# 2.2. AN INTRODUCTION TO SERIES

In everyday language, the words series and sequence mean the same thing. However, in mathematics, it is vital to recognize the difference. A *series* is the result of adding a sequence of numbers together. While you may never have thought of it this way, we deal with series all the time when we write expressions like

$$\frac{1}{3} = 0.333\ldots,$$

since this means that

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$$

In general we are concerned with infinite series such as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

for various sequences $\{a_n\}$. First though, we need to decide what it means to add an infinite sequence of numbers together. Clearly we can't just add the numbers together until we reach the end (like we do with finite sums), because we won't ever get to the end. Instead, we adopt the following limit-based definition.

**Convergence and Divergence of Series.** If the sequence $\{s_n\}$ of *partial sums* defined by

$$s_n = a_1 + a_2 + \cdots + a_n$$

has a limit as $n \to \infty$ then we say that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n,$$

and in this case we say that $\sum_{n=1}^{\infty} a_n$ *converges*. Otherwise, $\sum_{n=1}^{\infty} a_n$ *diverges*.

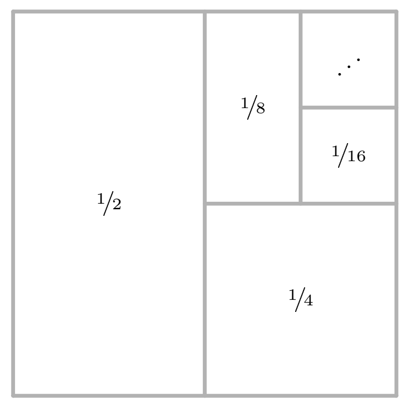
We begin with a particularly simple example.

**Example 1 (Powers of 2).** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to $1$.

**Solution.** We begin by computing a few partial sums:

$$\begin{array}{rclcccc} s_1 & = & 1/2 & = & 1/2 & = & 1 - 1/2 \\ s_2 & = & 1/2 + 1/4 & = & 3/4 & = & 1 - 1/4 \\ s_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 & = & 1 - 1/8 \\ s_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 & = & 1 - 1/16 \end{array}$$

These partial sums suggest that $s_n = 1 - 1/2^n$. Once we have guessed this pattern, it is easy to prove. If $s_n = 1 - 1/2^n$, then $s_{n+1} = 1 - 1/2^n + 1/2^{n+1} = 1 - 1/2^{n+1}$, so the formula is correct for all values of $n$ (this technique of proof is known as mathematical induction). With this formula, we see that $\lim_{n\to\infty} s_n = 1$, so $\sum_{n=1}^{\infty} 1/2^n = 1$. ●



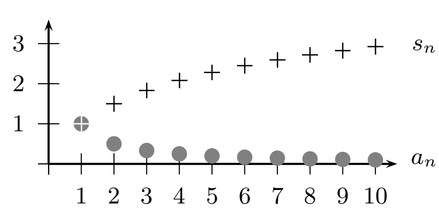

There is an alternative, more geometrical, way to see that this series converges to $1$. Divide the unit square in half, giving two squares of area $1/2$. Now divide one of these squares in half, giving two squares of area $1/4$. Now divide one of these in half, giving two squares of area $1/16$. If we continue forever, we will subdivide the unit square (which has area $1$) into squares of area $1/2$, $1/4$, $1/8$, ..., verifying that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Note that although we were able to find an explicit formula for the partial sums $s_n$ in Example 1, this is *not* possible in general.

Our next example shows a series $\sum a_n$ which diverges even though the sequence $\{a_n\}$ gets arbitrarily small, thereby demonstrating that the difference between convergent and divergent series is quite subtle.

**Example 2 (The Harmonic Series).** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.




**Solution.** The proof we give is due to the French philosopher Nicolas Oresme (1323–1382), and stands as one of the pinacles of medieval mathematical achievement. We simply group the terms together so that each group sums to at least $1/2$:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geqslant 2\cdot\frac{1}{4}=\frac{1}{2}} \\
&\quad + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geqslant 4\cdot\frac{1}{8}=\frac{1}{2}} \\
&\quad + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geqslant 8\cdot\frac{1}{16}=\frac{1}{2}} \\
&\quad + \cdots \\
&\geqslant 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,
\end{aligned}
$$

and therefore the series diverges. ●

The name of this series is due to Pythagoras, and his first experiments with music. Pythagoras noticed that striking a glass half-full of water would produce a note one octave higher than striking a glass full of water. A glass one-third full of water similarly produces a note at a "perfect fifth" of a whole glass, while a glass one-quarter full produces a note two octaves higher, and a glass one-fifth full produces a "major third." These higher frequencies are referred to as *harmonics*, and all musical instruments produce harmonics in addition to the *fundamental frequency* which they are playing (the instrument's "timbre" describes the amounts in which these different harmonics occur). This is what led Pythagoras to call the series $1 + 1/2 + 1/3 + \cdots$ the harmonic series.

**Example 3.** Suppose that scientists measure the total yearly precipitation at a certain point for 100 years. On average, how many of those years will have record high precipitation?

**Solution.** Suppose that the data is uncorrelated from year to year (in particular, that the amount of precipitation one year has no effect on the precipitation the next), and that the data shows no long-term trends (such as might be suggested if the climate were changing). In other words, suppose that the precipitations by year are *independent identically distributed random variables*.

Letting $a_n$ denote the amount of precipitation in the $n$th year, the question asks: for how many values $n$ can we expect $a_n$ to be the maximum of $a_1, \ldots, a_n$? By definition, $a_1$ is a maximum. The second year precipitation, $a_2$, then has a $1/2$ of being the maximum of $a_1, a_2$, while in general there is a $1/n$ chance that $a_n$ is the maximum of $a_1, \ldots, a_n$. This shows that the expected number of record high years of precipitation is

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100}.$$

The value of this sum is approximately 5.18738, so this is the expected number of record high years of precipitation. ●

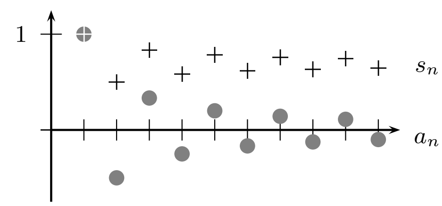
Example 2 provides a bit of intuition as to why the harmonic series diverges. Suppose that the precipitation data is collected *forever*. Then the expected number of record years is $\sum 1/n$. On the other hand, it seems natural that we should expect to see new record years no matter how long the data has been collected, although the record years will occur increasingly rarely. Therefore we should expect the harmonic series to diverge.

By making half the terms of the harmonic series negative, we obtain a convergent series:

**Example 4 (The Alternating Harmonic Series).** The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

converges.




**Solution.** Let $s_n$ denote the $n$th partial sum of this series. If we group the first $2n$ terms in pairs, we have

$$s_{2n} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right).$$

Since $1 > 1/2$, $1/3 > 1/4$, and so on, each of these groups is positive. Therefore $s_{2n+2} > s_{2n}$, so the even partial sums are monotonically increasing.

Moreover, by grouping the terms in a different way,

$$s_{2n} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right) - \frac{1}{2n},$$

we see that $s_{2n} < 1$ for all $n$. This shows that the sequence $\{s_{2n}\}$ is bounded and monotone. The Monotone Convergence Theorem (from the previous section) therefore implies that the sequence $\{s_{2n}\}$ has a limit; suppose that $\lim_{n\to\infty} s_{2n} = L$.

Now we consider the odd partial sums: $s_{2n+1} = s_{2n} + 1/2n+1$, so

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} \frac{1}{2n+1} = L.$$

Since both the even and odd partial sums converge to the same value, the sum of the series exists and is at most 1. ●

This leaves open a natural question: what is the sum of the alternating harmonic series? Our proof shows that the sum of this series is sandwiched between its even partial sums (which are under-estimates) and its odd partial sums (which are over-estimates), so

$$\frac{1}{2} = 1 - \frac{1}{2} \leqslant \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \leqslant 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6},$$

and we could get better estimates by including more terms. Back in Euler's time it would be difficult to guess what this series converges to, but with computers and the web, it is quite easy. In only a few seconds, a computer can compute that the $5$ millionth partial sum of the alternating harmonic series is approximately $0.693147$, and the *Inverse Symbolic Calculator* at

`http://oldweb.cecm.sfu.ca/projects/ISC/`

(which attempts to find "nice" expressions for decimal numbers) lists $\ln 2$ as its best guess for $0.693147$. Exercises 46 and 47 in Section 2.4 and Exercise 24 in Section 3.1 prove that this is indeed the sum.

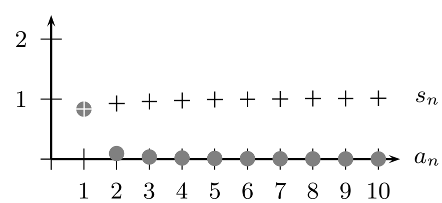
We next consider the result of adding the same terms in a different order.

**Example 5 (A Troublesome Inequality).** The series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right)$$

does not equal $\sum (-1)^{n+1}/n$, despite having the same terms.

**Solution.** Simplifying the inside of this series,




$$\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = \frac{8n-3}{(4n-3)(4n-1)(2n)}$$

shows that it contains only positive terms. Therefore its value is at least its first two terms added together,

$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right) \geqslant \left( 1 + \frac{1}{3} - \frac{1}{2} \right) + \left( \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \right) = \frac{389}{420} > \frac{9}{10}.$$

However, if we remove the parentheses then it is clear that the fractions we are adding in this series are precisely the terms of the alternating harmonic series, whose value is strictly less than $9/10$. To repeat,

$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right) \neq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

even though both sides contain the same fractions with the same signs! ●

We will explore this phenomenon more in Section 2.8. Until then, just remember that *the order in which you sum a series which has both negative and positive terms might affect the answer.* In other words, when you are adding infinitely many numbers, some of which are positive and some of which are negative, addition is *not necessarily* commutative.

It is because of examples such as this that we need to be extremely careful when dealing with series. This is why we will make sure to prove every tool we use, even when those tools are "obvious."

Next we consider a series which is sometimes referred to as Grandi's series, after the Italian mathematician, philosopher, and priest Guido Grandi (1671–1742), who studied the series in a 1703 work.

**Example 6 (An Oscillating Sum).** The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges.

**Solution.** While the harmonic series diverges to infinity, the series $1-1+1-1+\cdots$ diverges because its partial sums oscillate between $0$ and $1$:

$$s_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

so $\displaystyle\lim_{n\to\infty} s_n$ does not exist. ●

This is an example of a series which can be shown to diverge by the first general test in our toolbox:

**The Test for Divergence.** If $\displaystyle\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

**Proof.** It is easier to prove the contrapositive: if $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$. Since we are assuming that $\sum a_n$ converges, $\displaystyle\lim_{n\to\infty} s_n$ exists. Suppose that $\displaystyle\lim_{n\to\infty} s_n = L$. Then

$$a_n = s_n - s_{n-1} \to L - L = 0$$

as $n \to 0$, proving the theorem. □

It is important to remember that the converse to the Test for Divergence is false, i.e., even if the terms of a series tend to $0$, the series may still diverge. Indeed, the harmonic series is just such a series: $1/n \to 0$ as $n \to \infty$, but $\sum 1/n$ diverges.

Before concluding the section, we make one more general observation. The convergence of a series depends only on how small its "tail" is. Thus it does not matter from the point of view of convergence/divergence if we ignore the first $10$ (or the first $10$ octovigintillion) terms of a series:

**Tail Observation.** The series $\sum_{n=1}^{\infty} a_n$ converges if and only if its "tail"

$$\sum_{n=N}^{\infty} a_n$$

converges for some value of $N$.

While our techniques in this section have mostly been ad hoc, our goal in this chapter is to develop several tests which we can apply to a wide range of series. Our list of tests will grow to include†:

- The Test for Divergence
- The Integral Test
- The $p$-Series Test
- The Comparison Test
- The Ratio Test
- The Absolute Convergence Theorem
- The Alternating Series Test

It is important to realize that each test has distinct strengths and weaknesses, so if one test is inconclusive, you need to push onward and try more tests until you find one that can handle the series in question.

†This is in addition to the numerous more specialized tests developed in the exercises.

## EXERCISES FOR SECTION 2.2

In all of the following problems, $s_n$ denotes the $n$th partial sum of $\sum a_n$, that is,

$$s_n = a_1 + a_2 + \cdots + a_n.$$

**1.** If $\sum_{n=1}^{\infty} a_n = 3$ then what are $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} s_n$?

**2.** If $\lim_{n\to\infty} s_n = 4$ then what are $\lim_{n\to\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$?

**3.** Compute the 4th partial sum of $\sum_{n=1}^{\infty} \frac{2}{n+2}$.

**4.** Compute the 4th partial sum of $\sum_{n=1}^{\infty} \frac{2}{n^2+2}$.

In Exercises 5–10, write down a formula for $a_n$ and sum the series if it converges.

**5.** $s_n = \dfrac{3n+2}{n-4}$

**6.** $s_n = \dfrac{3}{n-4}$

**7.** $s_n = (-1)^n$

**8.** $s_n = \frac{n(n+1)}{2}$

**9.** $s_n = \sin n$

**10.** $s_n = \arctan n$

Determine if the series in Exercises 11–17 diverge by the Test for Divergence. (Note that if they do not diverge by the Test for Divergence, then we don't yet know if they converge or not.)

**11.** $\sum_{n=1}^{\infty} \dfrac{(-1)^n n^2}{3n^2+2n+1}$

**12.** $\sum_{n=1}^{\infty} \dfrac{\sin n}{n}$

**13.** $\sum_{n=1}^{\infty} \cos {}^1\!/\!_{n^2}$

**14.** $\sum_{n=1}^{\infty} \left(1+\dfrac{1}{n}\right)^n$

**15.** $\sum_{n=1}^{\infty} \left(\dfrac{1}{n}\right)^n$

**16.** $3+5+7+11+13+17+\cdots$, the sum of the primes.

**17.** $1/3 + 1/5 + 1/7 + 1/11 + 1/13 + 1/17 + \cdots$, the sum of the reciprocals of the primes.

In Exercises 18–21, reindex the series so that they begin at $n = 1$.

**18.** $\sum_{n=2}^{\infty} \dfrac{n^2}{2^n}$

**19.** $\sum_{n=4}^{\infty} \dfrac{n^2-n}{(n+5)^3}$

**20.** $\sum_{n=-4}^{\infty} \dfrac{n^2-n}{(n+5)^3}$

**21.** $\sum_{n=0}^{\infty} \dfrac{n\sin n}{(n+2)^3}$

Determine if the series in Exercises 22–25 converge at $x = -2$ and at $x = 2$.

**22.** $\sum_{n=1}^{\infty} \dfrac{x^n}{2^n}$

**23.** $\sum_{n=1}^{\infty} \dfrac{x^n}{n2^n}$

**24.** $\sum_{n=1}^{\infty} \dfrac{(-1)^n x^n}{2^n}$

**25.** $\sum_{n=1}^{\infty} \dfrac{(-1)^n x^n}{n2^n}$

Use the rules of limits described in Exercises 31–34 of Section 2.1 to prove the statements in Exercises 26–29.

**26.** If $\sum a_n$ and $\sum b_n$ are both convergent series, prove that

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n.$$

**27.** If $\sum a_n$ and $\sum b_n$ are both convergent series, prove that

$$\sum_{n=1}^{\infty}(a_n-b_n)=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n.$$

**28.** If $\sum a_n$ is a convergent series, prove that

$$\sum_{n=1}^{\infty}ca_n=c\sum_{n=1}^{\infty}a_n$$

for any number $c$.

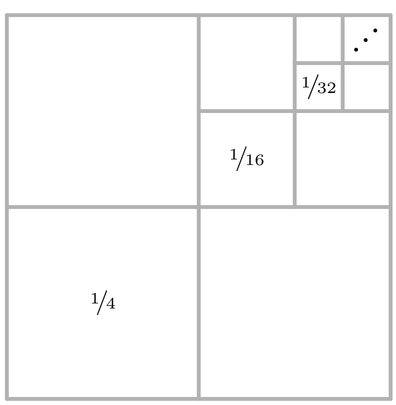
**29.** If $\sum a_n$ is a divergent series, prove that

$$\sum_{n=1}^{\infty}ca_n$$

diverges for any number $c$.

**30.** Archimedes (circa 287 BC–212 BC) was one of the first mathematicians to consider infinite series. In his treatise *The Quadrature of the Parabola*, he uses the figure shown below to prove that

$$1/4+1/4^2+1/4^3+\cdots=1/3.$$




Explain his proof in words.

**31.** Prove that if $\sum a_n$ converges, then its partial sums $s_n$ are bounded.

**32.** Give an example showing that the converse to Exercise 31 is false, i.e., give a sequence $\{a_n\}$ whose partial sums are bounded but such that $\sum a_n$ does not converge.

◆ **33.** Suppose that $a_n \to 0$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty}(a_{2n}+a_{2n+1})$ converges.

◆ **34.** Give an example showing that the hypothesis $a_n \to 0$ in Exercise 33 is necessary.

Sometimes a series can be rewritten in a such a way that nearly every term cancels with a succeeding or preceding term, for example,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots\\ &= 1.\end{aligned}$$

These series are called *telescoping*. In Exercises 35–36, express the series as telescoping series to compute their sums.

**35.** $\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$.

**36.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}}$, where $f_n$ is the $n$th Fibonacci number.

**37.** Explain why the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\cdots$$

is smaller than

$$1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots,$$

and then show that this second series $\sum 1/n(n+1)$ is a convergent telescoping series.

Exercises 38–41 concern the situation described in Example 2.

**38.** How many record years should we expect in 10 years of observations?

**39.** What is the probability that in 100 years of observations, the first year is the only record?

**40.** What is the probability that in 100 years of observations, there are only two record years?

**41.** What is the probability that in 100 years of observations, there are at most three record years?

When a musical instrument produces a sound, it also produces harmonics of that sound to some degree. For example, if you play a middle A on an instrument, you are playing a note at 440 Hz, but the instrument also produces sounds at 880 Hz, 1320 Hz, 1760 Hz, and so on. In a remarkable effect known as *restoration of the fundamental*, if the sound at 440 Hz is artificially removed, most listeners' brains will "fill it in," and perceive the collection of sounds as a middle A nonetheless. Indeed, your brain will perform the same feat even if several of the first few frequencies are removed. In Exercises 42–45, determine what fundamental frequency your brain will perceive the given collection of frequencies as.

**42.** 646 Hz, 969 Hz, 1292 Hz, 1615 Hz, . . .

**43.** 784 Hz, 1176 Hz, 1568 Hz, 1960 Hz, . . .

**44.** 789 Hz, 1052 Hz, 1315 Hz, 1578 Hz, . . .

**45.** Discuss how restoration of the fundamental can be used to play music on a speaker which can't produce low notes.

Exercises 46–48 concern the following procedure, which was brought to my attention by Professor Pete Winkler. Start with $a_1 = 2$. At stage $n$, choose an integer $m$ from 1 to $a_n$ uniformly at random (i.e., each number has a $1/a_n$ chance of occurring). If $m = 1$ then stop. Otherwise, set $a_{n+1} = a_n + 1$ and repeat. For example, this procedure has a $1/2$ probability of stopping after only one step, while otherwise it goes on to the second step, with $a_2 = 3$.

◆ **46.** Compute the probability that this procedure continues for at least 2 steps, at least 3 steps, and at least $n$ steps.

◆ **47.** Compute the probability that this procedure *never* terminates.

◆ **48.** The expected (or average) number of steps that this procedure takes is

$$\sum_{n=1}^{\infty} n \cdot \Pr(\text{the procedure takes precisely } n \text{ steps}).$$

Verify that another way to write this is

$$\sum_{n=1}^{\infty} \Pr(\text{the procedure takes at least } n \text{ steps}),$$

and use this to compute the expected number of steps that this procedure takes.

Cesàro summability, named for the Italian mathematician Ernesto Cesàro (1859–1906), is a different notion of sums, given by averaging the partial sums. Define

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

We say that the series $\sum a_n$ is *Cesàro summable* to $L$ if $\lim_{n\to\infty} \sigma_n = L$. Exercise 52 shows that if $\sum a_n$ converges to $L$ then $\sum a_n$ is Cesàro summable to $L$, however, the converse does not hold. We explore Cesàro summability in Exercises 49 and 50. In both of these exercises, you should use the formulas for $s_n$ found in the text. Exercises 52 and 53 explore the more theoretical aspects of Cesàro summability.

◆ **49.** Show that the Cesàro sum of $\sum_{n=1}^{\infty} 2^{-n}$ is 1.

◆ **50.** Show that the Cesàro sum of $\sum_{n=1}^{\infty} (-1)^{n+1}$ is $1/2$.

◆ **51.** Show that $\sum_{n=1}^{\infty} (-1)^{n+1} n$ is not Cesàro summable. (Compare this with Exercise 22 in Section 3.2.)

◆ **52.** Prove that if $\sum a_n = L$ then $\{a_n\}$ is Cesàro summable to $L$.

◆ **53.** Prove the result, due to Alfred Tauber (1866–1942), that if $\{a_n\}$ is a positive sequence and is Cesàro summable to $L$, then $\sum a_n = L$.

## ANSWERS TO SELECTED EXERCISES, SECTION 2.2

1. $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} s_n = 3$
3. $2/3 + 2/4 + 2/5 + 2/6$
5. $a_n = \dfrac{3n+2}{n-4} - \dfrac{3(n-1)+2}{n-5}$ and the sum of the series is $3$.
7. $a_n = (-1)^n - (-1)^{n-1}$ and the series diverges.
9. $a_n = \sin n - \sin(n-1)$ and the series diverges.
11. Diverges by the Test for Divergence
13. Diverges by the Test for Divergence
15. The terms do go to $0$, so the Test for Divergence does not apply.
17. The terms do go to $0$, so the Test for Divergence does not apply.
19. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)^2 - (n+3)}{((n+3)+5)^3}$
21. The $n = 0$ term of this series is already $0$, so one answer is simply $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin n}{(n+2)^3}$. Another correct answer is $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)\sin(n-1)}{((n-1)+2)^3}$.
23. Converges at $x = -2$ (alternating harmonic series) and diverges at $x = 2$ (harmonic series).
25. Diverges at $x = -2$ (harmonic series) and converges at $x = 2$ (alternating harmonic series).