1. Find the Taylor series for the function

$$f(x) = x^5$$

around the point $x = 1$.

(Your answer will have only finitely many terms, because after some point, all the terms are zero.)

Solution: The Taylor series for $f(x)$ around the point $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For our function, we have

$$f(x) = x^5 \qquad f'(x) = 5x^4 \qquad f''(x) = 20x^3 \qquad f^{(3)}(x) = 60x^2$$

$$f^{(4)}(x) = 120x \qquad f^{(5)}(x) = 120 \qquad f^{(6)}(x) = 0,$$

and $f^{(n)}(x) = 0$ for all $n \geq 6$. Plugging in our a, which is 1, we get

$$f(1) = 1 \qquad f'(1) = 5 \qquad f''(1) = 20 \qquad f^{(3)}(1) = 60$$

$$f^{(4)}(1) = 120 \qquad f^{(5)}(1) = 120 \qquad f^{(6)}(1) = 0,$$

and $f^{(n)}(1) = 0$ for all $n \geq 6$. Therefore, our Taylor series is

$$\frac{1}{0!} + \frac{5}{1!}(x-1) + \frac{20}{2!}(x-1)^2 + \frac{60}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4 + \frac{120}{5!}(x-1)^5 =$$

$$1 + 5(x-1) + 10(x-1)^2 + 10(x-1)^3 + 5(x-1)^4 + (x-1)^5.$$

(You can also do this problem by writing $x^5 = ((x-1)+1)^5$, and applying the Binomial Theorem.)

2. (a) What is the Maclaurin series (the Taylor series about $x = 0$) for the function $f(x) = e^x$?

You do not need to show any work for this part of the problem, so if you remember the answer, you can just write it down.

Solution:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(b) Find the Maclaurin series for $g(x) = e^{-x^2}$.

Solution: Substituting $-x^2$ for x in the Maclaurin series for e^x, we get

$$\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

(c) Use the series in part (b) to approximate e^{-1} with an error of at most .01.

Solution: You can use Taylor's formula, but it's easier to use the alternating series test: We know the Maclaurin series for e^x converges to e^x for every x, and so the series in part (b) converges to e^{-x^2} for every x. In particular

$$e^{-1} = e^{-(1^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}.$$

Since this is an alternating series, the error in using the partial sum

$$\sum_{n=0}^{c} \frac{(-1)^n}{n!}$$

to approximate the actual sum is at most the absolute value of the next term, or

$$\left| \frac{(-1)^{(c+1)}}{(c+1)!} \right|.$$

This will be less than .01 if $(c+1)!$ is greater than 100. Since $5! = 120$, $c = 4$ will do, and our approximation is

$$e^{-1} \approx \sum_{n=0}^{4} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = \frac{3}{8}.$$

3. Find an equation in the form $Ax + By + C = D$ for the plane containing the line

$$\langle x,\, y,\, z\rangle = \langle 1,\, -1,\, 2\rangle + t\,\langle 2,\, 1,\, 3\rangle$$

and the point $C = (2,\, 0,\, 3)$.

Solution: Vector parallel to line, therefore to plane:

$$\langle 2,\, 1,\, 3\rangle$$

Another vector parallel to plane, between points $\langle 1,\, -1,\, 2\rangle$ (on line) and $\langle 2,\, 0,\, 3\rangle$:

$$\langle 2,\, 0,\, 3\rangle - \langle 1,\, -1,\, 2\rangle = \langle 1,\, 1,\, 1\rangle$$

Vector normal to plane:

$$\vec{n} = \langle 2,\, 1,\, 3\rangle \times \langle 1,\, 1,\, 1\rangle = \langle -2,\, 1,\, 1\rangle$$

Equation of plane:

$$-2(x-2) + 1(y) + 1(z-3) = 0$$

$$\boxed{-2x + y + z = -1}$$

4. Consider the lines L_1 and L_2 with vector equations

$$\langle x,\, y,\, z\rangle = \langle 1,\, 2,\, 3\rangle + t\,\langle a,\, 1,\, 0\rangle \quad \text{and} \quad \langle x,\, y,\, z\rangle = \langle 2,\, 0,\, 1\rangle + s\,\langle 1,\, 1,\, 0\rangle$$

respectively. Is it possible to choose the constant a so that the lines intersect? (This is not simply a "YES or NO" question. You must explain how you arrived at your conclusion.)

Solution:

Rewrite the equations of the lines:

$$\langle x,\, y,\, z\rangle = \langle 1 + at,\, 2 + t,\, 3\rangle \quad \text{and} \quad \langle x,\, y,\, z\rangle = \langle 2 + s,\, s,\, 1\rangle$$

We see that, whatever the value of a, every point on L_1 has z-coordinate 3 and every point on L_2 has z-coordinate 1. Therefore no point can be on both lines, and the answer to the question is NO: It is not possible to choose the constant a so that the lines intersect.

5. Suppose that $\vec{u} \times \vec{v} = \langle 5, 1, 1 \rangle$, that $\vec{u} \cdot \vec{u} = 4$, and that $\vec{v} \cdot \vec{v} = 9$. Find $|\vec{u} \cdot \vec{v}|$.

Solution: From the given information, we see that

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}} = 4 \quad \& \quad |\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} = 3.$$

From the given information we also see that

$$|\vec{u} \times \vec{v}| = \sqrt{27} = 3\sqrt{3}.$$

Since $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ where θ is the angle between the vectors,

$$3\sqrt{3} = |\vec{u}||\vec{v}| \sin\theta = (2)(3)\sin\theta$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \pm\frac{1}{2}.$$

From this we see $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta = \pm 3$. Since the problem asks for the absolute value of $\vec{u} \cdot \vec{v}$,

$$\boxed{|\vec{u} \cdot \vec{v}| = 3.}$$

6. Give a set of parametric equations for the line of intersection of the planes $x+2y-3z=5$ and $5x+5y-z=1$.

Solution: The normal vectors for the two planes are $\mathbf{n_1} = \langle 1,2,-3\rangle$ and $\mathbf{n_2} = \langle 5,5,-1\rangle$. Since the line lies in both planes, its direction must be orthogonal to both normal vectors. So we can find the direction of the line by taking the cross product

$$\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 5 & 5 & -1 \end{bmatrix} = (-2+15)\hat{\imath} - (-1+15)\hat{\jmath} + (5-10)\hat{k}$$
$$= \langle 13, -14, -5\rangle$$

A point (x, y, z) lies in both planes if it satisfies both equations. So we solve the system by multiplying the first equation by -5 and adding it to the first to eliminate x:

$$\begin{aligned} -5x - 10y + 15z &= -25 \\ 5x + 5y - z &= 1 \\ -5y + 14z &= -24 \end{aligned}$$

So we can take $z = 0$, $y = 24/5$ and $x = -23/5$, and $(-23/5, 24/5, 0)$ is a point on the line. Hence the parametric equations are

$$\begin{aligned} x &= 13t - 23/5 \\ y &= -14t + 24/5 \\ z &= -5t \end{aligned}$$

7. Find the radius of convergence and the interval of convergence for the series

$$\sum_{n=2}^{\infty} \frac{n^2(x-2)^n}{3^n}$$

Solution: Let a_n be the terms of this series. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2(x-2)^{n+1}}{3^{n+1}} \frac{3^n}{n^2(x-2)^n}\right| = \left(\frac{n+1}{n}\right)^2 \frac{1}{3}|x-2| \to \frac{|x-2|}{3}$$

as $n \to \infty$. So by the Ratio Test the series diverges if $|x-2| > 3$ and converges absolutely, and hence converges, if $|x-2| < 3$. Hence the radius of convergence is 3 and the series converges for $-1 < x < 5$. To find the interval of convergence, we must decide whether the series converges when $x = -1$ and $x = 5$. In the first case, the series

$$\sum_{n=2}^{\infty} \frac{n^2(-3)^n}{3^n} = \sum_{n=2}^{\infty} n^2(-1)^n$$

diverges by the Divergence Test. Similarly, when $x = 5$ the series

$$\sum_{n=2}^{\infty} \frac{n^2 3^n}{3^n} = \sum_{n=2}^{\infty} n^2$$

the series diverges by the Divergence Test. Thus the interval of convergence is $(-1, 5)$.

8. (a) What is the area of the triangle with corners (0, 0, 0), (0, 1, −1) and (1, 0, 1)?

Solution:

$$\boxed{\frac{\sqrt{3}}{2}}$$

(The triangle is half of the parallelogram whose edges are vectors from (0, 0, 0) to (0, 1, −1) and to (1, 0, 1), so the triangle has half the area of the parallelogram. The area of the parallelogram is the magnitude of the cross product of those vectors.)

(b) An object moves with constant velocity of $\mathbf{v} = \langle 4, 2, 0\rangle$ units per second, while a constant force $\mathbf{F} = \langle 1, 1, 1\rangle$ is acting on the object. Find the work done by the force after the object has been travelling for 5 seconds.

Solution: The displacement vector is given by $\mathbf{D} = 5\mathbf{v} = \langle 20, 10, 0\rangle$. The work is given by the product of the component of the force in the direction of motion and the distance moved. Hence if θ is the angle between the force vector and displacement vector, $W = (|\mathbf{F}|\cos\theta)|\mathbf{D}| = \mathbf{F}\cdot\mathbf{D}$. Thus

$$W = \langle 1, 1, 1\rangle \cdot \langle 20, 10, 0\rangle = 20 + 10 = 30$$

(c) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ where $\mathbf{a} = \langle -2, 3, -6\rangle$ $\mathbf{b} = \langle 5, -1, 4\rangle$.

Solution: First calculating $|\mathbf{a}|^2 = 4 + 9 + 36 = 49$, the vector projection formula gives

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \frac{-10 - 3 - 24}{49}\langle -2, 3, -6\rangle = \frac{-37}{49}\langle -2, 3, -6\rangle$$