2.3. GEOMETRIC SERIES

One of the most important types of infinite series are geometric series. A *geometric series* is simply the sum of a geometric sequence,

$$\sum_{n=0}^{\infty} ar^n.$$

Fortunately, geometric series are also the easiest type of series to analyze. We dealt a little bit with geometric series in the last section; Example 1 showed that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

while Exercise 26 presented Archimedes' computation that

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}.$$

(Note that in this section we will sometimes begin our series at $n = 0$ and sometimes begin them at $n = 1$.)

Geometric series are some of the only series for which we can not only determine convergence and divergence easily, but also find their sums, if they converge:

Geometric Series. The geometric series

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$$

converges to

$$\frac{a}{1-r}$$

if $|r| < 1$, and diverges otherwise.

Proof. If $|r| \geqslant 1$, then the geometric series diverges by the Test for Divergence, so let us suppose that $|r| < 1$. Let s_n denote the sum of the first n terms,

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1},$$

so

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Subtracting these two, we find that

$$\begin{array}{rcccccccccccc} s_n & = & a & + & ar & + & ar^2 & + & \cdots & + & ar^{n-1} & & \\ -rs_n & = & & - & ar & - & ar^2 & - & \cdots & - & ar^{n-1} & - & ar^n \\ \hline (1-r)s_n & = & a & & & & & & & & & - & ar^n. \end{array}$$

This allows us to solve for the partial sums s_n,

$$s_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now we know (see Example 5 of Section 2.1) that for $|r| < 1$, $r^n \to 0$ as $n \to \infty$, so

$$\frac{ar^n}{1-r} \to 0$$

as $n \to \infty$ as well, and thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{1-r} - \frac{ar^n}{1-r} = \frac{a}{1-r},$$

proving the result. □

An easy way to remember this theorem is

$$\text{geometric series} \sum = \frac{\text{first term}}{1 - \text{ratio between terms}}.$$

We begin with two basic examples.

Example 1. Compute $12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots$.

Solution. The first term is 12 and the ratio between terms is $1/3$, so

$$12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots = \frac{\text{first term}}{1 - \text{ratio between terms}} = \frac{12}{1 - \frac{1}{3}} = 18,$$

solving the problem. ●

Example 2. Compute $\displaystyle\sum_{n=6}^{\infty} (-1)^n \frac{2^{n+3}}{3^n}$.

Solution. This series is geometric with common ratio

$$r = \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}\dfrac{2^{n+4}}{3^{n+1}}}{(-1)^n\dfrac{2^{n+3}}{3^n}} = -\frac{2}{3},$$

and so it converges because $|-{}^2\!/\!_3| < 1$. Its sum is

$$\sum_{n=6}^{\infty}(-1)^n\frac{2^{n+3}}{3^n} = \frac{\text{first term}}{1-\text{ratio between terms}} = \frac{2^9/3^6}{1+2/3},$$

which simplifies to $512/1215$. ●

The use of the geometric series formula is of course not limited to single geometric series, as our next example demonstrates.

Example 3. Compute $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n+1}+9^{n/2}}{5^n}$.

Solution. We break this series into two:

$$\sum_{n=1}^{\infty}\frac{2^{n+1}+9^{n/2}}{5^n} = \sum_{n=1}^{\infty}\frac{2^{n+1}}{5^n} + \sum_{n=1}^{\infty}\frac{9^{n/2}}{5^n}$$

(if both series converge). The first of these series has common ratio $2/5$, so it converges. To analyze the second series, note that $9^{n/2} = \left(9^{1/2}\right)^n = \sqrt{9}^n = 3^n$, so this series has common ratio $3/5$. Since both series converge, we may proceed with the addition:

$$\sum_{n=1}^{\infty}\frac{2^{n+1}+9^{n/2}}{5^n} = \sum_{n=1}^{\infty}\frac{2^{n+1}}{5^n} + \sum_{n=1}^{\infty}\frac{3^n}{5^n} = \frac{2^2/5}{1-2/5} + \frac{3/5}{1-3/5} = \frac{4}{3}+\frac{3}{2}.$$

This answer simplifies to $17/6$. ●

If a geometric series involves a variable x, then it may only converge for certain values of x. Where the series does converge, it defines a function of x, which we can compute from the summation formula. Our next example illustrates these points.

Example 4. For which values of x does the series $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$ converge? For those values of x, which function does this series define?

Solution. The common ratio of this series is

$$r = \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}x^{2n+2}}{(-1)^n x^{2n}} = -x^2.$$

Since geometric series converge if and only if $|r| < 1$, we need $|-x^2| < 1$ for this series to converge. This expression simplifies to $|x| < 1$.

Where this series does converge (i.e., for $-1 < x < 1$), its sum can be found by the geometric series formula:

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{\text{first term}}{1-\text{ratio between terms}} = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Although note that this only holds for $|x| < 1$. ●

The geometric series formula may also be used to convert repeating decimals into fractions, as we show next.

Example 5. Express the number $4.342342342\ldots$ as a fraction in the form p/q where p and q have no common factors.

Solution. Our first step is to express this number as the sum of a geometric series. Since the decimal seems to repeat every 3 digits, we can write this as

$$4.342342342\cdots = 4 + \frac{342}{1000} + \frac{342}{1000^2} + \cdots = 4 + \sum_{n=1}^{\infty}\frac{342}{1000^n}.$$

This series is geometric, so we can use the formula to evaluate it:

$$\sum_{n=1}^{\infty}\frac{342}{1000^n} = \frac{\text{first term}}{1-\text{ratio between terms}} = \frac{342/1000}{1-1/1000} = \frac{342}{1000}\cdot\frac{1000}{999} = \frac{342}{999}.$$

Our initial answer is therefore

$$4 + \frac{342}{999} = \frac{3996+342}{999} = \frac{4338}{999}.$$

Since we were asked for a fraction with no common factors between the numerator and denominator, we now have to factor out the 9 which divides both 4338 and 999, leaving

$$4.342342342 = \frac{482}{111}.$$

To be sure that 482 and 111 have no common factors, we need to verify that $111 = 3\cdot 37$, and the prime 37 does not divide 482. ●

Our next example in some sense goes in the other direction. Here we are given a fraction and asked to use geometric series to approximate its decimal expansion.

Example 6. Use geometric series to approximate the decimal expansion of $1/48$.

Solution. First we find a number near $1/48$ with a simple decimal expansion; $1/50$ will work nicely. Now we express $1/48$ as $1/50$ times a fraction of the form $1/(1-r)$:

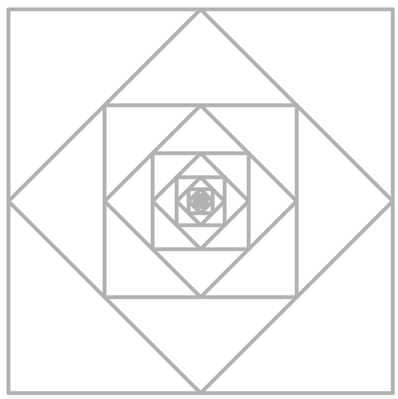
$$\frac{1}{48} = \frac{1}{50-2} = \frac{1}{50}\cdot\frac{1}{1-\frac{2}{50}}.$$

Now we can expand the fraction on the righthand side as a geometric series,

$$\frac{1}{48} = \frac{1}{50}\left(1 + \frac{2}{50} + \left(\frac{2}{50}\right)^2 + \left(\frac{2}{50}\right)^3 + \cdots\right).$$

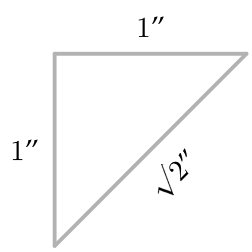
Using the first two terms of this series, we obtain the approximation $1/48 \approx 0.02(1+0.02) = 0.0204$. ●

Example 7. Suppose that you draw a 2″ by 2″ square, then you join the midpoints of its sides to draw another square, then you join the midpoints of that square's sides to draw another square, and so on, as shown below.



Would you need infinitely many pencils to continue this process forever?

Solution. If we look just at the upper left-hand corner of this figure, we see a triangle with two sides of length 1″ and a hypotenuse of length $\sqrt{2}$″:



So the sequence of side lengths of these rectangles is geometric with ratio $\sqrt{2}/2$: 2″, $\sqrt{2}$″, $1/2$″, ... Since $\sqrt{2}/2 < 1$, the sum of the side lengths of all these (infinitely many) squares therefore converges to

$$\frac{2}{1 - \frac{\sqrt{2}}{2}} = \frac{2}{\frac{2-\sqrt{2}}{2}} = 2(2+\sqrt{2}).$$

As the perimeter of a square is four times its side length, the total perimeter of this infinite construction is $8(2+\sqrt{2})''$, so we would only need finitely many pencils to draw the figure forever. ●

Our last two examples are significantly more advanced that the previous examples. On the other hand, they are also more interesting.

Example 8 (A Game of Chance). A gambler offers you a proposition. He carries a fair coin, with two different sides, heads and tails ("fair" here means that it is equally likely to land heads or tails), which he will toss. If it comes up heads, he will pay you \$1. If it comes up tails, he will toss the coin again. If, on the second toss, it comes up heads, he will pay you \$2, and if it comes up tails again he will toss it again. On the third toss, if it comes up heads, he will pay you \$3, and if it comes up tails again, he will toss it again... How much should you be willing to pay to play this game?

Solution. Would you pay the gambler \$1 to play this game? Of course. You've got to win at least a dollar. Would you pay the gamble \$2 to play this game? Here things get more complicated, since you have a 50% chance of losing money if you pay \$2 to play, but you also have a 25% chance to get your \$2 back, and a 25% chance to win money. Would you pay the gambler one million dollars to play the game? The gambler asserts that there are infinitely many positive integers greater than one million. Thus (according to the gambler, at least) you have infinitely many chances of winning more than one million dollars!

To figure out how much you should pay to play this game, we compute its average payoff (its *expectation*). In the chart below, we write H for heads and T for tails.

Outcome	Probability	Payoff	Expected Winnings
H	$1/2$	$\$1$	$1/2 \cdot \$1 = \$0.50$
TH	$1/4$	$\$2$	$1/4 \cdot \$2 = \$0.50$
TTH	$1/8$	$\$3$	$1/8 \cdot \$3 = \$0.375$
$TTTH$	$1/16$	$\$4$	$1/16 \cdot \$4 = \$0.25$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underbrace{TT\cdots T}_{n-1}H$	$1/2^n$	$\$n$	$1/2^n \cdot \$n = \$n/2^n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

There are several observations we can make about this table of payoffs. First, the sum of the probabilities of the various outcomes is

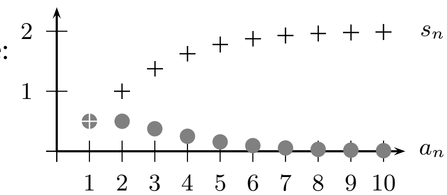
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

which indicates that we have indeed listed all the (non-negligible) outcomes. But what if the coin *never* lands heads? The probability of this happening is $\lim_{n\to\infty} (1/2)^n = 0$, and so it is safely ignored. Therefore, we ignore this possibility.

To figure out how much you should pay to play this game, it seems reasonable to compute the sum of the expected winnings, over all possible outcomes. This sum is

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

We demonstrate two methods to compute this sum. The first is elementary but uses a clever trick, while the second uses calculus.



Applying the geometric series formula, we can write:

$$\begin{array}{ccccccccccc} 1 & = & 1/2 & + & 1/2^2 & + & 1/2^3 & + & 1/2^4 & + & \cdots \\ 1/2 & = & & & 1/2^2 & + & 1/2^3 & + & 1/2^4 & + & \cdots \\ 1/2^2 & = & & & & & 1/2^3 & + & 1/2^4 & + & \cdots \\ 1/2^3 & = & & & & & & & 1/2^4 & + & \cdots \\ \vdots & = & & & & & & & & & \vdots \end{array}$$

If we now add this array *vertically*, we obtain an equation whose left-hand side is 2 (it is the sum of a geometric series itself), and whose right-hand side, $1/2 + 2/2^2 + 3/2^3 + \cdots$, is the sum of the expected winnings, so

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

Because the expected winnings are \$2, you should be willing to pay anything less than \$2 to play the game, because in the long-run, you will make money.

Now we present a method using calculus. Since we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

we can differentiate this formula to get

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

Now multiply both sides by x:

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots .$$

Finally, if we set $x = 1/2$ we obtain the sum we are looking for:

$$2 = \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \cdots = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

One might (quite rightly) complain that we don't know that we can take the derivative of an infinite series in this way. We consider these issues in Section 3.2. ●

Example 9 (The St. Petersburg Paradox†). Impressed with the calculation of the expected winnings, suppose that the gambler offers you a different wager. This time, he will pay \$1 for the outcome H, \$2 for the outcome TH, \$4 for the outcome TTH, \$8 for the outcome $TTTH$, and in general, $\$2^n$ if the coin lands tails n times before landing heads. Computing the expected winnings as before, we obtain the following chart.

Outcome	Probability	Payoff	Expected Winnings
H	$1/2$	\$1	\$0.50
TH	$1/4$	\$2	\$0.50
TTH	$1/8$	\$4	\$0.50
$TTTH$	$1/16$	\$8	\$0.50
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underbrace{TT\cdots T}_{n}H$	$1/2^n$	$\$2^n$	\$0.50
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The gambler now suggests that you should be willing to pay *any* amount of money to play this game since the expected winnings, $\$0.50 + \$0.50 + \$0.50 + \cdots$, are infinite!

Solution. That is why this game is referred to as a paradox. Seemingly, there is no "fair" price to pay to play this game. Bernoulli, who introduced this paradox, attempted to resolve it by asserting that money has "declining marginal utility." This is not in dispute; imagine what your life would be like with \$1 billion, and then imagine what it would be like with \$2 billion. Clearly the first \$1 billion makes a much bigger difference than the second \$1 billion, so it has higher "utility." However, the gambler can simply adjust his payoffs, for example, he could offer you $\$2^{2^{n-1}}$ if the coin lands tails $n-1$ times before landing heads, so even taking into account the declining marginal utility of money, there is always some payoff function which results in a paradox.

A practical way out of this paradox is to note that the gambler can't keep his promises. If we assume generously that the gambler has \$1 billion, then the gambler cannot pay you the full amount if the coin lands tails 30 times before landing heads ($2^{29} = 536,870,912$, but $2^{30} = 1,073,741,824$). The payouts in this case will be as above up to $n = 29$, but then beginning at $n = 30$, all you can win is \$1 billion. This gives an expected winnings of

$$\sum_{n=1}^{n=29} \$0.50 + \sum_{n=30}^{\infty} \frac{\$1{,}000{,}000{,}000}{2^{n-1}} = \$14.50 + \frac{\left(\frac{\$1{,}000{,}000{,}000}{2^{30}}\right)}{1-\frac{1}{2}} \approx \$16.36.$$

†This problem is known as the St. Petersburg Paradox because it was introduced in a 1738 paper of Daniel Bernoulli (1700–1782) published in the *St. Petersburg Academy Proceedings*.

So by making this rather innocuous assumption, the game goes from being "priceless" to being worth the same as a new T-shirt.

Another valid point is that *you* don't have unlimited money, so there is a high probability that by repeatedly playing this game, you will go broke before you hit the rare but gigantic jackpot which makes you rich.

One amusing but nevertheless accurate way to summarize the St. Petersburg Paradox is therefore: If both you and the gambler had an infinite amount of time and money, you could earn another infinite amount of money playing this game forever. But wouldn't you have better things to do with an infinite supply of time and money? ●

EXERCISES FOR SECTION 2.3

Determine which of the series in Exercises 1–8 are geometric series. Find the sums of the geometric series.

1. $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$

4. $\sum_{n=1}^{\infty} \frac{3^n}{4^{2n-1}}$

5. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$

6. $\sum_{n=0}^{\infty} \frac{5^n}{5^{n+4}}$

7. $\sum_{n=1}^{\infty} \frac{5^n}{25^{n+4}}$

8. $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

Determine which of the geometric series (or sums of geometric series) in Exercises 9–12 converge. Find the sums of the convergent series.

9. $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^n}$

10. $\sum_{n=1}^{\infty} \frac{4^{n/2+1}}{3^n}$

11. $\sum_{n=1}^{\infty} \frac{2^n + 5^n}{4^{n+9}}$

12. $\sum_{n=0}^{\infty} (-1)^n \frac{3^{n+2}}{5^{n-1}}$

Find the sums of the series in Exercises 13–16.

13. $\sum_{n=1}^{\infty} \frac{3^n + 5^n}{7^n}$

14. $\sum_{n=1}^{\infty} e^n \pi^{-n}$

15. $\sum_{n=1}^{\infty} \frac{(9/2)^{n+2}}{3^{2n}}$

16. $\sum_{n=1}^{\infty} \frac{3^{n+2} + 4^{n/2}}{6^n}$

Determine which values of x the series in Exercises 17–20 converge for. When these series converge, they define functions of x. What are these functions?

17. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

18. $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{3^n}$

19. $\displaystyle\sum_{n=1}^{\infty} 5^{n+2}x^n$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^{n-1}}$

For Exercises 21–24, use geometric series to approximate these reciprocals, as in Example 6.

21. $1/99$

22. $1/102$

23. $-2/99$

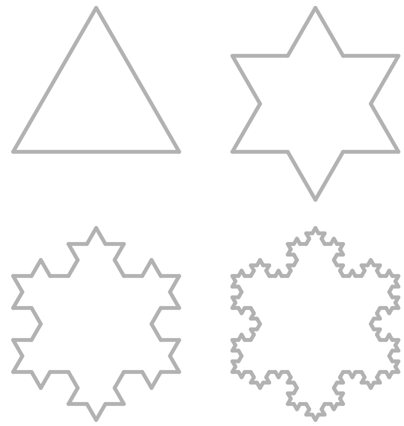
24. $1/24$

For Exercises 25 and 26, suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is a geometric series such that the sum of the first 3 terms is 3875 and the sum of the first 6 terms is 3906.

25. What is a_6?

26. What is $\displaystyle\sum_{n=1}^{\infty} a_n$?

Exercises 27 and 28 consider the *Koch snowflake*. Introduced in 1904 by the Swedish mathematician Helge von Koch (1870–1924), the Koch snowflake is one of the earliest fractals to have been described. We start with an equilateral triangle. Then we divide each of the three sides into three equal line segments, and replace the middle portion with a smaller equilateral triangle. We then iterate this construction, dividing each of the line segments of the new figure into thirds and replacing the middle with an equilateral triangle, and then iterate this again and again, forever. The first four iterations are shown below.



◆ **27.** Write a series representing the area of the Koch snowflake, and find its value.

◆ **28.** Write a series representing the perimeter of the Koch snowflake. Does this series converge?

29. Two 50% marksmen decide to fight in a duel in which they exchange shots until one is hit. What are the odds in favor of the man who shoots first?

30. In the decimal system, some numbers have more than one expansion. Verify this by showing that $2.35999\ldots = 2.36$.

◆ **31.** The *negadecimal* number system is like the decimal system except that the base is -10. So, for example, $(12.43)_{-10} = 1(-10)^2 + 2(-10) + 4(-10)^{-1} + 3(-10)^{-2} = 97.63$ in base 10. Prove that (like the decimal system) there are non-unique expansions in the negadecimal system by showing that $(1.909090\ldots)_{-10} = (0.090909\ldots)_{-10}$.

Exercises 32 and 33 ask you to prove that there are infinitely many primes, following a proof of Euler. Exercise 34 provides the classic proof, due to Euclid.

◆ **32.** Prove, using the fact that every positive integer n has a unique prime factorization†, that

$$\sum \frac{1}{n} = \left(\sum \frac{1}{2^n}\right)\left(\sum \frac{1}{3^n}\right)\left(\sum \frac{1}{5^n}\right)\cdots$$

where the right-hand side is the product of all series of the form $\sum \frac{1}{p^n}$ for primes p.

†This refers to the fact that every positive integer n can be written as a product $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ for primes $p_1, p_2, \ldots, p_k$ and nonnegative integers $a_1, a_2, \ldots, a_k$.

◆ **33.** Use Exercise 32 to prove that there are infinitely many primes. *Hint:* the left hand-side diverges to ∞, while every sum on the right hand-side converges.

34. (Euclid's proof) Suppose to the contrary that there are only finitely many prime numbers, $\{p_1, p_2, \ldots, p_m\}$. Draw a contradiction by considering the number $n = p_1 p_2 \cdots p_m + 1$.

◆ **35.** Compute $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.
Hint: consider the derivative of $x/(1-x)^2$. For now, assume that you can differentiate this series as done in Example 8.

36. Suppose the gambler from Example 8 alters his game as follows. If the coin lands tails an even number of times before landing heads, you must pay him \$1. However, if the coin lands tails an odd number of times before landing heads, he pays you \$1. The gambler argues that there are just as many even numbers as odd numbers, so the game is fair. Would you be willing to play this game? Why or why not? (Note that 0 is an even number.)

37. Suppose the gambler alters his game once more. If the coin lands tails an even number of times before landing heads, you must pay him \$1. However, if the coin lands tails an odd number of times before landing heads, he pays you \$2. Would you be willing to play this game? Why or why not?

◆ **38.** Suppose that after a few flips, you grow suspicious of the gambler's coin because it seems to land heads more than 50% of the time (but less than 100% of the time). Design a procedure which will nevertheless produce 50-50 odds using his unfair coin. (The simplest solution to the problem is attributed to John von Neumann (1903–1957).)

◆ **39.** Consider the series $\sum 1/n$ where the sum is taken over all positive integers n which do not contain a 9 in their decimal expansion. Due to A.J. Kempner in 1914, series like this are referred to as *depleted harmonic series*. Show that this series converges. *Hint:* how many terms have denominators between 1 and 9? Between 10 and 99? More generally, between 10^{n-1} and $10^n - 1$?

ANSWERS TO SELECTED EXERCISES, SECTION 2.3

1. Geometric series, converges to $\dfrac{2/3}{1 - 2/3} = 2$.
3. Not a geometric series.
5. Geometric series, converges to $\dfrac{1}{1 - (-1/2)} = 2/3$.
7. Geometric series, converges to $\dfrac{5/25^5}{1 - 5/25} = 4/9765625$.
9. Diverges; the ratio is $9/2 > 1$.
11. Diverges.
13. The sum is $\dfrac{3/7}{1 - 3/7} + \dfrac{5/7}{1 - 5/7} = 13/4$.
15. The sum is $\dfrac{9^2/2^3}{1 - 1/2} = 9^2/2^2$.
17. Converges for $-3 < x < 3$, to the function $\dfrac{x/3}{1 - x/3}$.
19. Converges for $-1/5 < x < 1/5$, to the function $\dfrac{5^3 x}{1 - 5x}$.