx.$$

$u = e^{2x} \quad dv = \cos x \, dx$

$du = 2e^{2x} dx \quad v = \sin x$

$uv - \int v \, du = e^{2x} \sin x - \int 2e^{2x} \sin x \, dx$

$u = 2e^{2x} \quad dv = \sin x \, dx$

$du = 4e^{2x} dx \quad v = -\cos x$

$-2e^{2x} \cos x + \int 4e^{2x} \cos x \, dx$

$\int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x - \int 4e^{2x} \cos x \, dx$

$5 \int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x + C$

$\int e^{2x} \cos x \, dx = \frac{e^{2x} \sin x + 2e^{2x} \cos x}{5} + C$

3. (10) Evaluate

$$\int_1^2 \frac{1}{x^2\sqrt{x^2-1}}\,dx$$

$\tan^2\theta = \sec^2\theta - 1$

$x = \sec\theta$ $\qquad\qquad$ $\sec\theta = 1 \Rightarrow \theta = 0$

$dx = \sec\theta\tan\theta\,d\theta$ $\qquad\qquad$ $\sec\theta = 2 \Leftrightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

$$\int_0^{\pi/3} \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}\,d\theta = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta$$

$$= \int_0^{\pi/3} \cos\theta\,d\theta = \sin\theta\Big|_0^{\pi/3} = \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2}$$

4. (12) Evaluate

$$\int \sin^2 x \cos^2 x \, dx$$

double- and half-angle

$$= \int \left(\tfrac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4x)\, dx$$

$$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

half-angle only

$$= \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{2}(1+\cos 2x)\, dx$$

$$= \int \frac{1}{4}(1-\cos^2 2x)\, dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \cos 4x\right) dx$$

$\vdots$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

5. (12) Determine if the following series

$$\sum_{n=1}^{\infty} \frac{3^n \cos n}{\pi^{n-1}}$$

converges. Mention any test(s) that you might use and verify that it is applicable.

test for absolute convergence:

$\sum_{n=1}^{\infty} \frac{3^n |\cos n|}{\pi^{n-1}}$ terms are positive and bounded by $\frac{3^n}{\pi^{n-1}}$, so consider $\sum_{n=1}^{\infty} \frac{3^n}{\pi^{n-1}}$

$\sum_{n=1}^{\infty} \frac{3^n}{\pi^{n-1}} = \sum_{n=1}^{\infty} 3\left(\frac{3}{\pi}\right)^{n-1}$ geometric with $r = \frac{3}{\pi}$

since $\pi > 3$, this is < 1

so convergent.

By comparison, $\sum_{n=1}^{\infty} \frac{3^n |\cos n|}{\pi^{n-1}}$ converges, and

so $\sum_{n=1}^{\infty} \frac{3^n \cos n}{\pi^{n-1}}$ is (absolutely) convergent.

6. (14) Find the radius of convergence and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{8^n}{n\,3^{2n+1}} x^n$$

ratio test.

$$|a_n| = \frac{8^n |x|^n}{n\, 3^{2n+1}}$$

$$|a_{n+1}| = \frac{8^{n+1} |x|^{n+1}}{(n+1)\, 3^{2n+3}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{8^{n+1} |x|^{n+1}}{(n+1)\, 3^{2n+3}} \cdot \frac{n \cdot 3^{2n+1}}{8^n |x|^n} = \frac{8|x|\, n}{(n+1)\, 3^2}$$

$$= \frac{8|x| \cdot n}{9 \cdot (n+1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{8|x|}{9} \qquad \frac{8|x|}{9} < 1 \quad \text{for } |x| < \frac{9}{8}$$

center $= 0$, check endpts. $\pm\frac{9}{8}$

$x = \frac{9}{8}$: $\sum \frac{8^n \left(\frac{9}{8}\right)^n}{n\, 3^{2n+1}} = \sum \frac{9^n}{n 3 \cdot 9^n} = \sum \frac{1}{3n}$ divergent ($\frac{1}{3} \cdot$ harmonic series)

$x = -\frac{9}{8}$: $\sum \frac{8^n \left(-\frac{9}{8}\right)^n}{n\, 3^{2n+1}} = \sum \frac{(-1)^n 9^n}{3n \cdot 9^n} = \sum \frac{(-1)^n}{3n}$ convergent ($\frac{1}{3} \cdot$ alternating harmonic series)

7. (10) Find a power series representation for the following function and find its interval of convergence:

$$f(x) = \frac{3\sqrt{x}}{5-x}$$

$$= 3\sqrt{x}\left(\frac{1}{5-x}\right) = 3\sqrt{x} \cdot \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}}$$

$a = \frac{3\sqrt{x}}{5}$ $\quad r = \frac{x}{5}$ $\quad$ geometric

$$\sum_{n=1}^{\infty} \frac{3\sqrt{x}}{5}\left(\frac{x}{5}\right)^{n-1}$$

need $\left|\frac{x}{5}\right| < 1$ for convergence

$|x| < 5$

endpts are never included for geometric series

so interval of convergence is $(-5, 5)$

8. (20) For each of the following statements, fill in the blank with the letters **T** or **F** depending on whether the statement is true or false. You do not need to show your work and no partial credit will be given on this problem.

(a) The sequence $\{n^{10}e^{-n}\}$ is convergent.

$$\frac{n^{10}}{e^{n}} \to 0$$

(10 applications of L'Hôpital will prove)

ANS: T

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{2}{3}}}$ is conditionally convergent.

$\dfrac{(-1)^n}{n^{2/3}}$ satisfies AST

$\dfrac{1}{n^{2/3}}$ is divergent p-series

ANS: T

(c) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

converse of one-way implication
in Test for Divergence

ANS: F

(d) The sequence $\left\{\frac{(-1)^n}{n}\right\}$ is divergent.

sequence $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

ANS: F

(e) The series $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2+n+1}}{5n-3}$ is convergent.

compare to $\frac{n^{2/3}}{5n} = \frac{1}{5n^{1/3}}$

divergent p-series

ANS: F