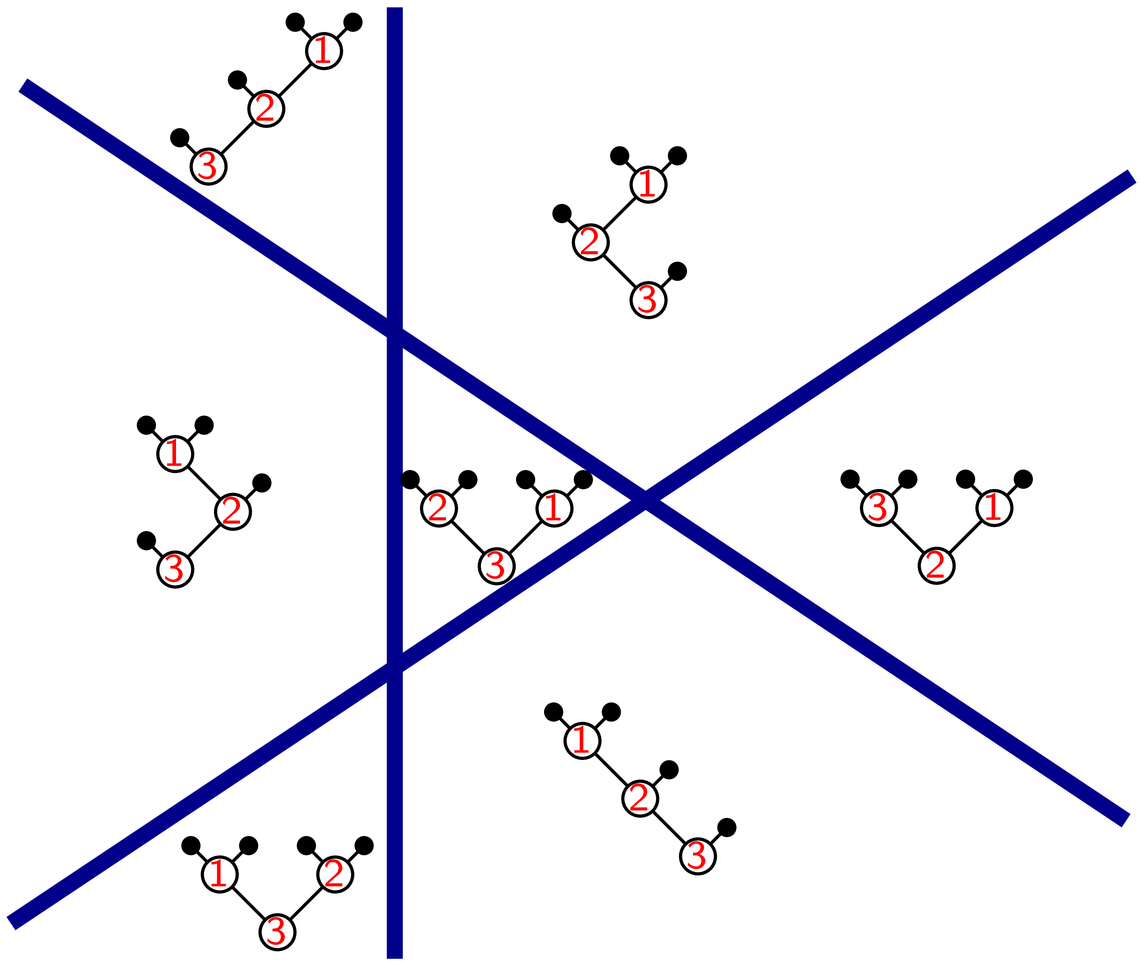
Bijections for the deformations of the braid arrangement

Olivier Bernardi - Brandeis University

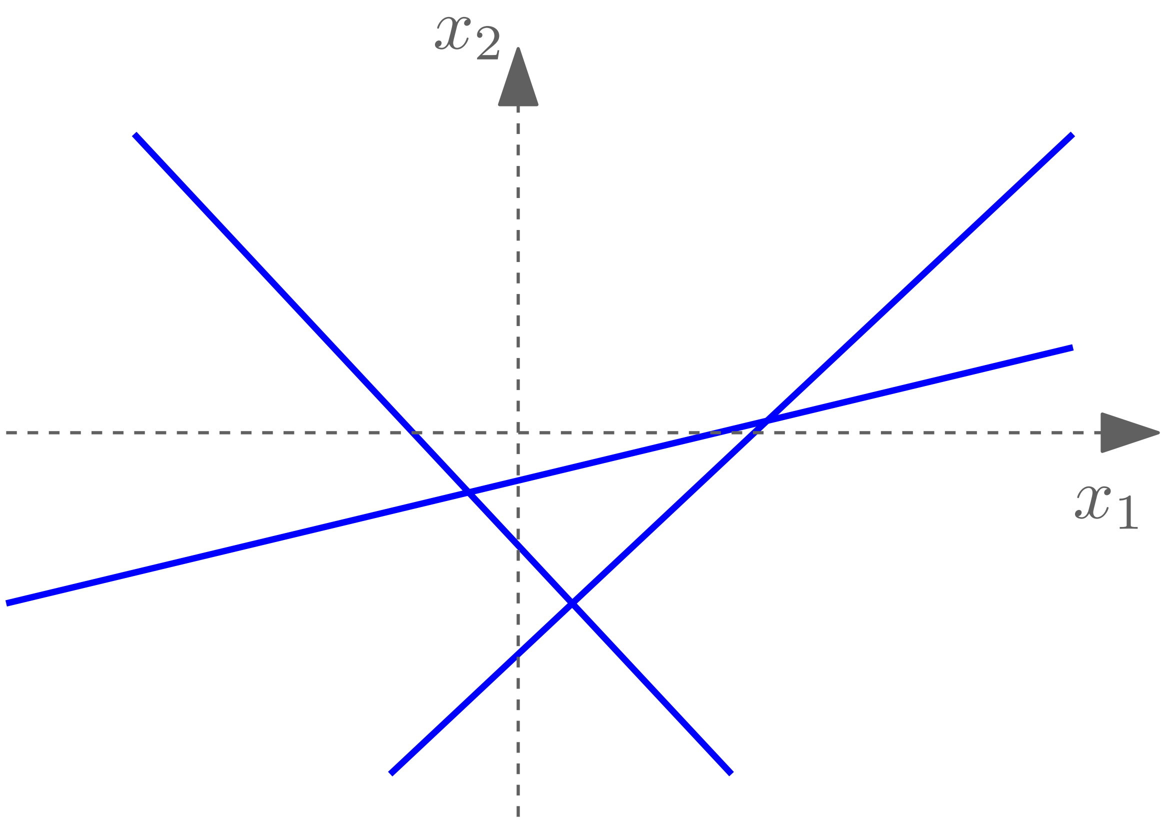


Discrete Math Days of the North-East, May 2017

Hyperplane arrangements

A **hyperplane arrangement** of dimension n is a finite collection of affine hyperplanes in $\mathbb{R}^n$.

Example:



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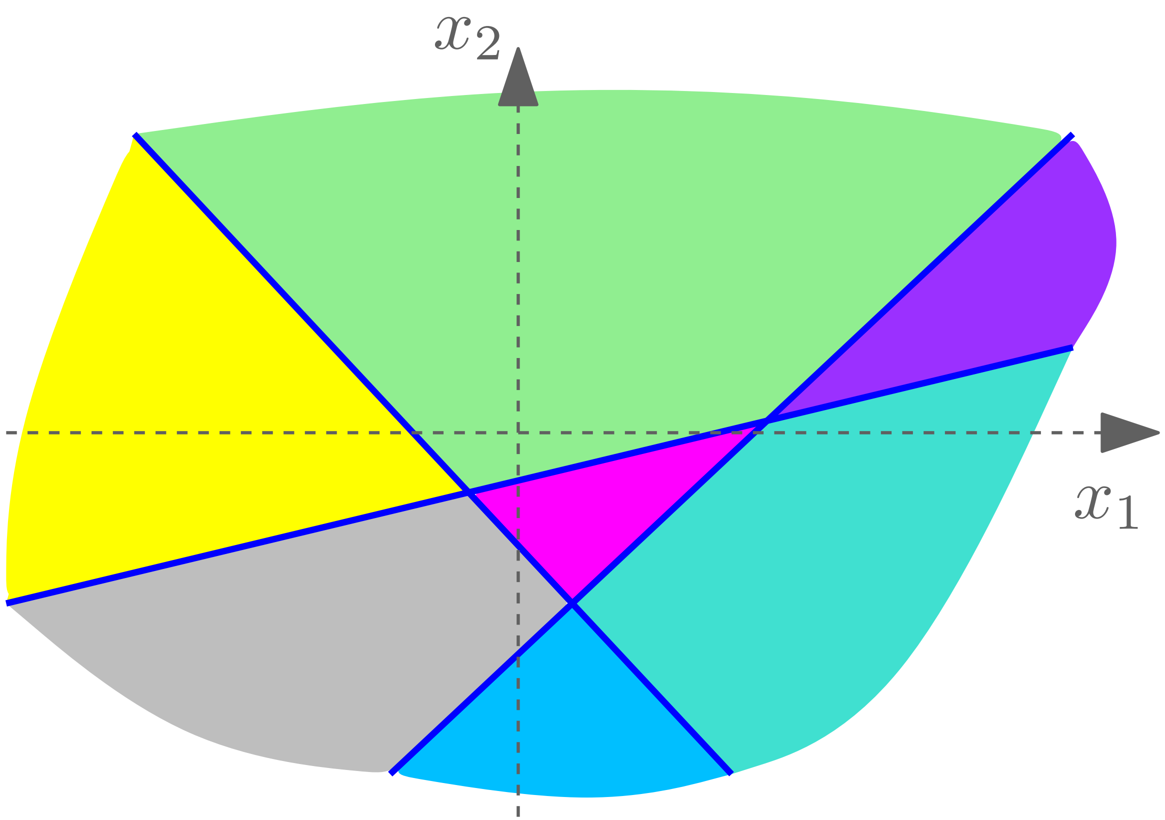

The complement of the hyperplanes is divided into **regions**.

Braid arrangement

Def: The **braid arrangement** of dimension n has hyperplanes

$$\{x_i - x_j = 0\}$$

for all $0 \leq i < j \leq n$.

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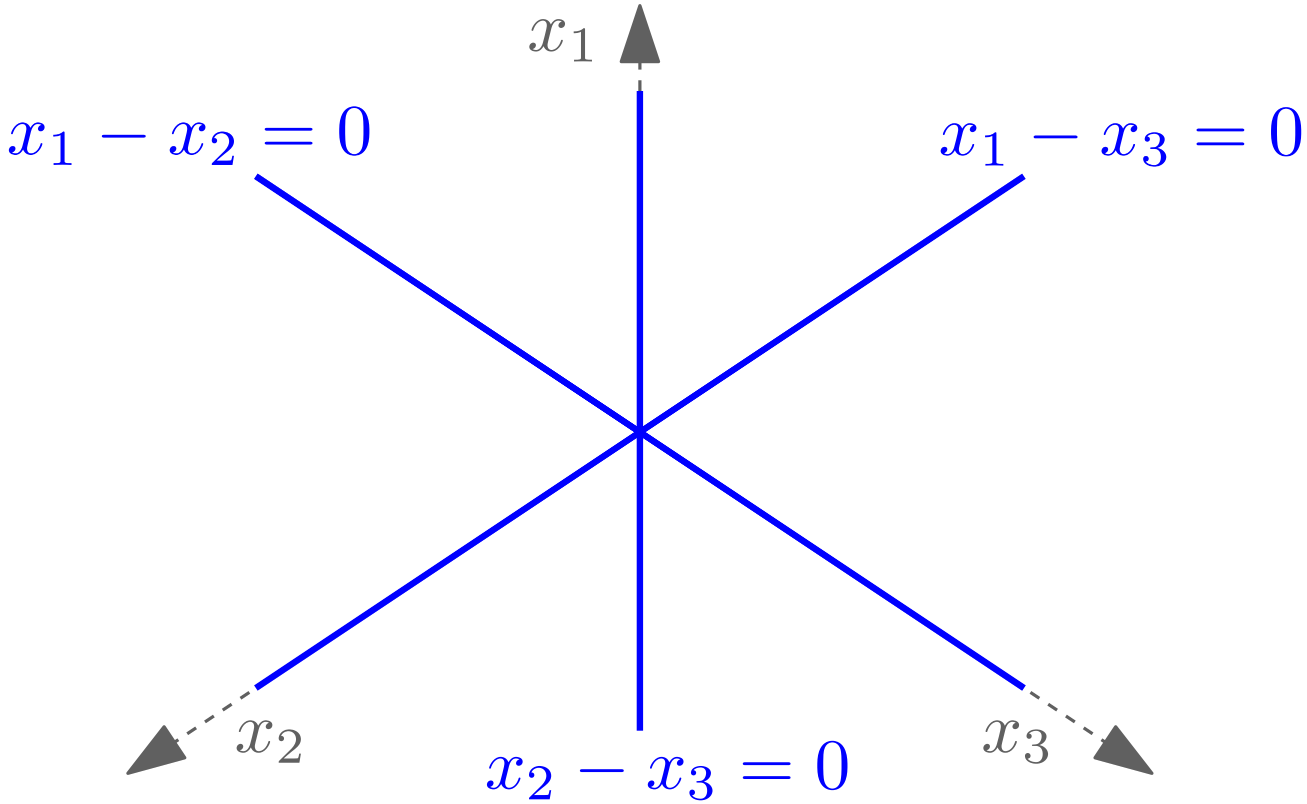
Example: $n = 3$



$n!$ regions

Deformations of the braid arrangement

Def: Fix $S \subset \mathbb{Z}$ finite.

The S-**deformed braid arrangement** $A_S(n) \subset \mathbb{R}^n$ has hyperplanes

$$\{x_i - x_j = s\}$$

for all $0 \leq i < j \leq n$, and all $s \in S$.

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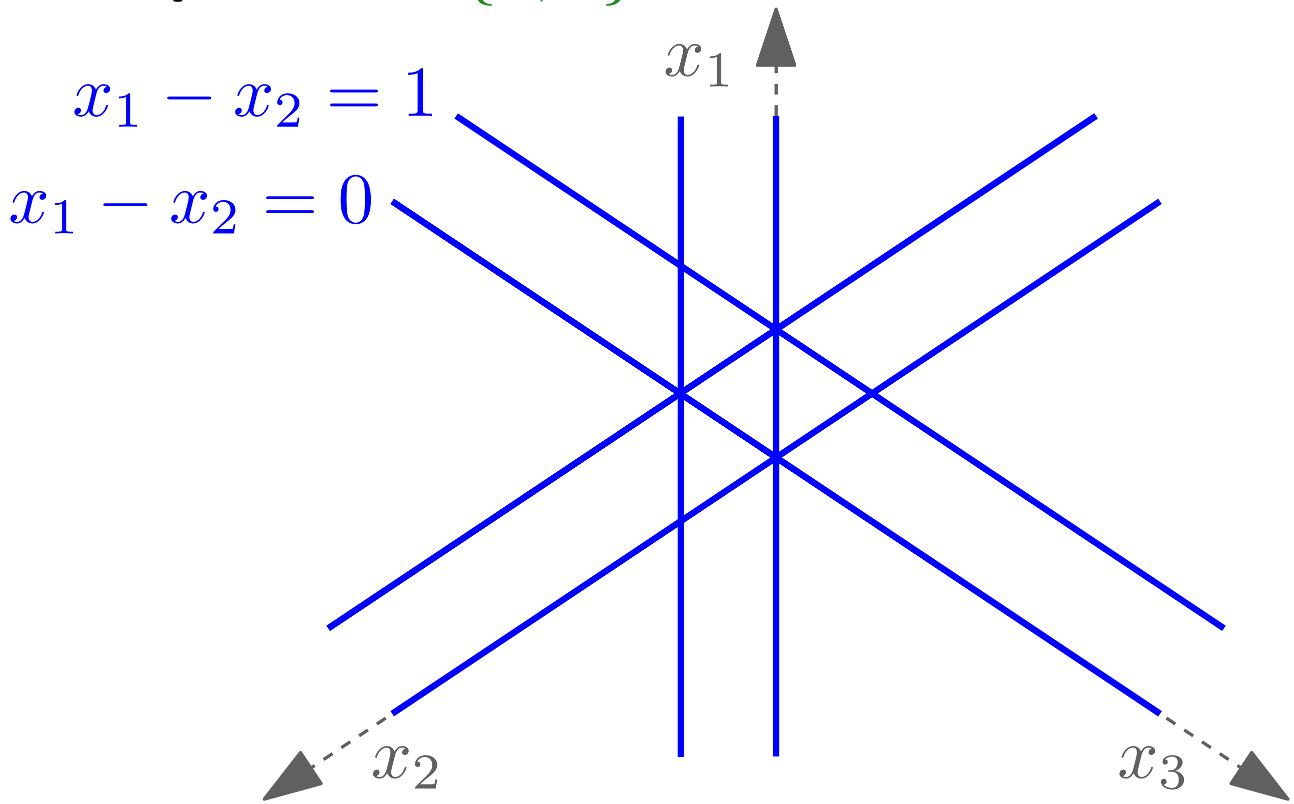
Example: $S = \{0, 1\}$ and $n = 3$.



$(n+1)^{n-1}$ regions

Known counting results for $S \subseteq \{-1, 0, 1\}$

[Stanley, Postnikov, Athanasiadis, . . .]

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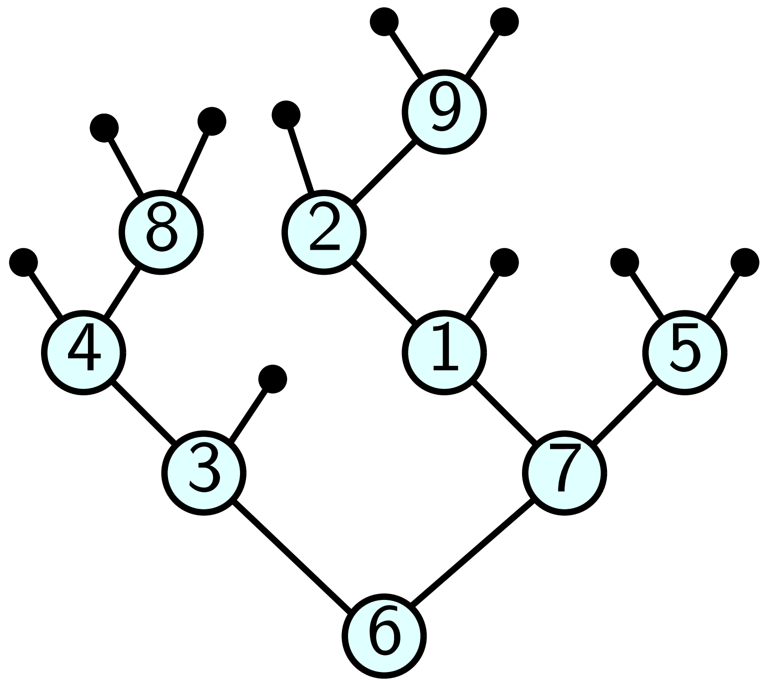
$\mathcal{B}(n)$ = set of rooted binary trees with n labeled nodes.



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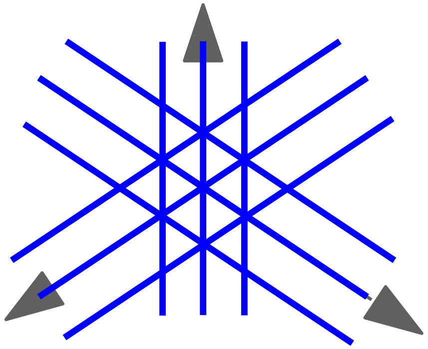	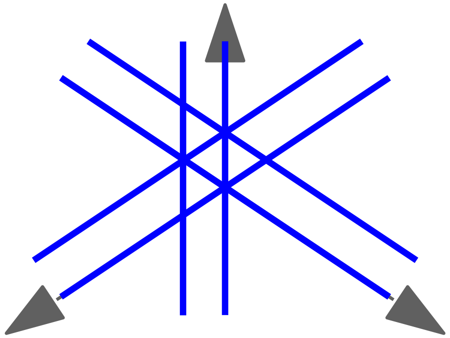	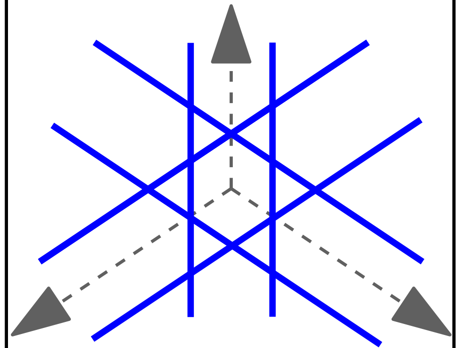	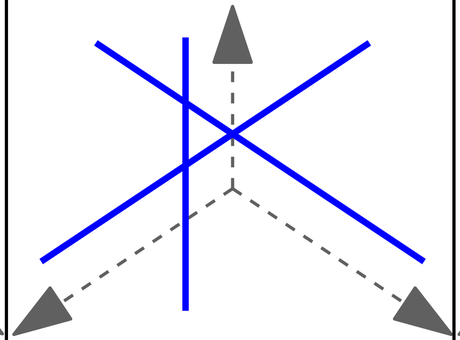	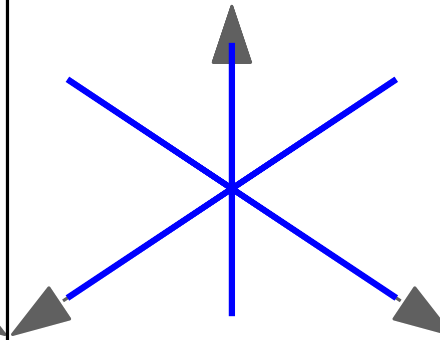
Catalan	**Shi**	**Semi-order**	**Linial**	**Braid**
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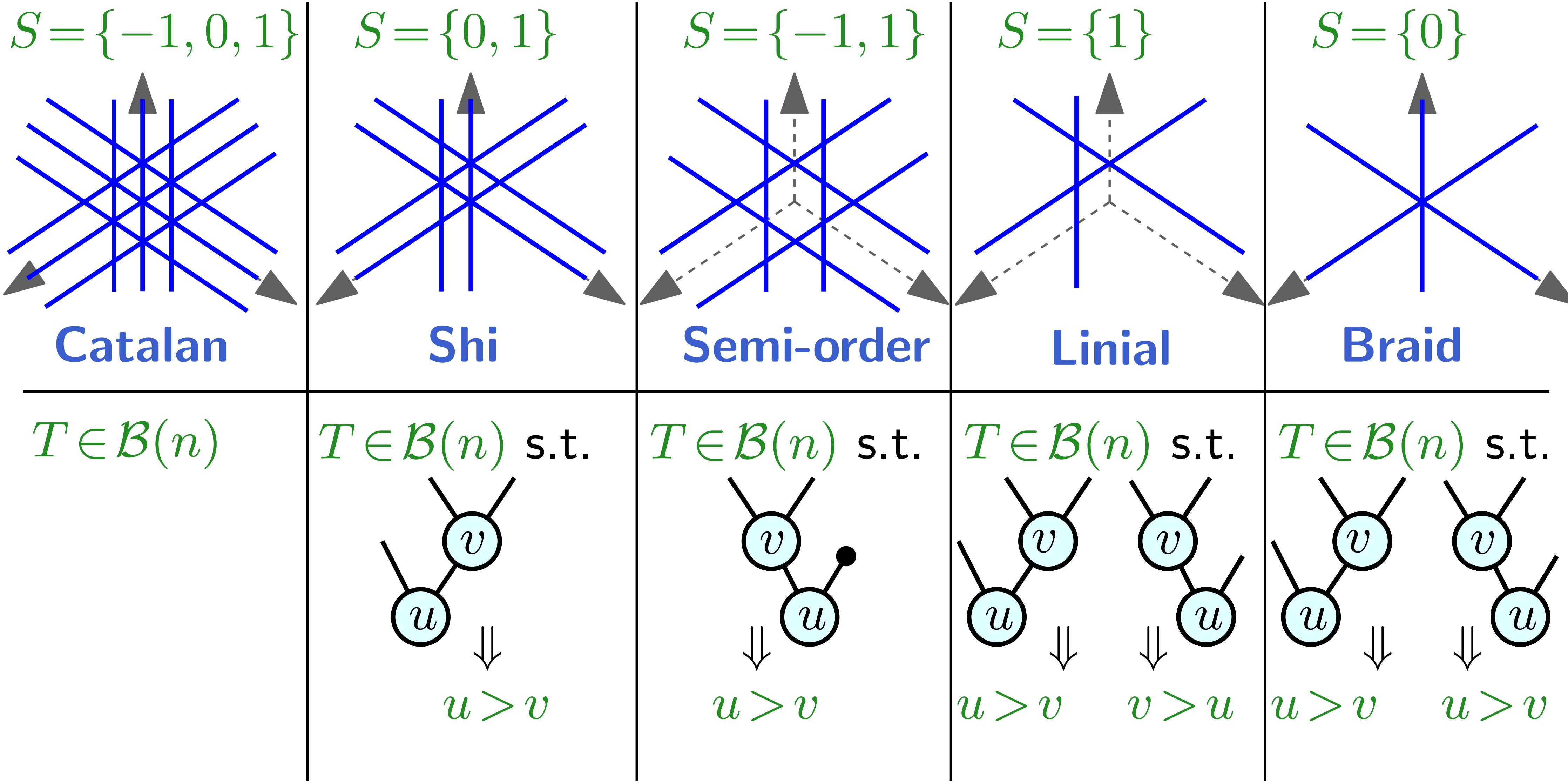

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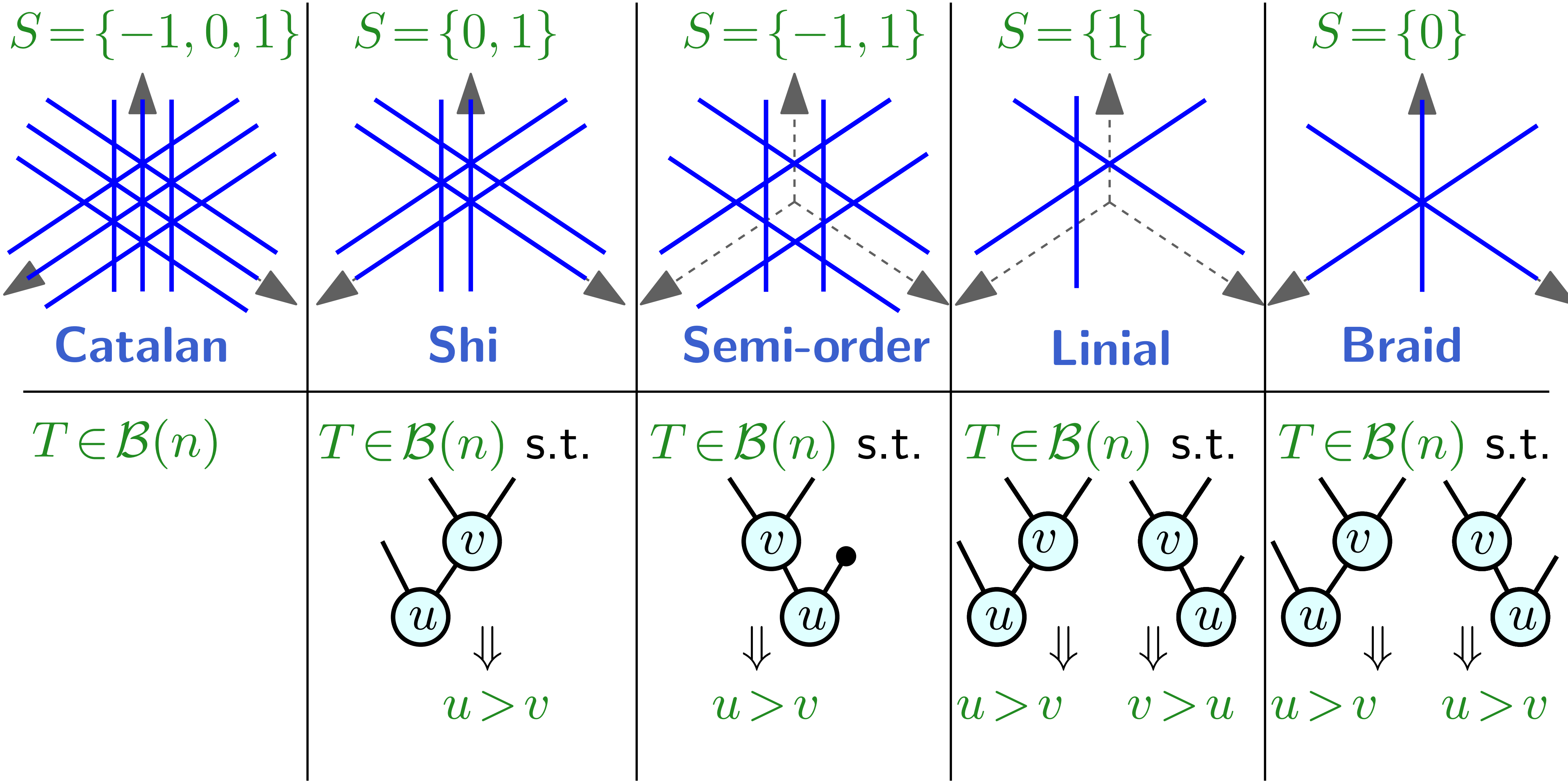

"*Why?*"
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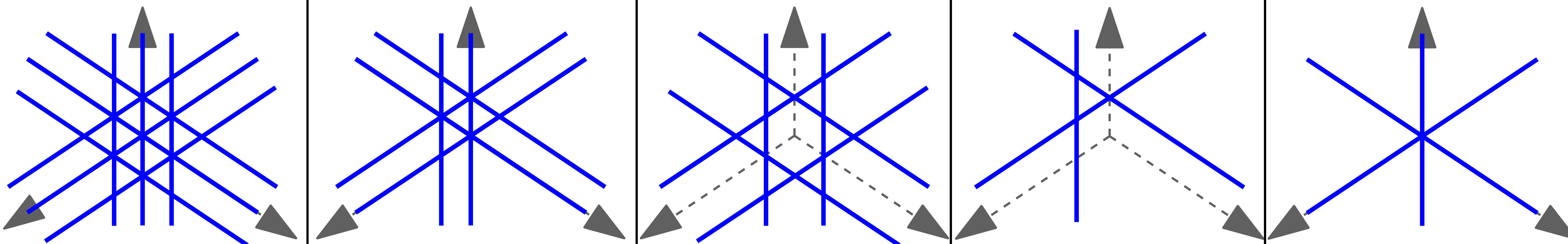

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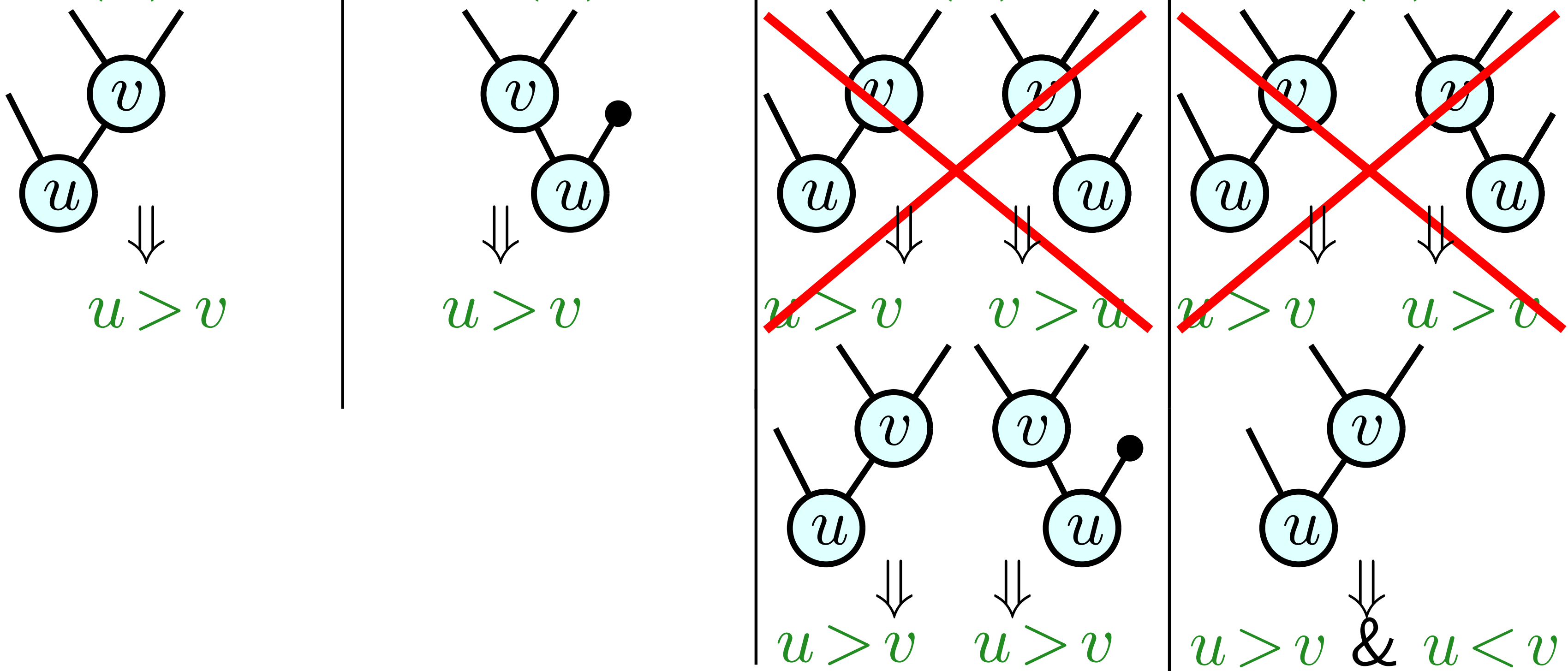

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Bijection

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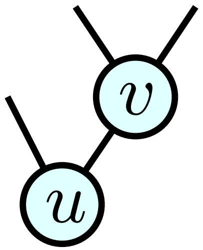
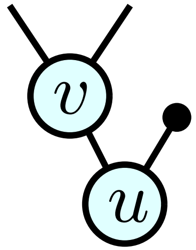
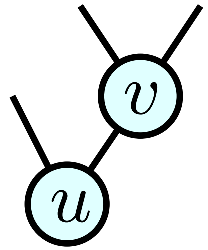
 $\Rightarrow u > v$	 $\Rightarrow u > v$	 $\Rightarrow u < v$

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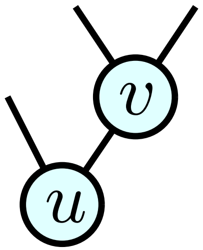
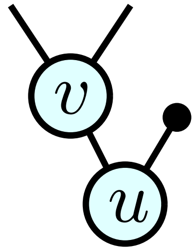
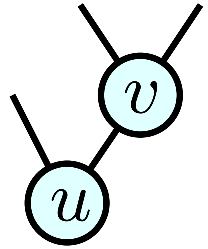
 $\Rightarrow u > v$	 $\Rightarrow u > v$	 $\Rightarrow u < v$

Map: $\Phi_S : \mathcal{T}_S(n) \mapsto$ regions of $\mathcal{A}_S(n)$

$$\Phi_S(T) = \bigcap_{\substack{s\in S,\ 1\leq i<j\leq n \\ (s,i,j)\in T^+}} \{x_i - x_j < s\} \bigcap_{\substack{s\in S,\ 1\leq i<j\leq n \\ (s,i,j)\notin T^+}} \{x_i - x_j > s\}$$

where T^+ is . . .

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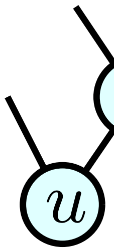
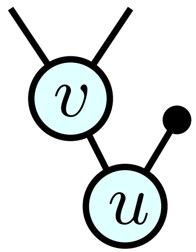
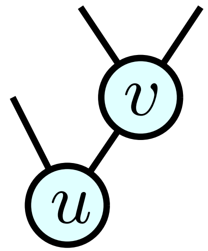
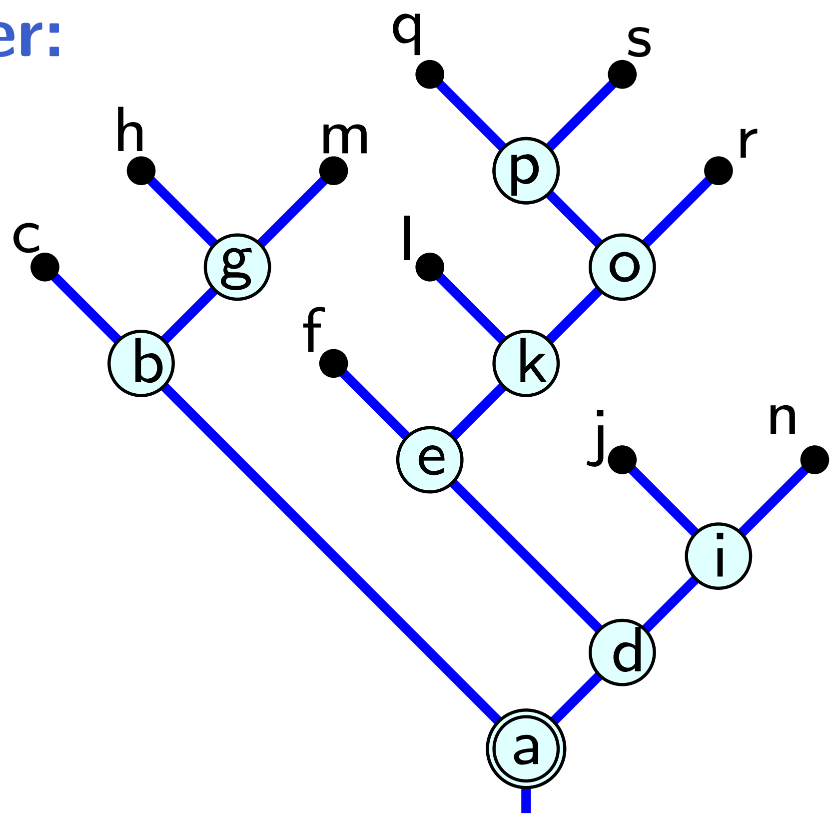
 $\Rightarrow u > v$	 $\Rightarrow u > v$	 $\Rightarrow u < v$

Tree order:



a $\prec_T$ b $\prec_T$ c $\prec_T$ d $\prec_T$ e $\cdots$

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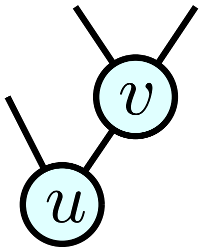
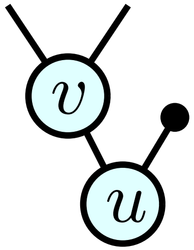
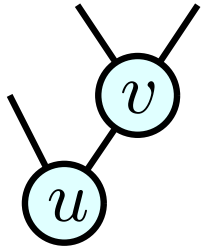
 $\Rightarrow u > v$	 $\Rightarrow u > v$	 $\Rightarrow u < v$

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where $(0, i, j) \in T^+$ if $i \prec_T j$,
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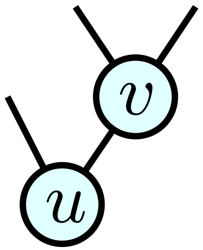
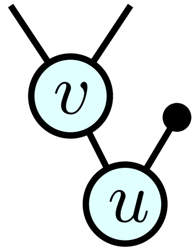
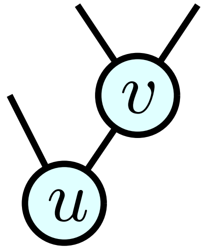
 $\Rightarrow u > v$	 $\Rightarrow u > v$	 $\Rightarrow u < v$

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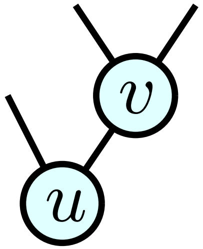
Thm: Φ_S is a bijection between $\mathcal{T}_S(n)$ and the regions of $\mathcal{A}_S(n)$.

Example: Linial $S = \{1\}$

$\mathcal{T}_S(n)$:



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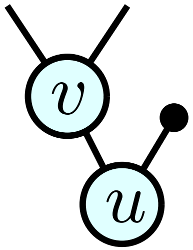

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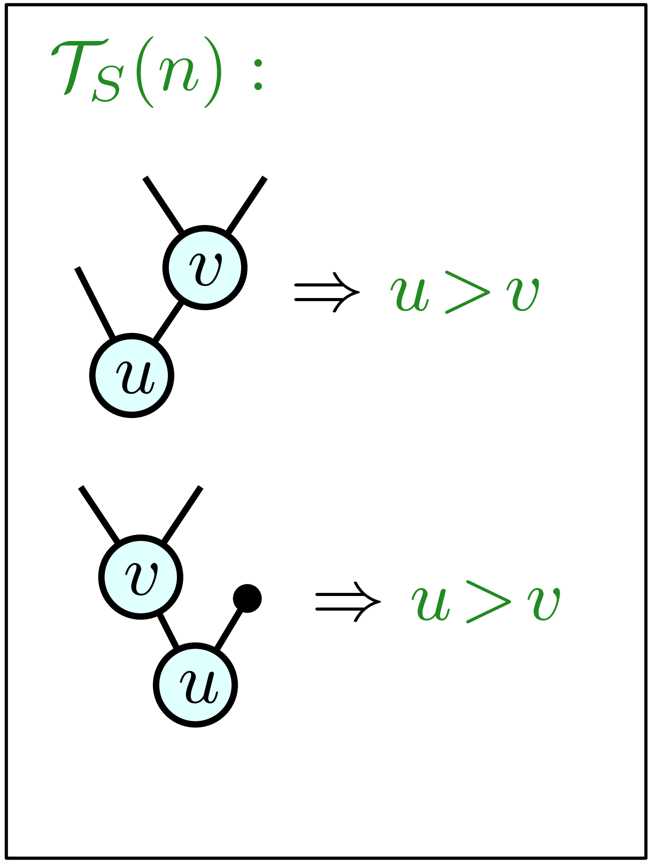

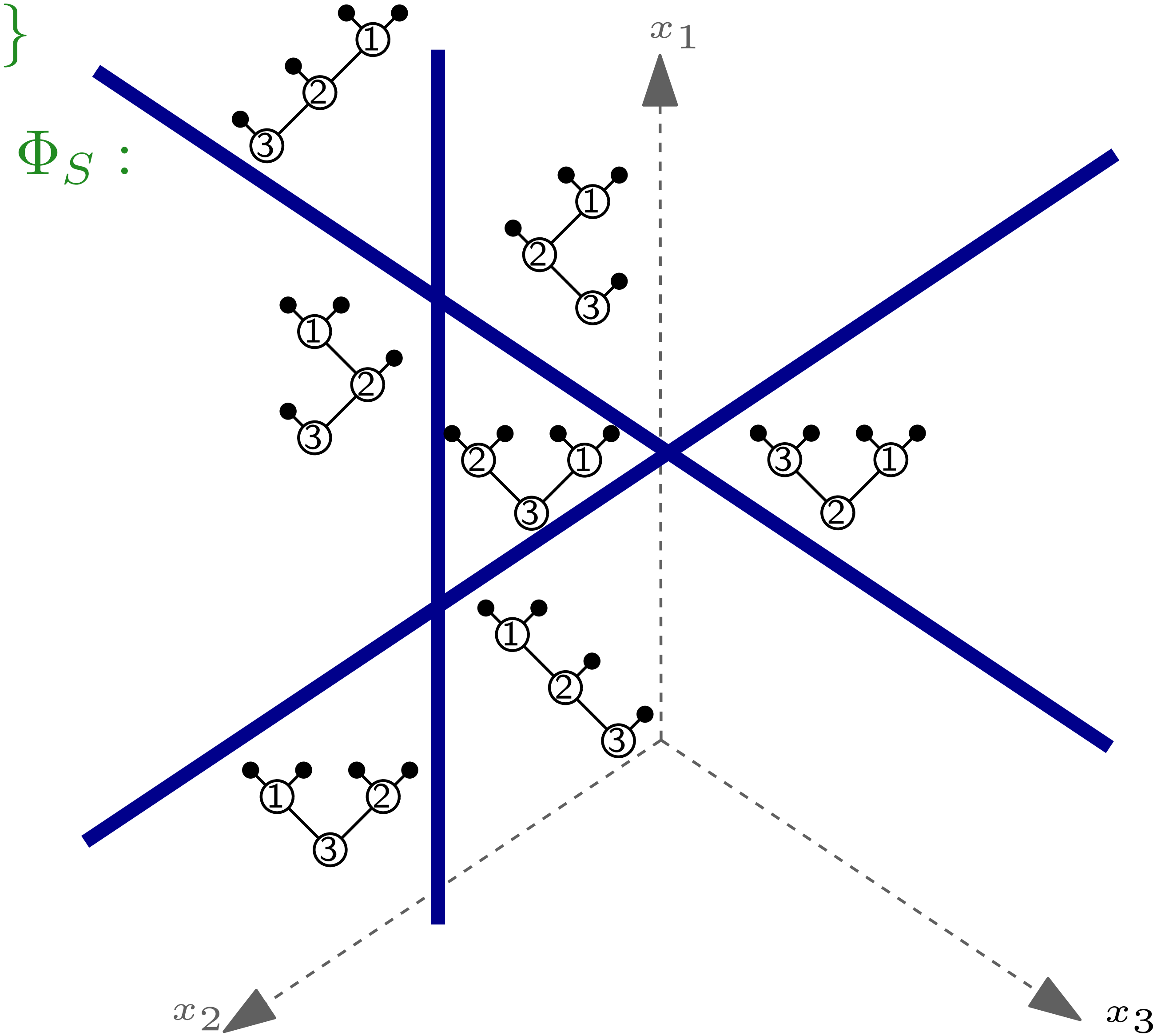

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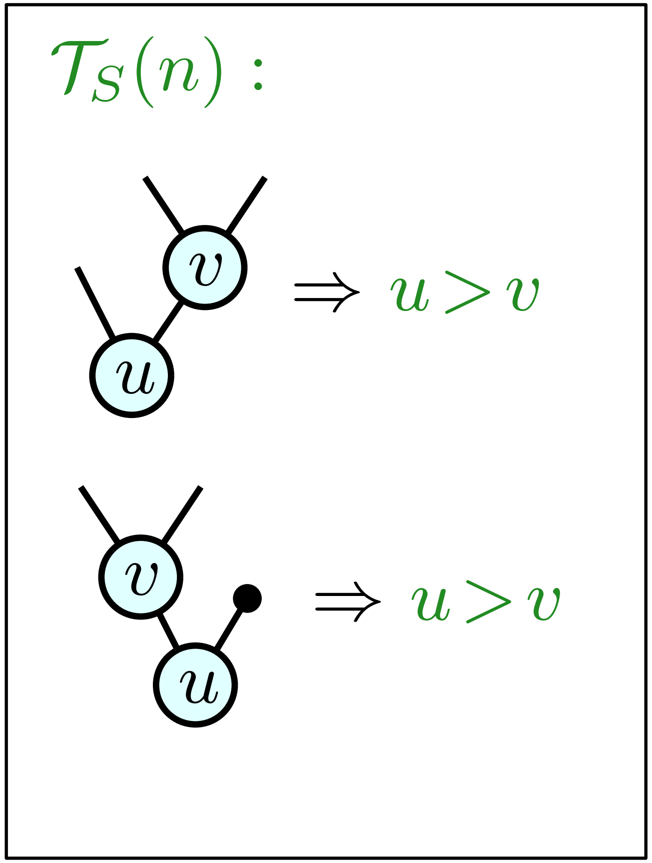

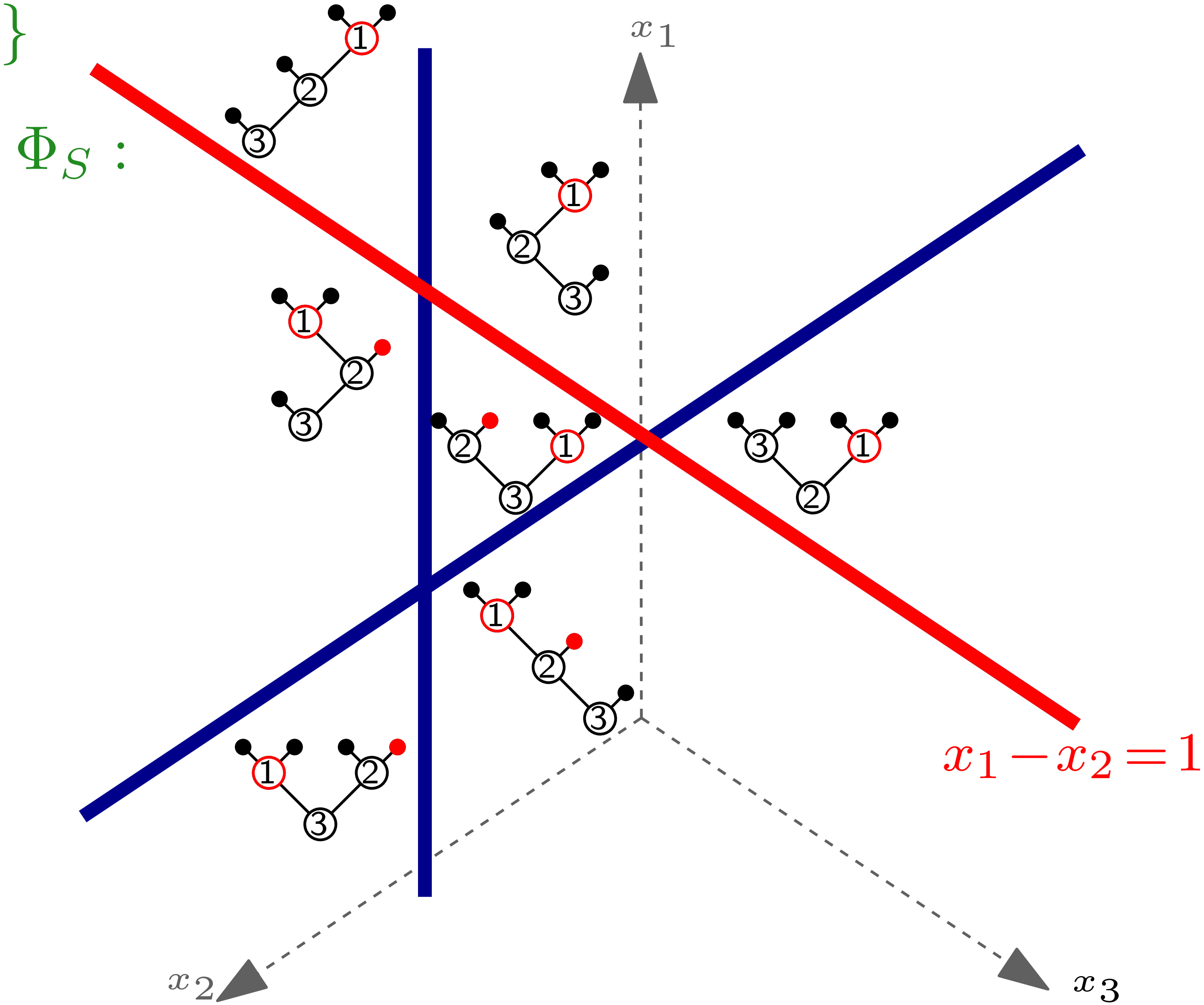

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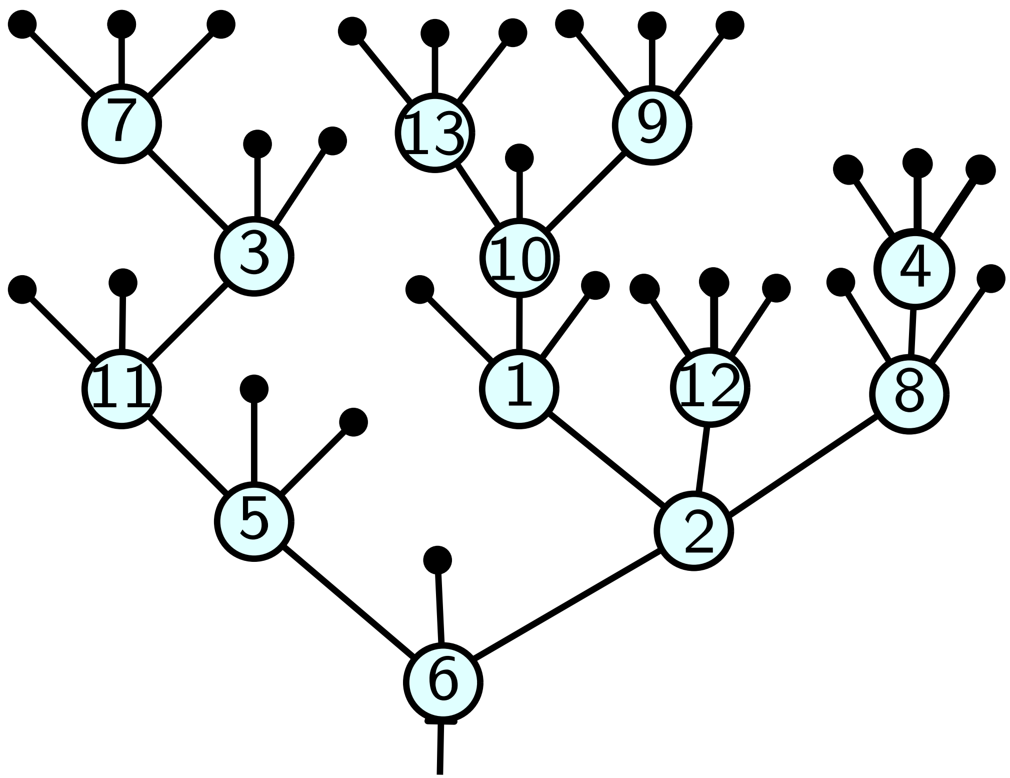
Generalization $S \subseteq [-m..m]$

- $\mathcal{T}^{(m)}$ = set of rooted $(m+1)$-ary trees with labeled nodes.



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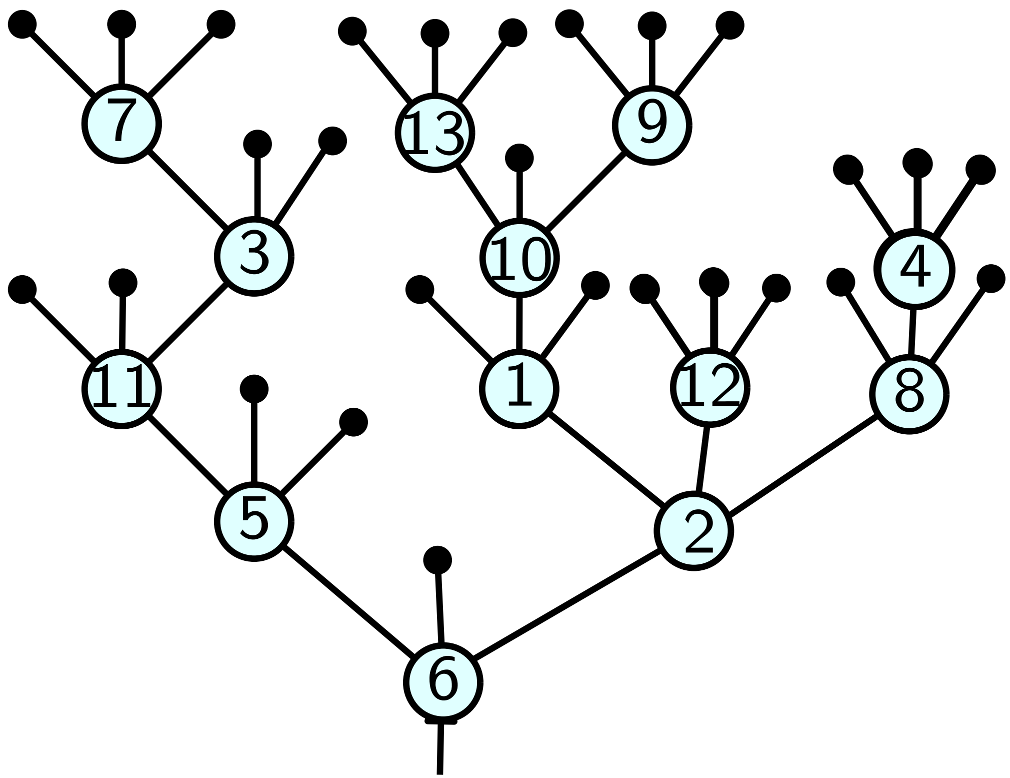

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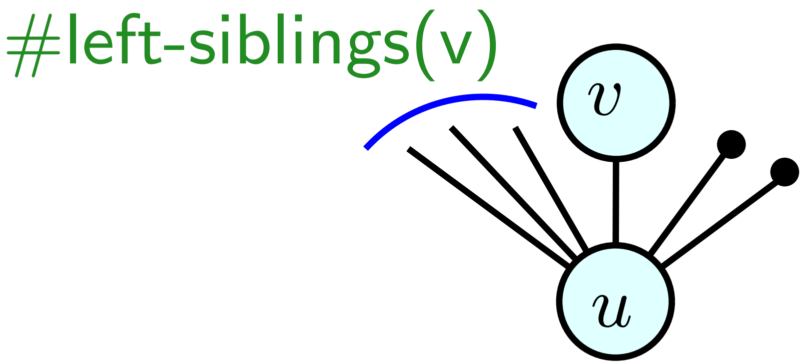
Def: $\mathcal{T}_S$ = set of trees in $\mathcal{T}^{(m)}$ such that for all $v = cadet(u)$,

- $\#\text{left-siblings}(v) \notin S \cup \{0\} \quad \Rightarrow \quad u < v$,
- $-\#\text{left-siblings}(v) \notin S \quad \Rightarrow \quad u > v$.



Generalization $S \subseteq [-m..m]$

Def: S is **transitive** if it satisfies:

- if $a, b \notin S$, with $ab > 0$, then $a + b \notin S$,
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Examples of transitive sets:

- Any subset of $\{-1, 0, 1\}$.
- Any interval of integers containing 1.
- S such that $[-k..k] \subseteq S \subseteq [-2k..2k]$ for some k.

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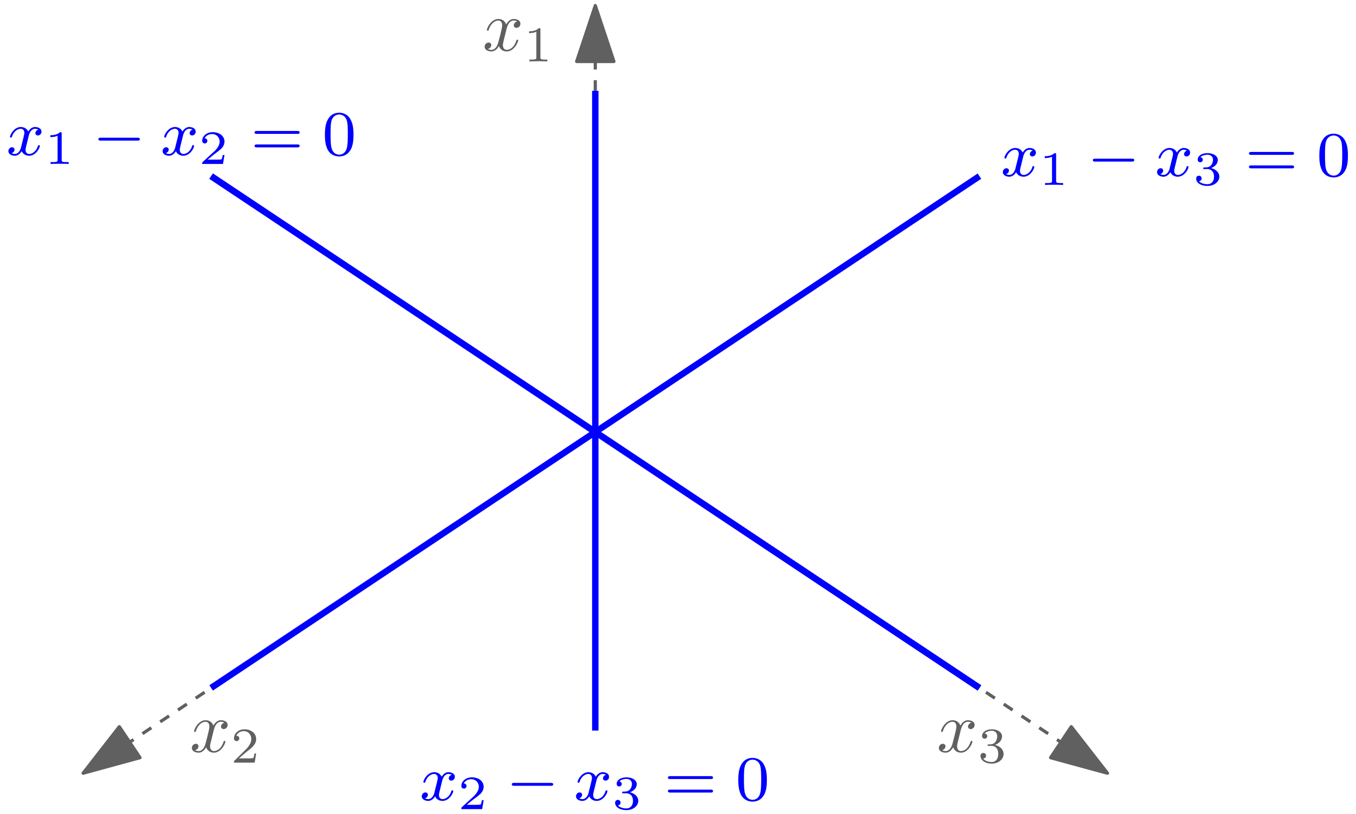
Direct proof for $S \subseteq \{-1, 0, 1\}$

Warm up: Braid arrangement



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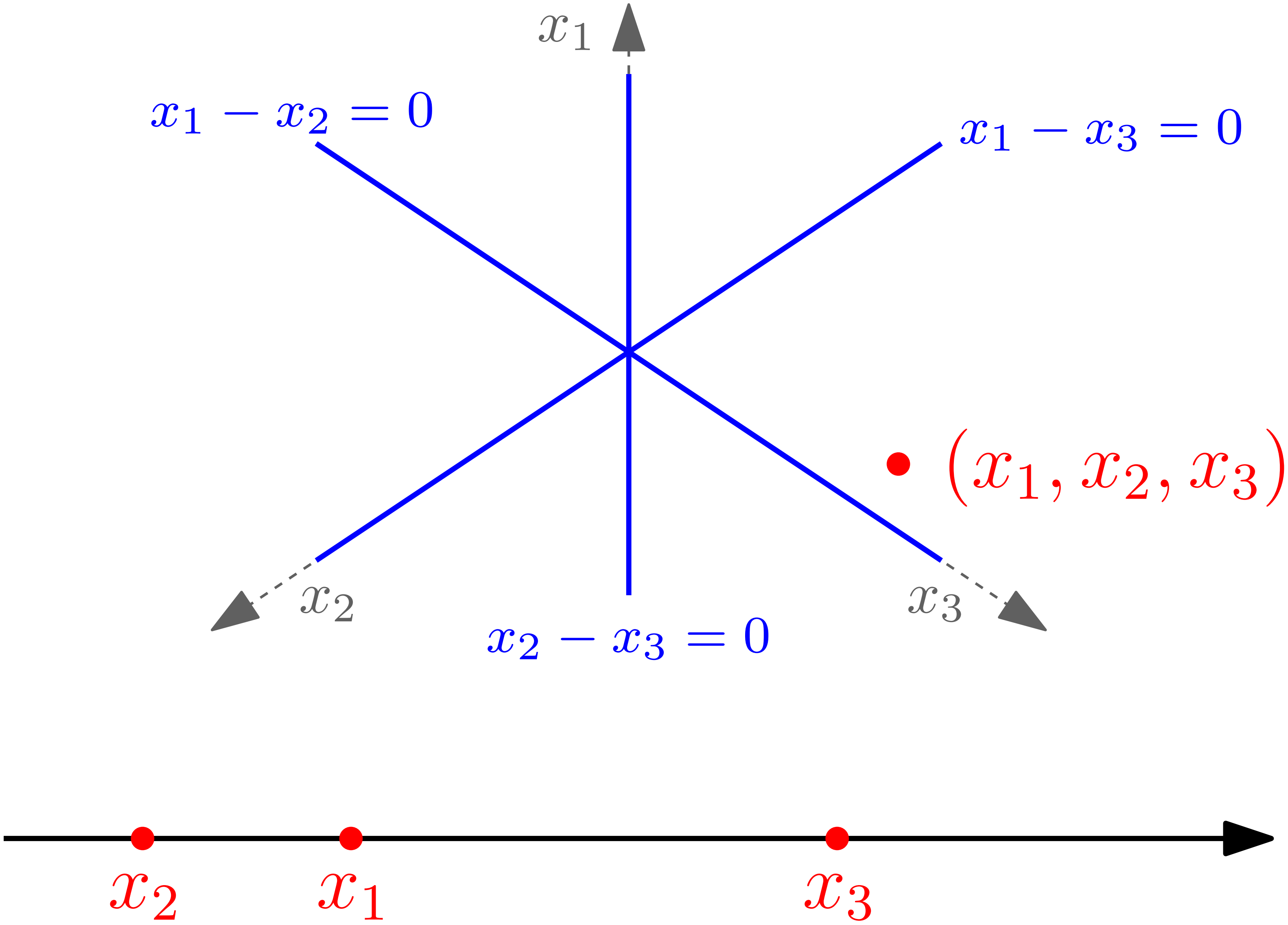

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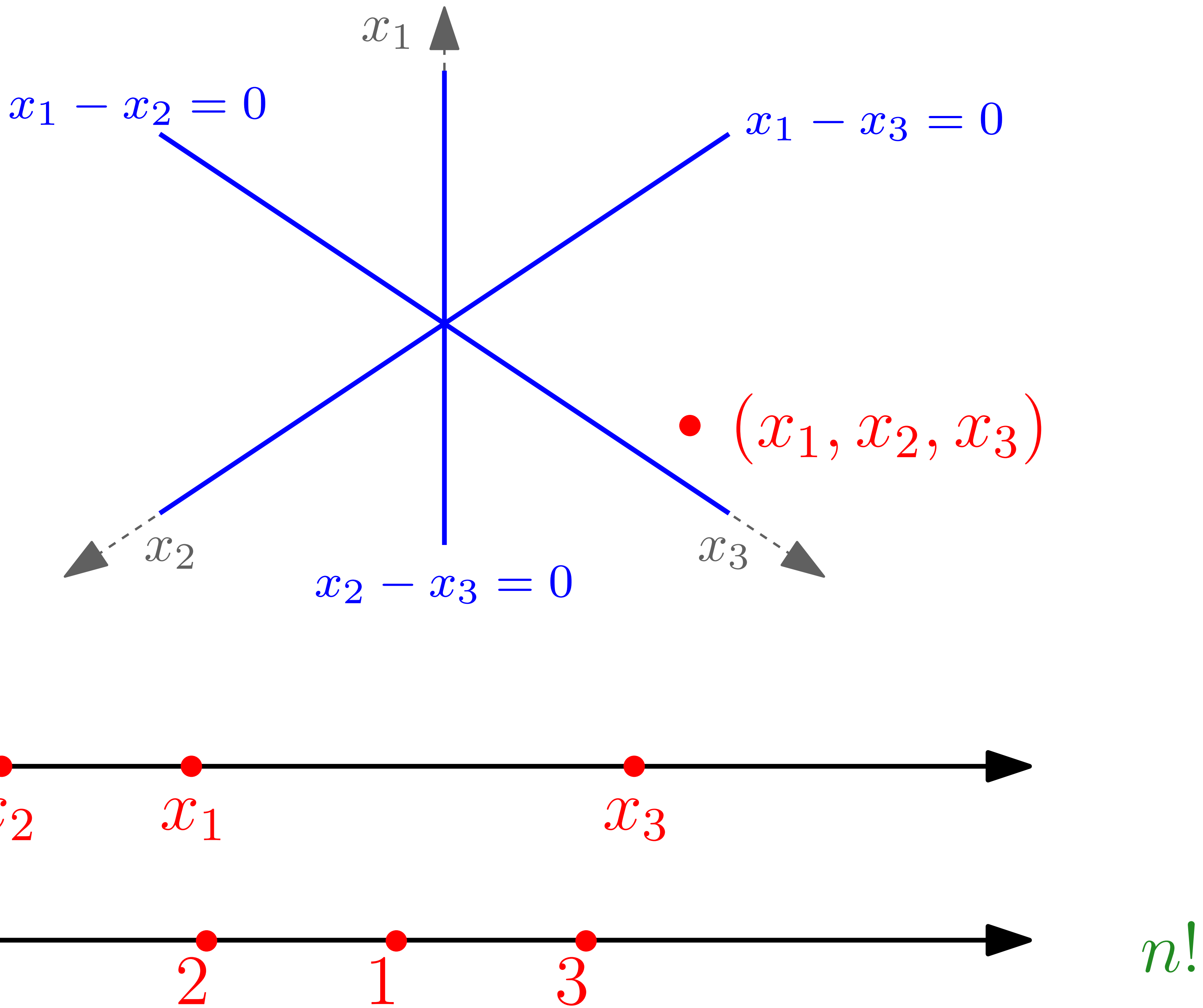
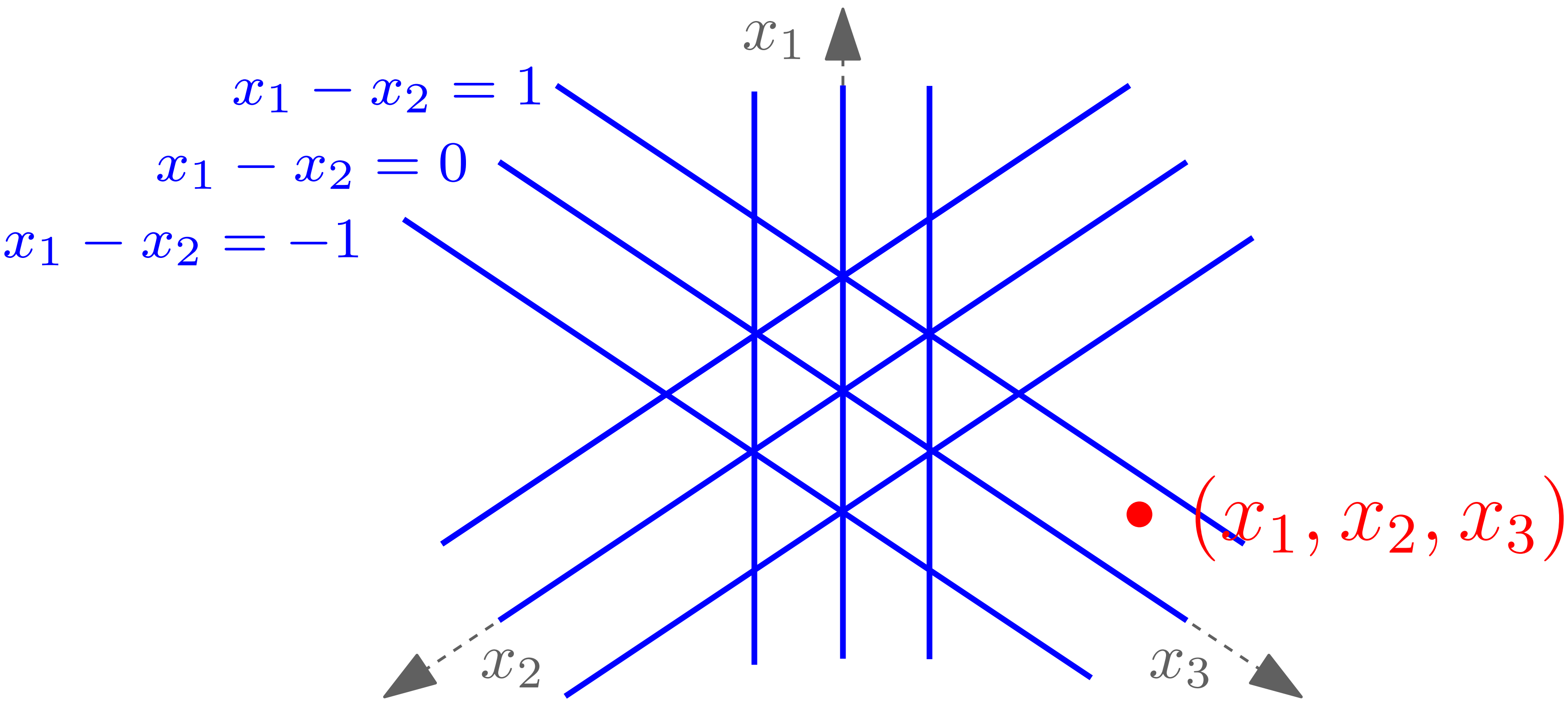
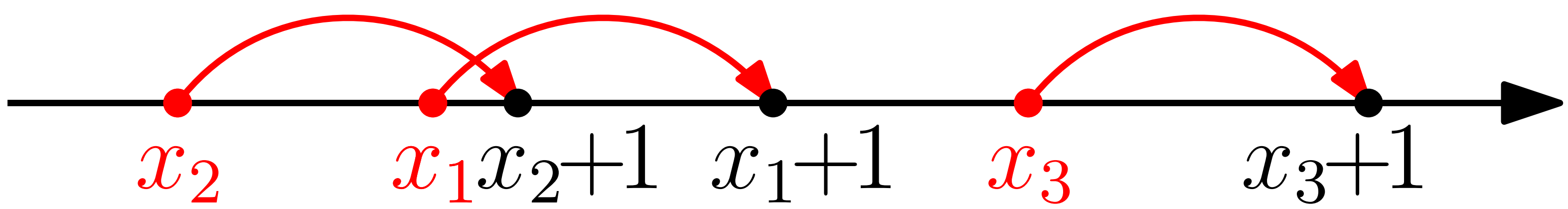


Catalan arrangement

$S = \{-1, 0, 1\}$





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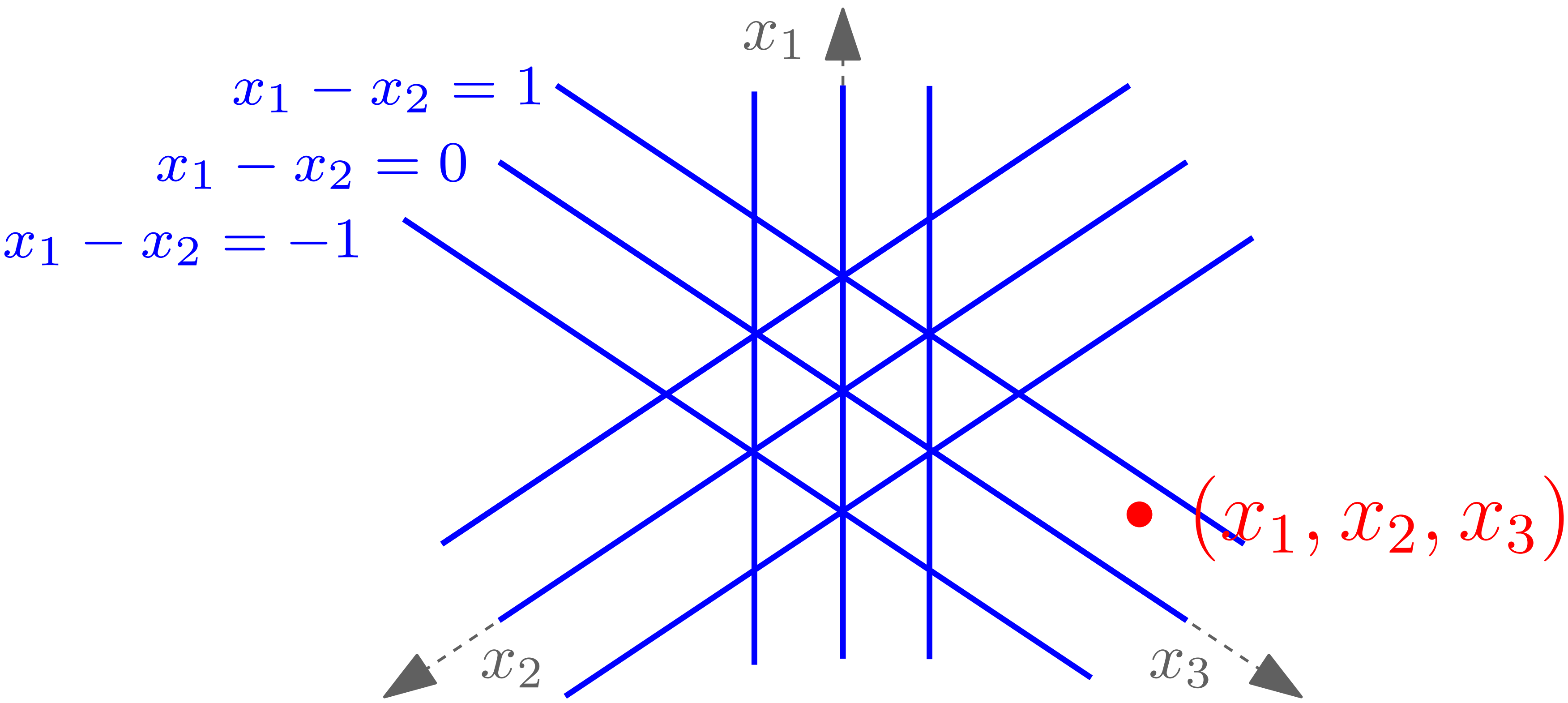
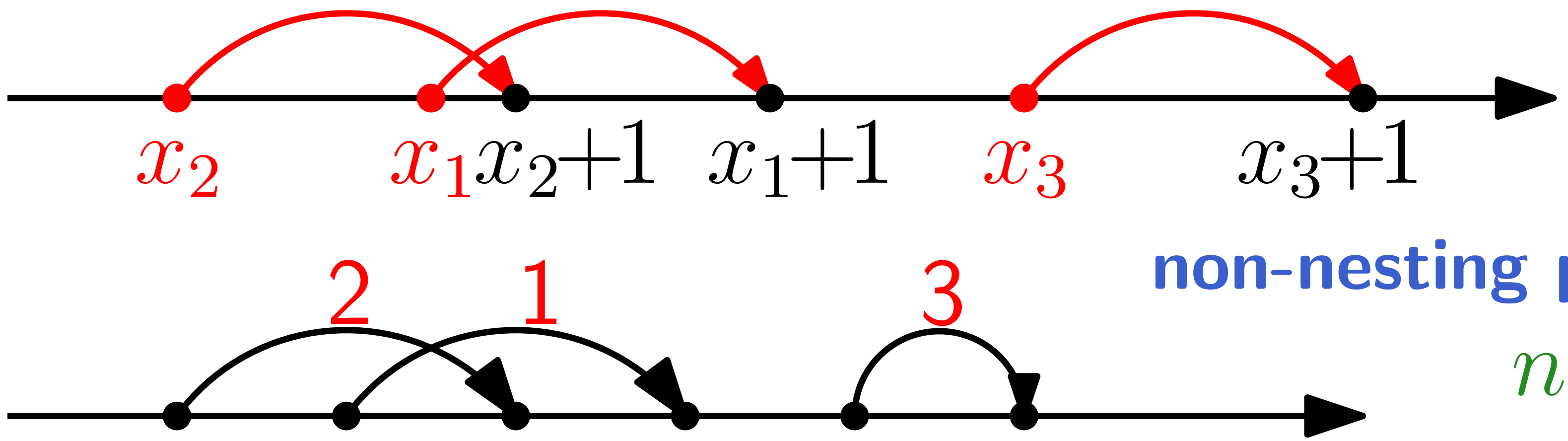
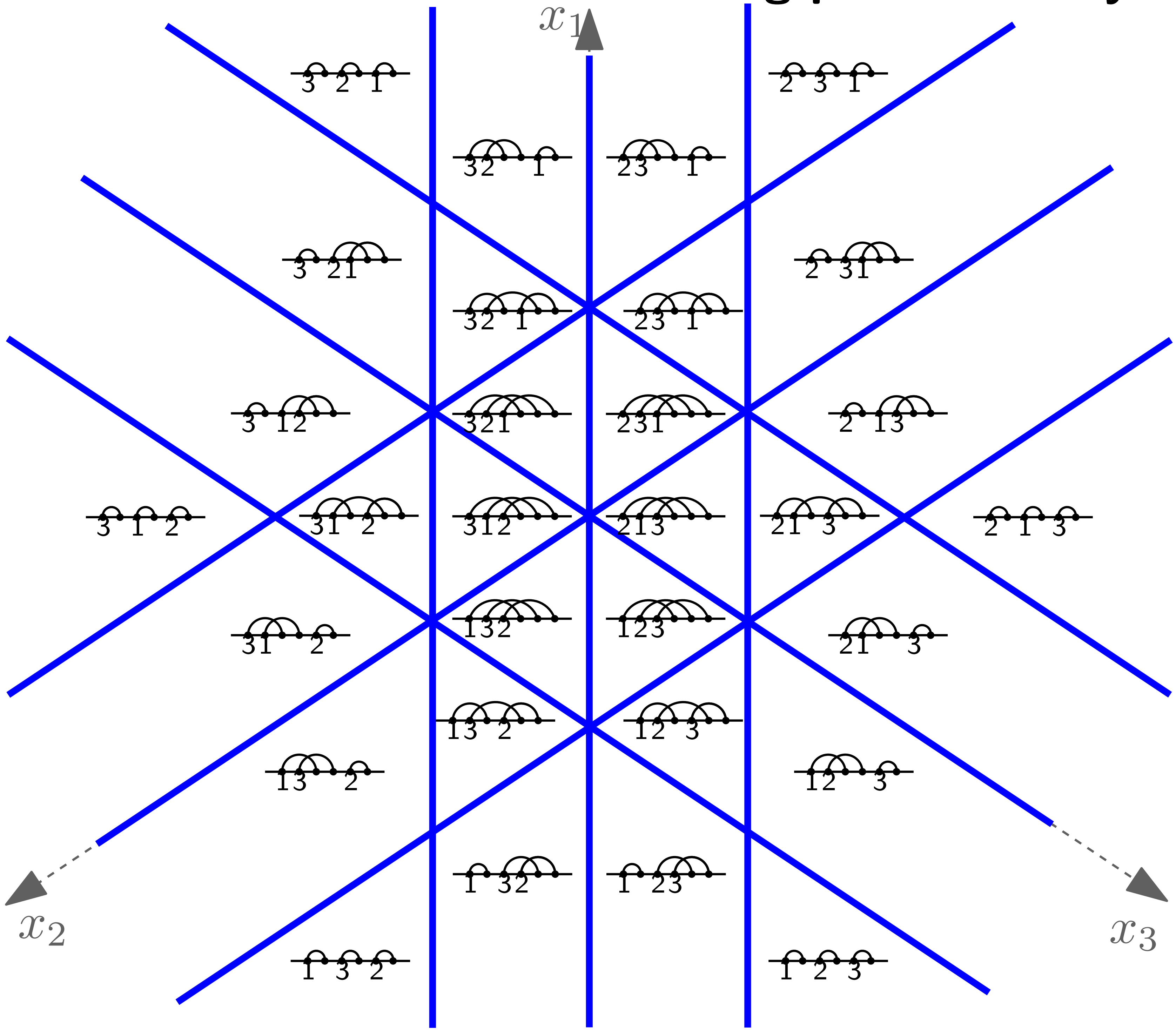




non-nesting parentheses

$n!\mathrm{Cat}(n)$

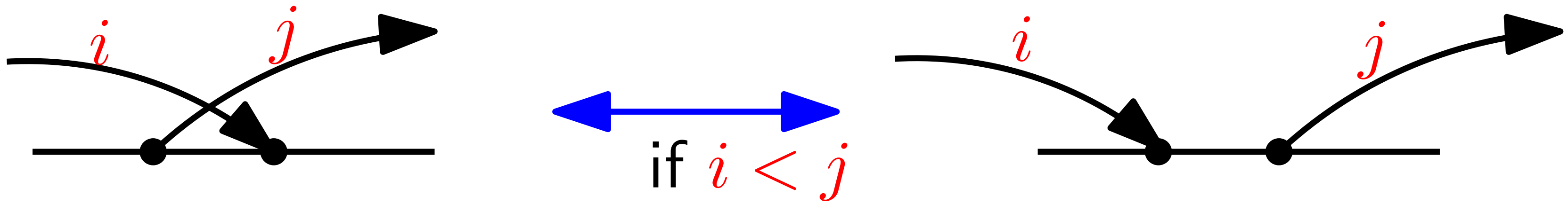
Catalan schemes = labeled non-nesting parenthesis systems



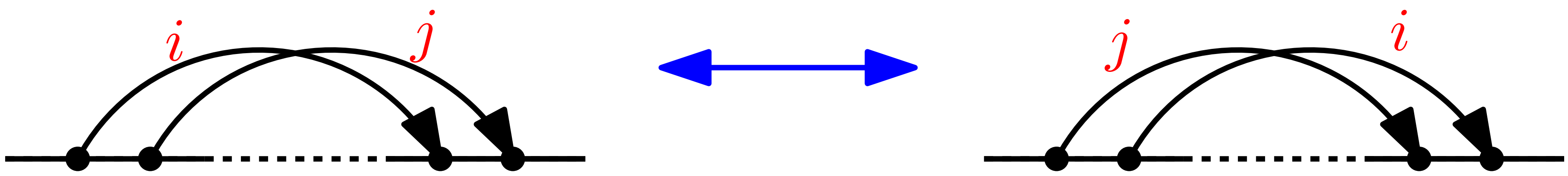
Shi/SO/Linial regions as equivalence classes of schemes

Definition:

- **Shi moves** ($S = \{0, 1\}$):



- **Semi-order moves** ($S = \{-1, 1\}$):

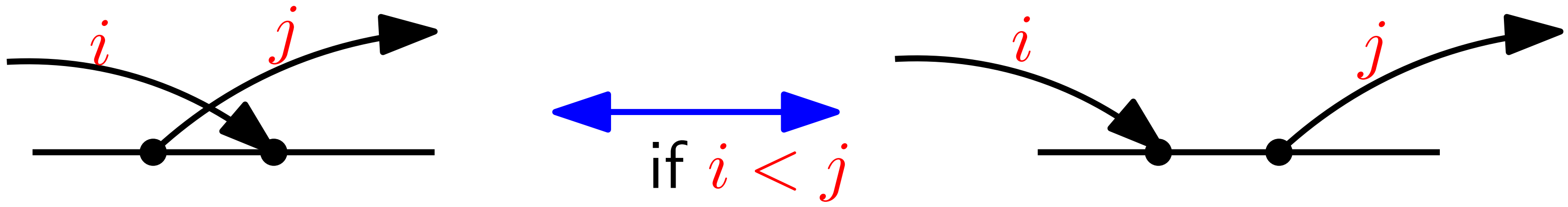


- **Linial moves** ($S = \{1\}$) $=$ Shi moves $+$ semi-order moves

Shi/SO/Linial regions as equivalence classes of schemes

Definition:

- **Shi moves** ($S = \{0, 1\}$):



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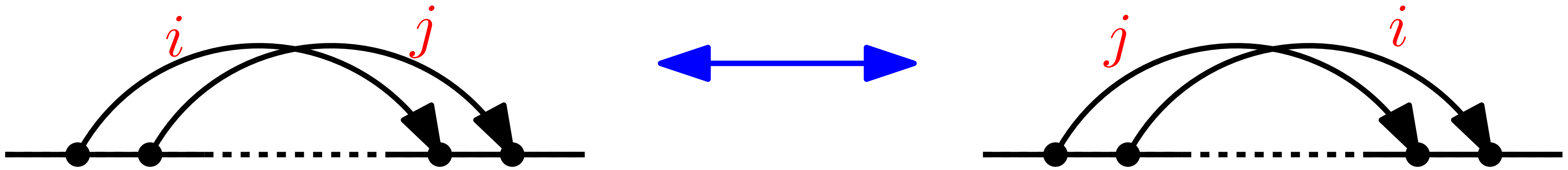
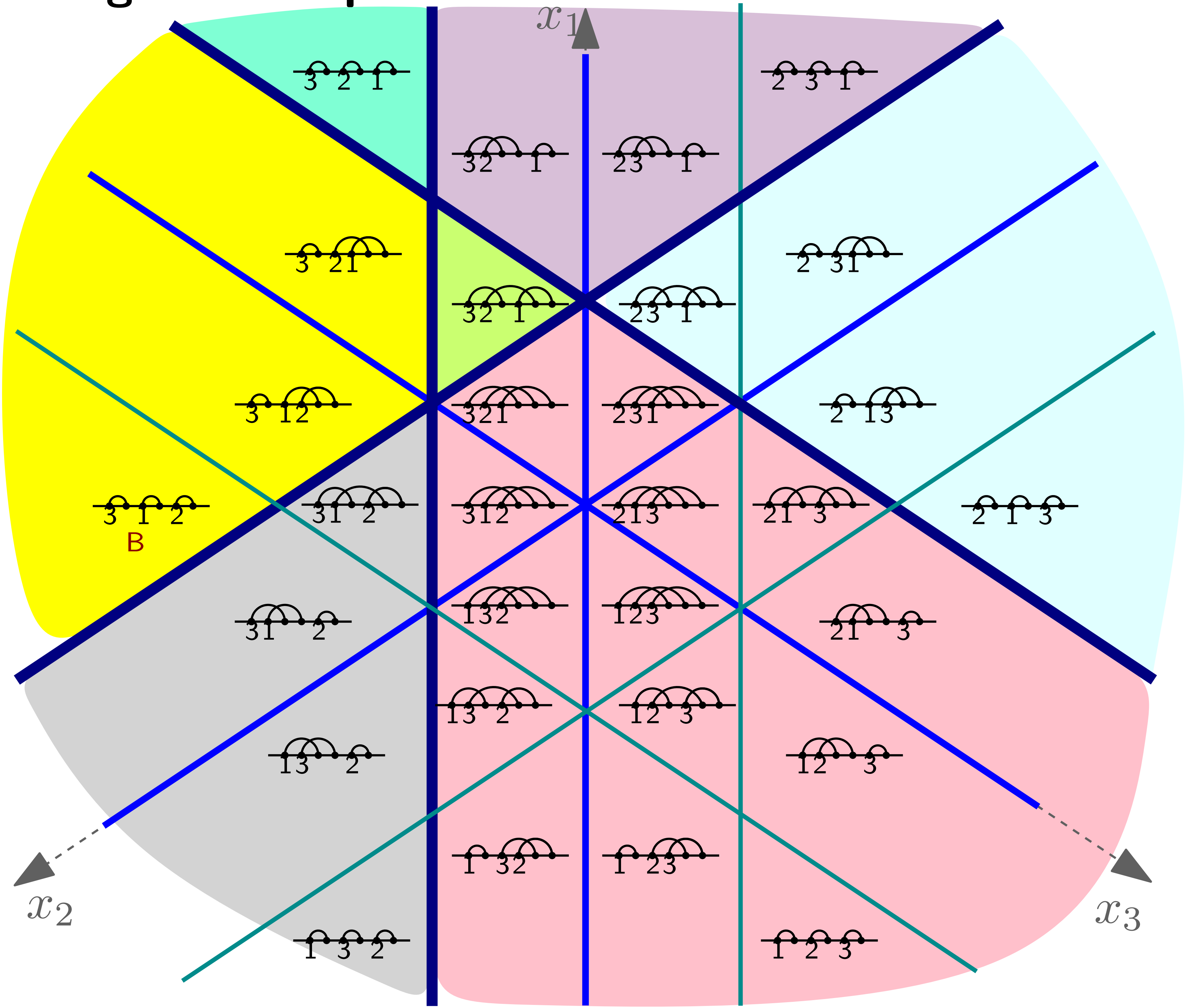


- **Linial moves** ($S = \{1\}$) = Shi moves + semi-order moves

Remark: Shi/SO/Linial regions are in bijection with equivalence classes of schemes under Shi/SO/Linial moves.

Linial regions as equivalence classes of schemes

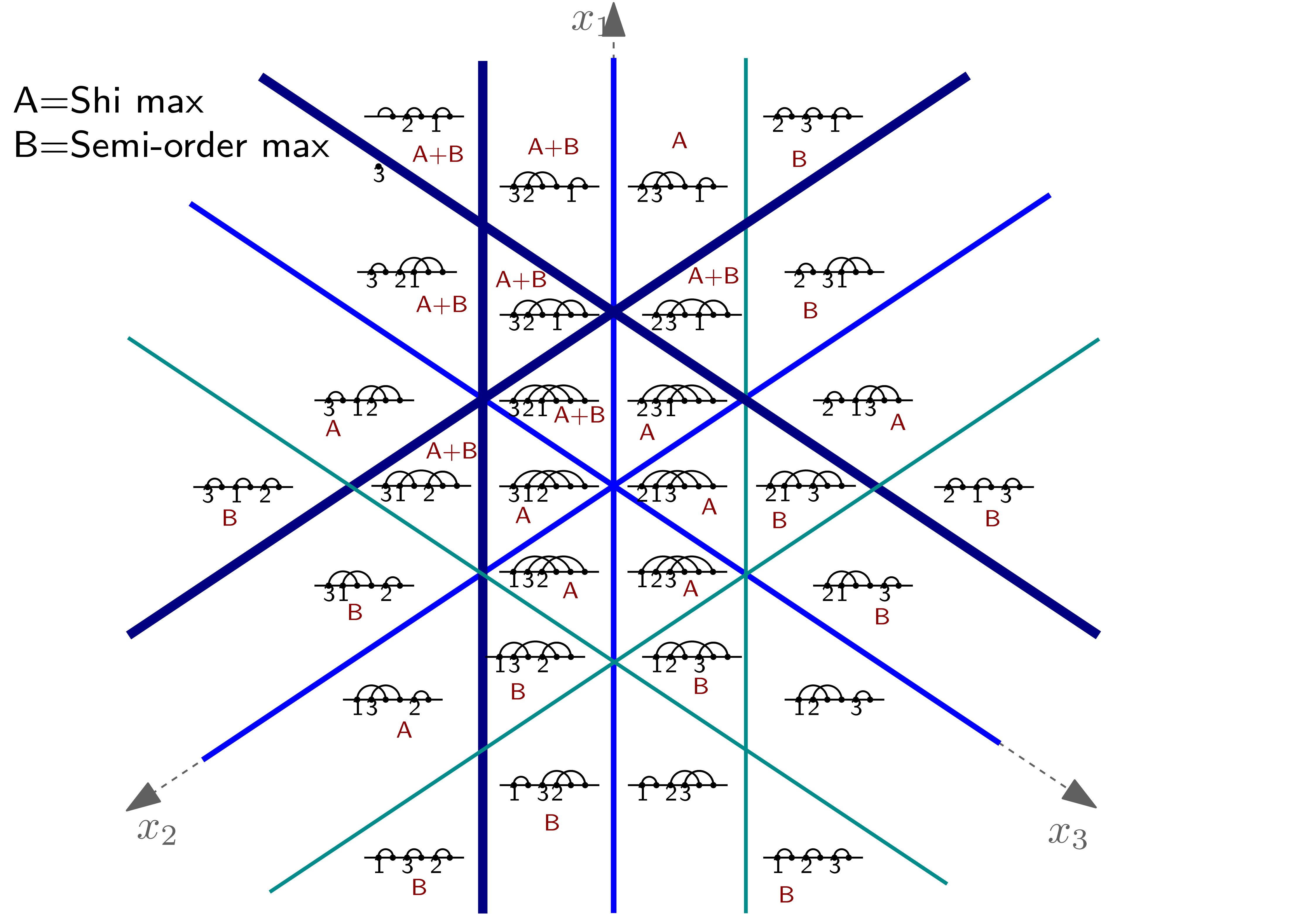


Shi/SO/Linial regions as equivalence classes of schemes

Total order on schemes: $C < C'$ if at first place they differ one has

- $\searrow$ in C and $\nearrow$ in C',
- or $\nearrow$ in both, but label in C < label in C'.

Remark: Shi/SO/Linial regions are in bijection with schemes which are **maximal** in their equivalence class.



A=Shi max
B=Semi-order max
x_1
x_2
x_3

Shi/SO/Linial regions as equivalence classes of schemes

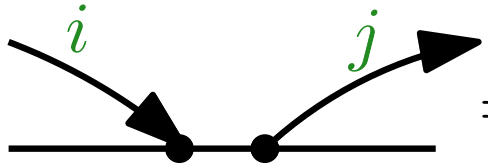
Lemma: Schemes are Shi/SO/Linial-**maximal** if and only if they are **locally maximal** (cannot increase by a single move).

Shi/SO/Linial regions as equivalence classes of schemes

Lemma: Schemes are Shi/SO/Linial-**maximal** if and only if they are **locally maximal** (cannot increase by a single move).

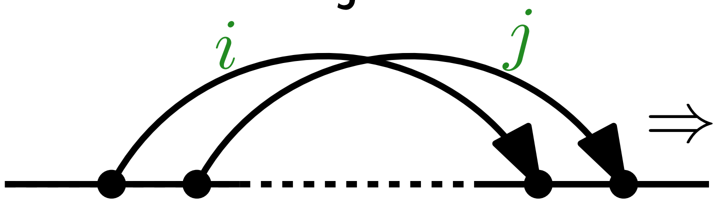
Corollary:

- **Shi regions** are in bijection with schemes such that



$\Rightarrow i > j$

- **SO regions** are in bijection with schemes such that



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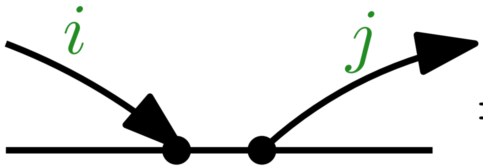

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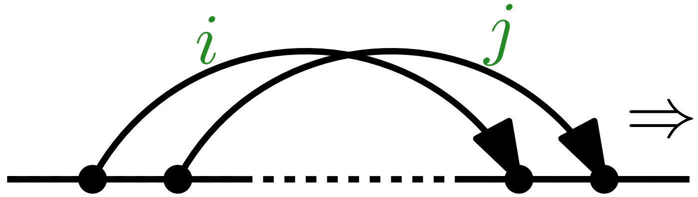

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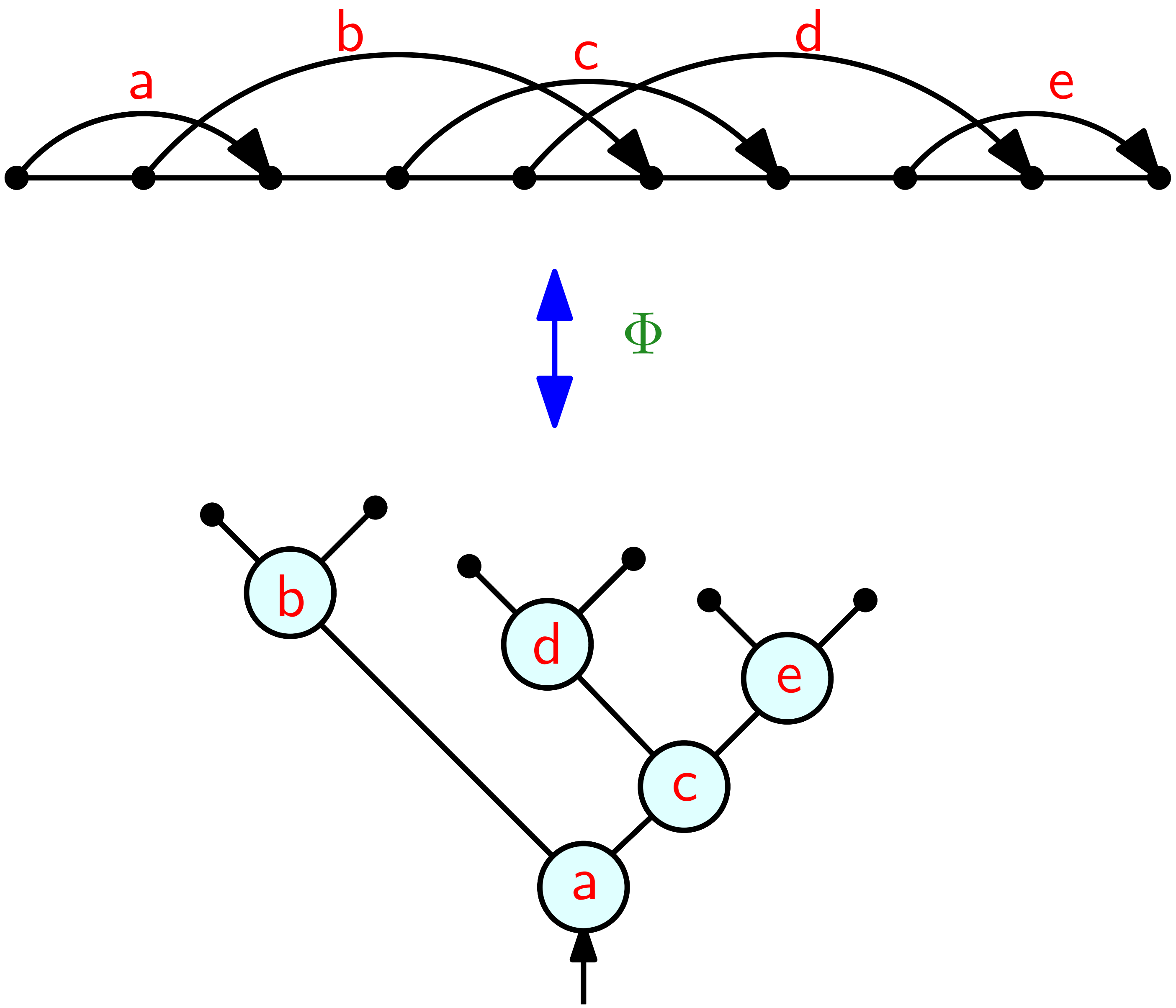
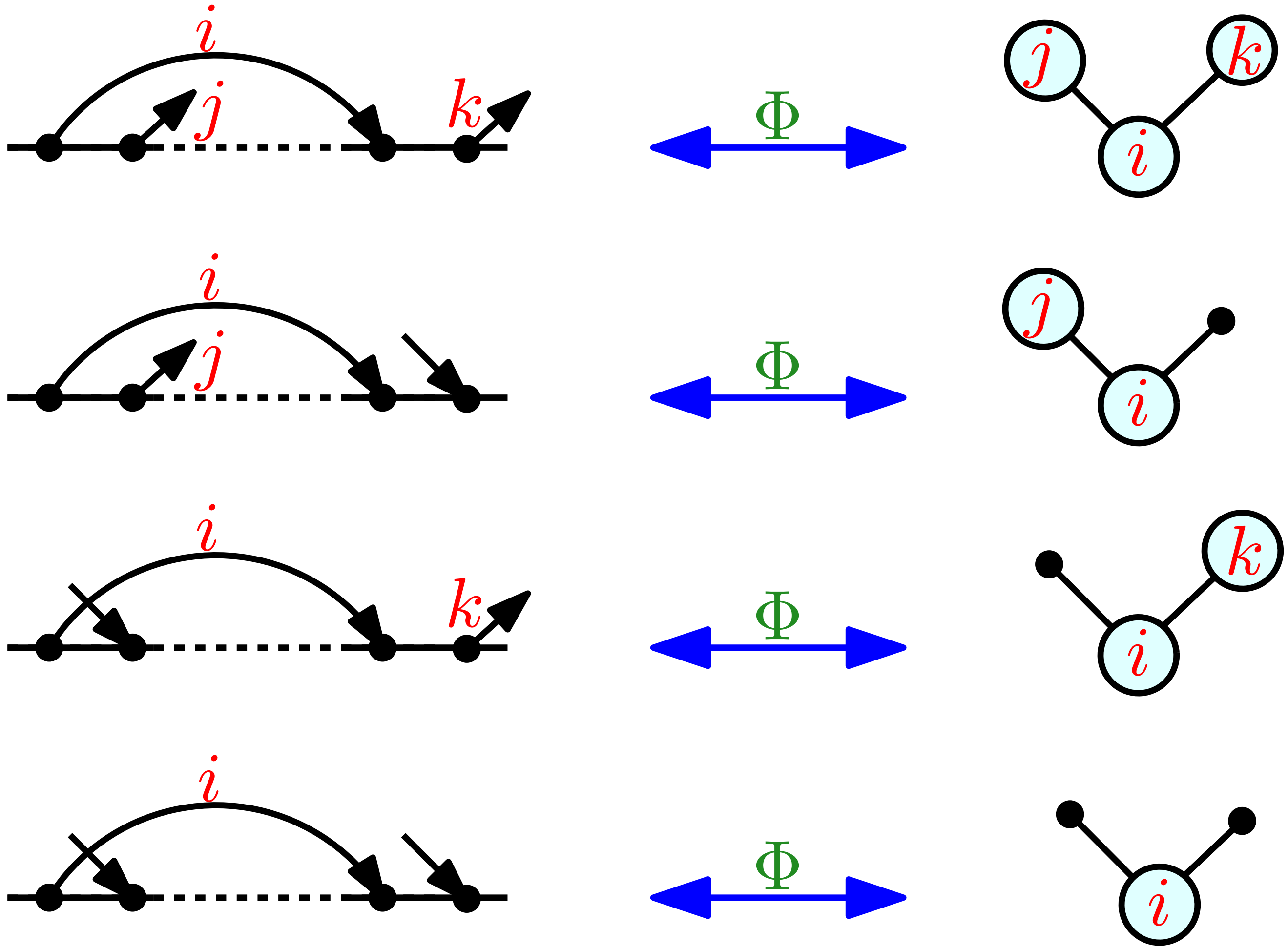
Bijection: Schemes ⟷ labeled binary trees



Bijection: Schemes $\longleftrightarrow$ labeled binary trees

Claim:



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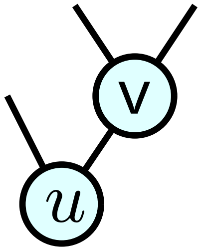

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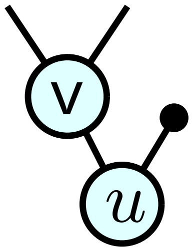

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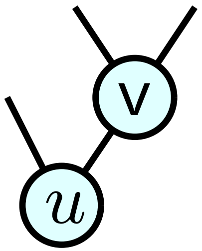

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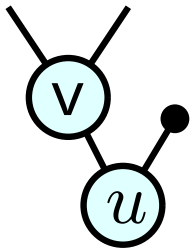

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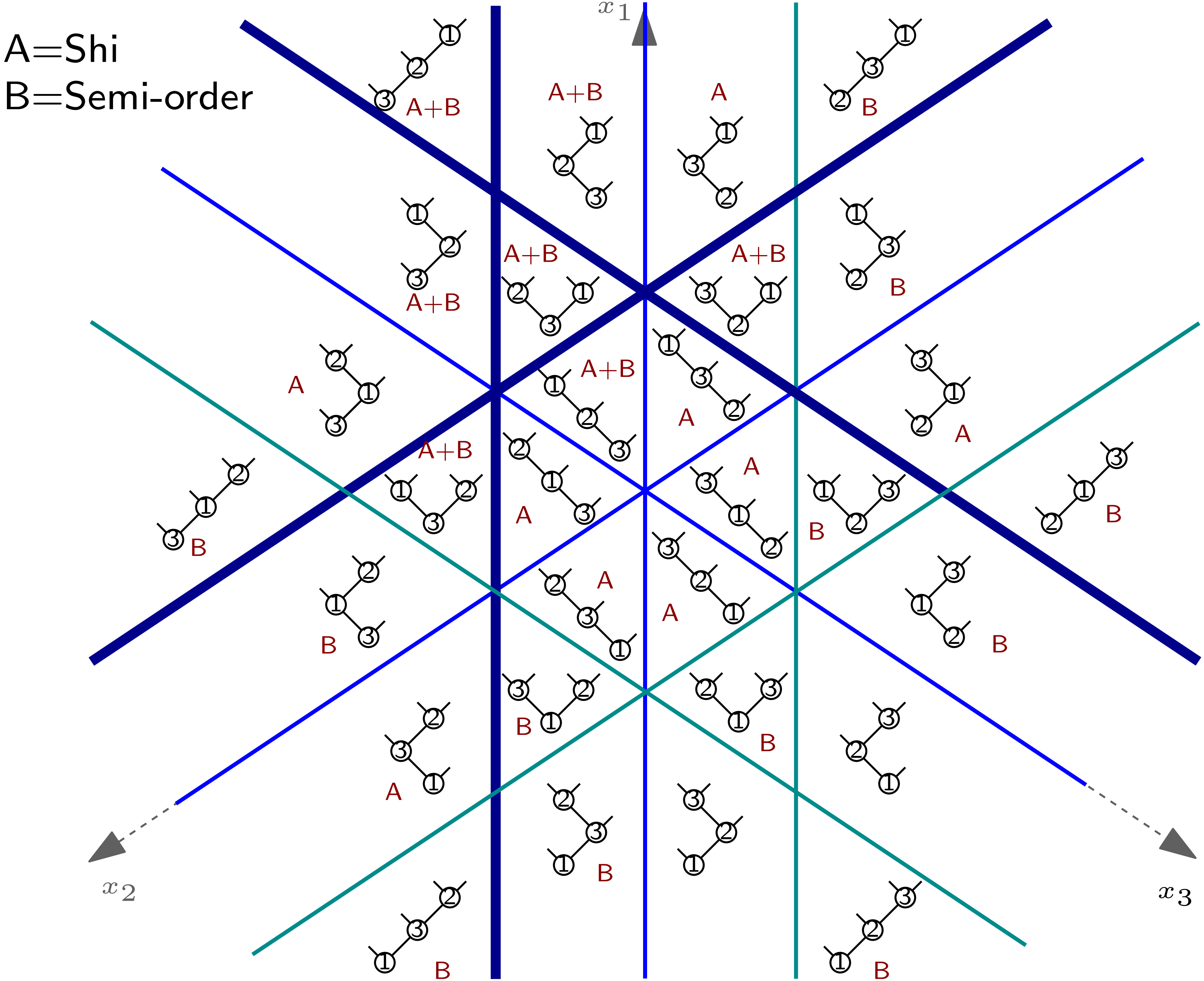
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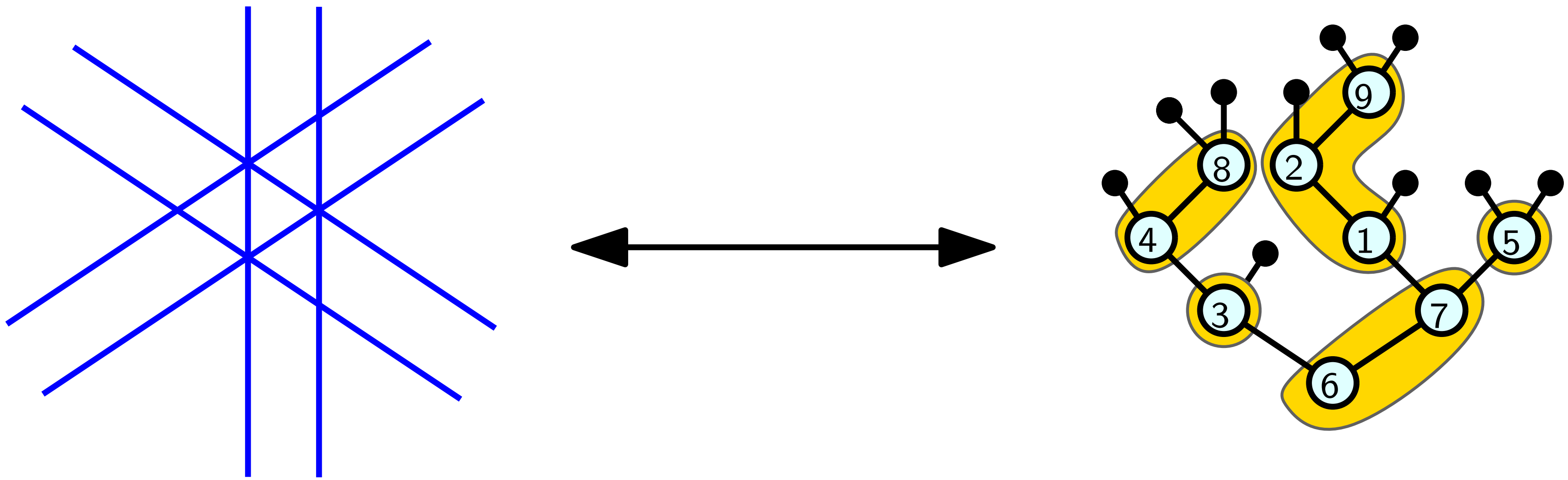
In general, we obtain **surjection** $\Phi_S : \mathcal{T}_S(n) \to$ regions of $\mathcal{A}_S(n)$

Problem: Not always true that locally-maximal schemes are maximal.

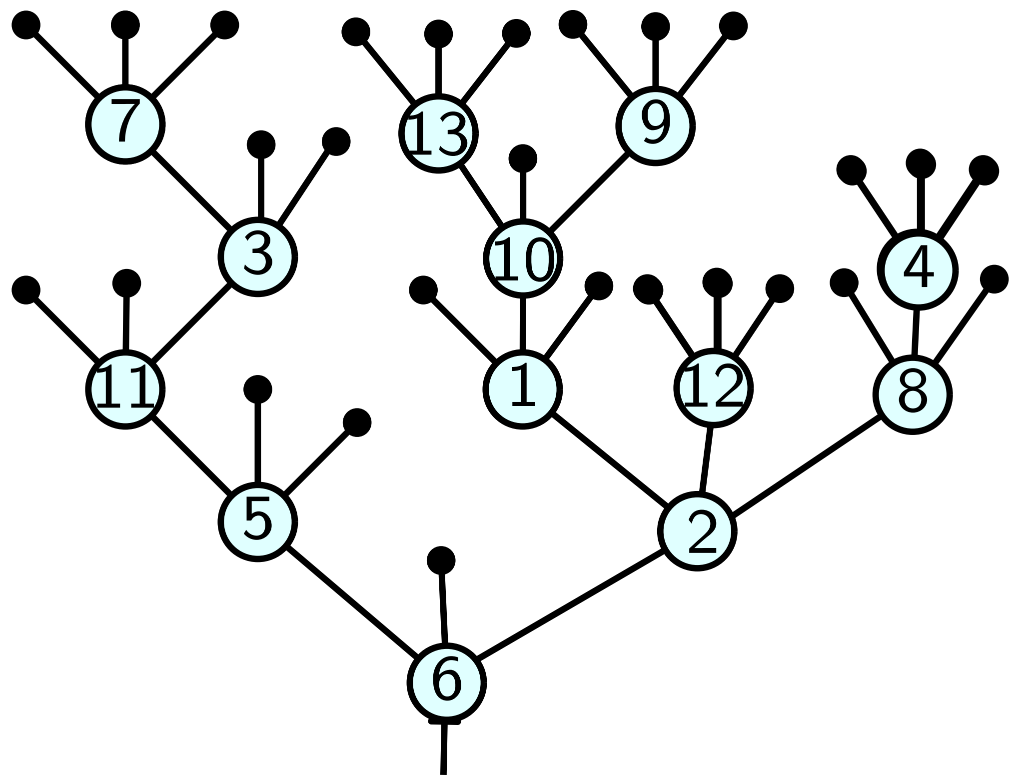
But it is true for **transitive sets** S. In this case Φ_S is **bijection**.
To prove it, it suffices to show that $|\mathcal{T}_S(n)| = \#$ regions of $\mathcal{A}_S(n)$.

Counting results



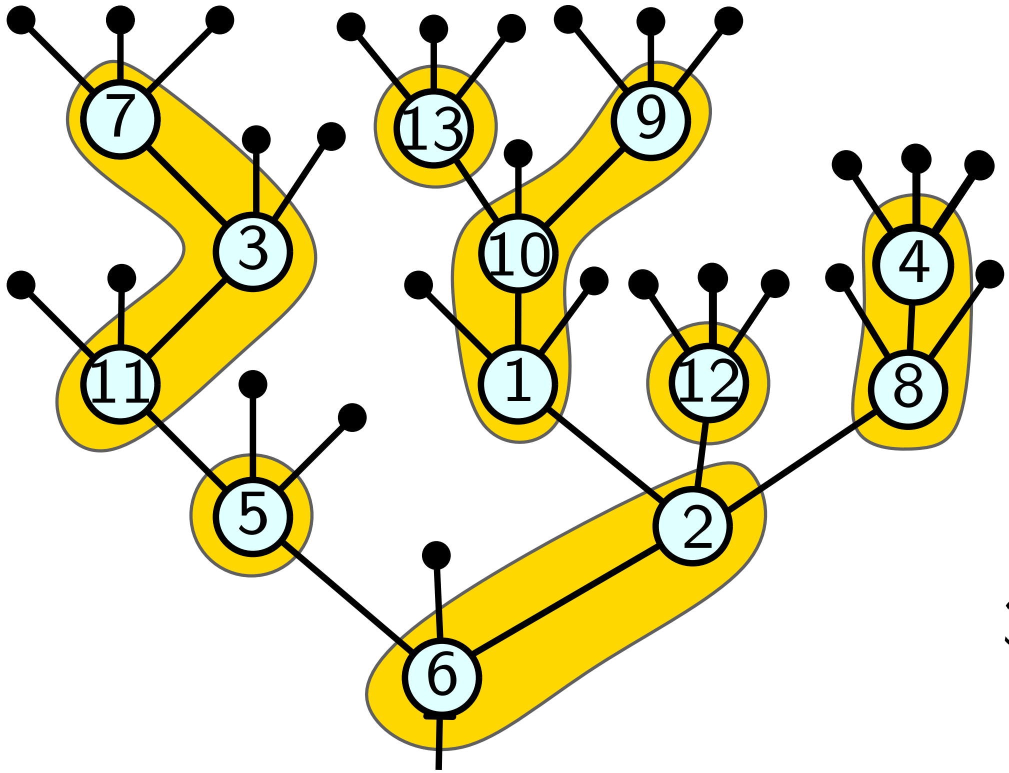
Boxed trees

- $\mathcal{T}^{(m)}$ = set of $(m+1)$-ary trees with labeled nodes.



Boxed trees

- $\mathcal{T}^{(m)}$ = set of $(m+1)$-ary trees with labeled nodes.
- A m-**boxed tree** is a tree in $\mathcal{T}^{(m)}$ decorated with boxes partitioning the nodes into cadet-sequences.



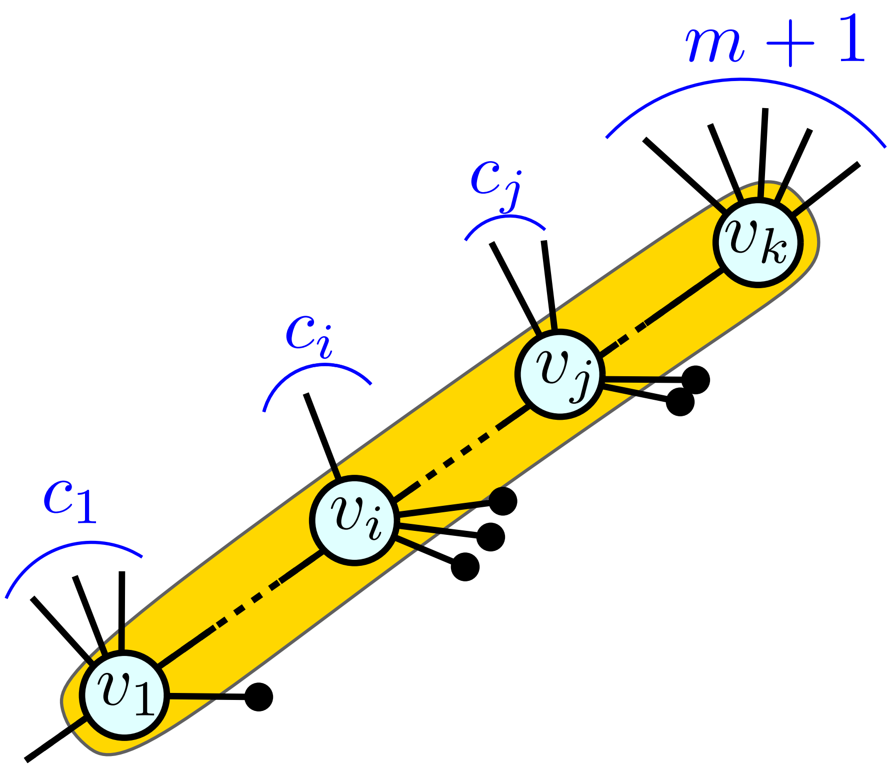
3-boxed tree

Main counting result

Let $S \subset \mathbb{Z}$. Let $m = \max(|s|,\ s \in S)$.

Def: S-**boxed** is m-boxed tree such that in each box

$$\forall i < j, \qquad (c_i + c_{i+1} + \cdots + c_{j-1}) \in S \cup \{0\} \quad \Rightarrow \quad v_i < v_j,$$
$$-(c_i + c_{i+1} + \cdots + c_{j-1}) \in S \quad \Rightarrow \quad v_i > v_j.$$



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Theorem: For any $S \subseteq [-m..m]$

$$\#\text{ regions of } A_S(n) = \sum_{T \in \mathcal{U}_{\mathrm{S}}(n)} (-1)^{n - \#\text{boxes}},$$

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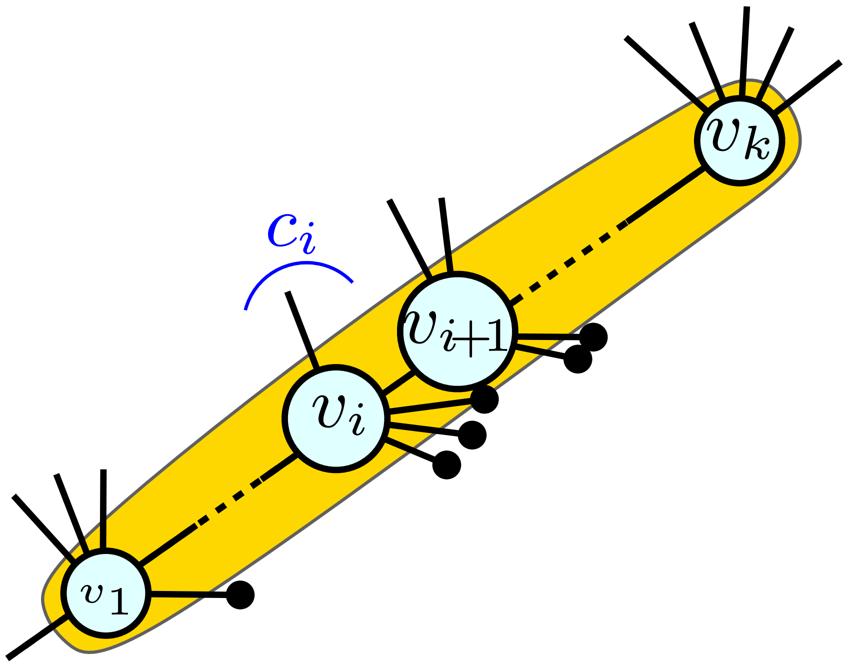
Corollary: If S is transitive, then $\#$ regions of $A_S(n) = |T_S(n)|$.
Thus, Φ_S is a bijection.

Proof of corollary.

Locality: For a transitive set S, a tree is S-boxed if

$$\forall i, \quad c_i \in S \cup \{0\} \Rightarrow v_i < v_{i+1}, \qquad -c_i \in S \Rightarrow v_i > v_{i+1}.$$



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Rk: For $v = \mathrm{cadet}(u)$,
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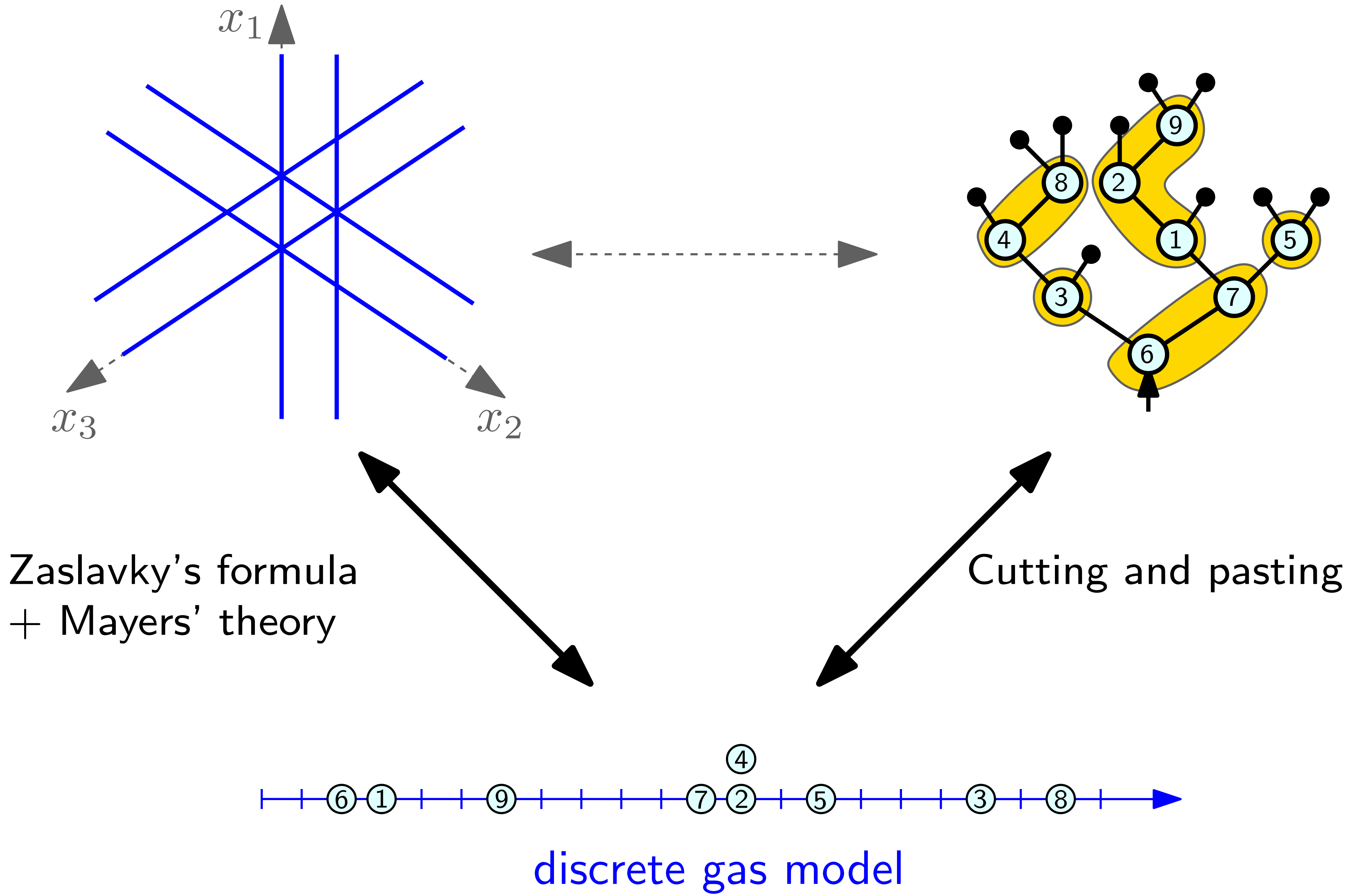
Killing involution:

$$\#\text{regions } \mathcal{A}_S(n) = \sum_{\substack{T \in \mathcal{U}_S \\ \text{satisfying condition } \mathcal{T}_S}} (-1)^{n-\#\text{boxes}} + \sum_{\substack{T \in \mathcal{U}_S \\ \text{not satisfying condition } \mathcal{T}_S}} (-1)^{n-\#\text{boxes}}$$

First sum: $|\mathcal{T}_S(n)|$ — Distinct box for each node.

Second sum: 0 — Merge/cut the box at $v = cadet(u)$ not satisfying condition of $\mathcal{T}_S$.

Proof of the counting result



Lemma 1: #regions $\mathcal{A}_S(n) = \sum_{G=([n],E)} (-1)^{|E|+c(G)-n} |W_S(G)|$,

where $c(G)$=#components, and

$W_S(G) = \{(x_1,\ldots,x_n) \mid$ $x_i - x_j \in S$, $\forall \{i,j\} \in E$ with $i<j$,
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Proof:

Zaslavsky formula: For any arrangement $\mathcal{A} \subset \mathbb{R}^n$,

$$\#\text{regions of } \mathcal{A} = \sum_{\mathcal{B} \subseteq \mathcal{A},\ \bigcap \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}| + \dim(\bigcap \mathcal{B}) - n}.$$

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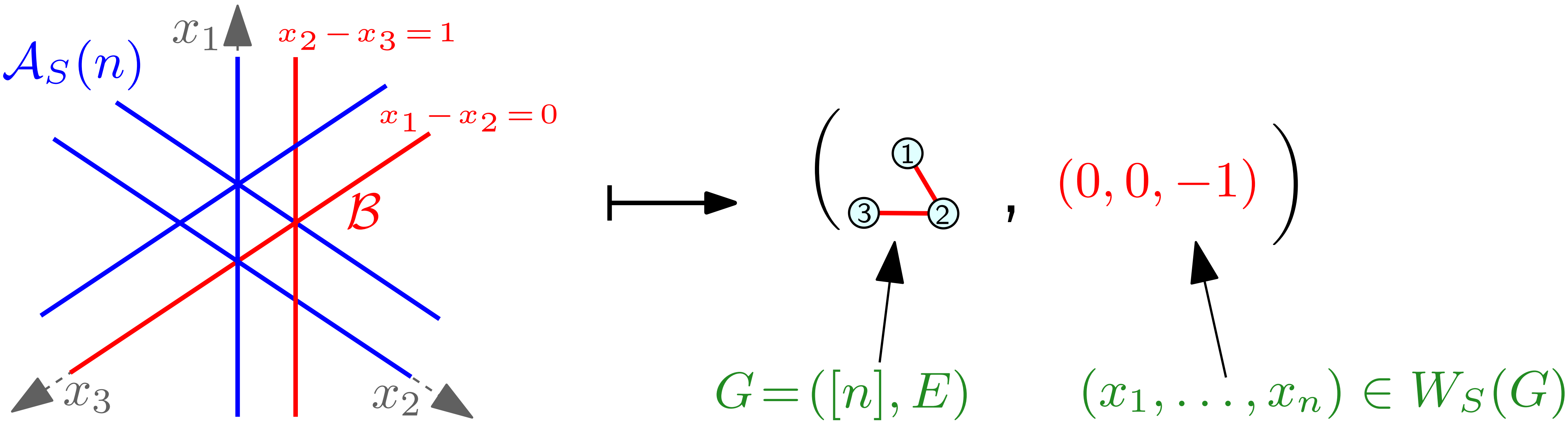

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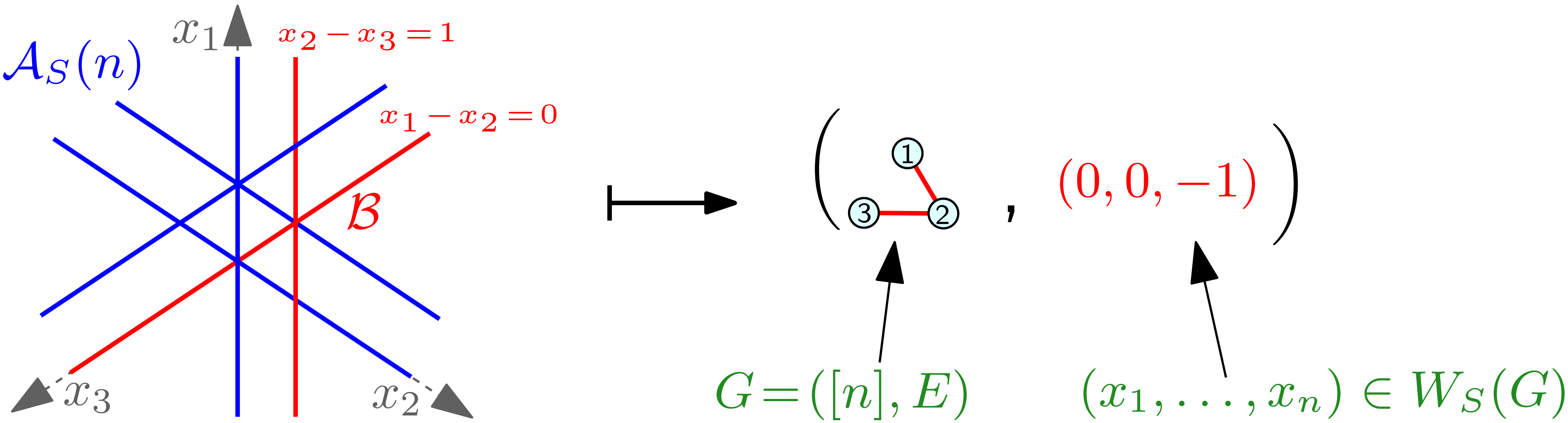

$$r_S(n) = \sum_{\mathcal{B} \subseteq \mathcal{A}_S(n),\ \bigcap \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+\dim(\bigcap \mathcal{B})-n} = \sum_{\substack{G=([n],E), \\ (x_1,..,x_n) \in \mathcal{W}_S(G)}} (-1)^{|E|+c(G)-n}.$$

□

Def: $\mathcal{Z}_{S,\delta}(n) = \{(x_1, \ldots, x_n) \in [\delta]^n \mid x_i - x_j \notin S, \ \forall i < j\}$.

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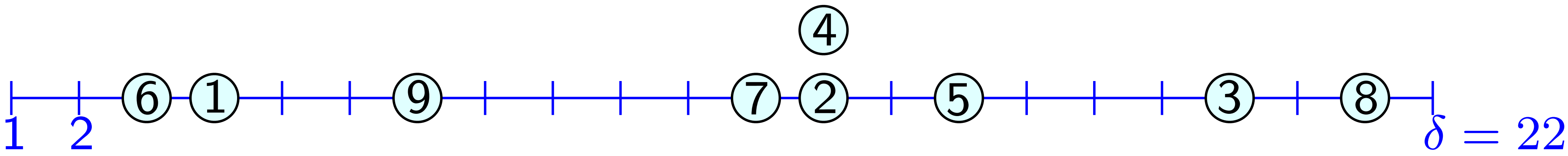
Example: $(4, 13, 19, 13, 15, 3, 12, 21, 7)$ is in $\mathcal{Z}_{\{-1,2\},22}(9)$.



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Lemma 2: $\log\left(R_S(t)\right) = \lim_{\delta\to\infty} -\frac{1}{\delta}\log(Z_{S,\delta}(-t))$,

where $R_S(t) = \sum_{n\geq 0} \#\text{regions}_S(n)\frac{t^n}{n!}$, and $Z_{S,\delta}(t) = \sum_{n\geq 0} |\mathcal{Z}_{S,\delta}(n)|\frac{t^n}{n!}$.

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Proof:

- $$|\mathcal{Z}_{S,d}(n)| = \sum_{x_1,\ldots,x_n \in [\delta]} \prod_{i<j} \mathbf{1}_{x_i - x_j \notin S}$$
$$= \sum_{x_1,\ldots,x_n \in [\delta]} \prod_{i<j} (1 - \mathbf{1}_{x_i - x_j \in S})$$
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- Exponential formula (log $\Rightarrow$ connected graphs), and limit $\delta \to \infty$. □

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where $U_S(t) = \sum_{S\text{-boxed tree}} (-1)^{\#\text{boxes}} \frac{t^v}{v!}$, and $U_S^\bullet(t) =$related series.

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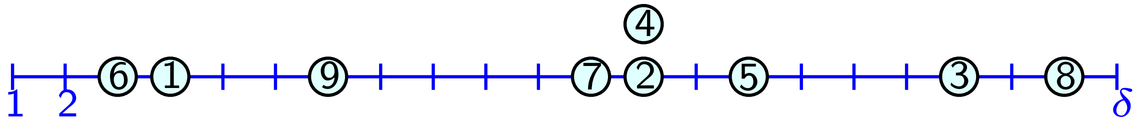

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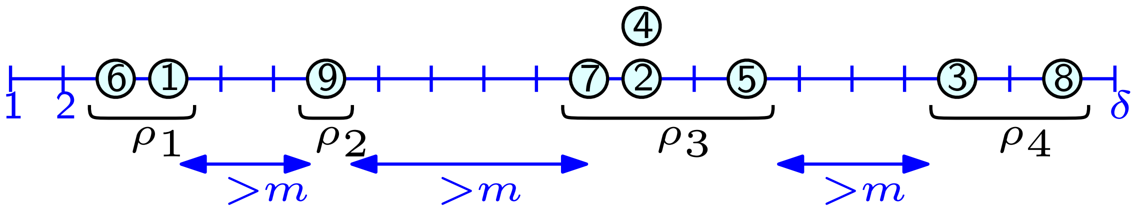

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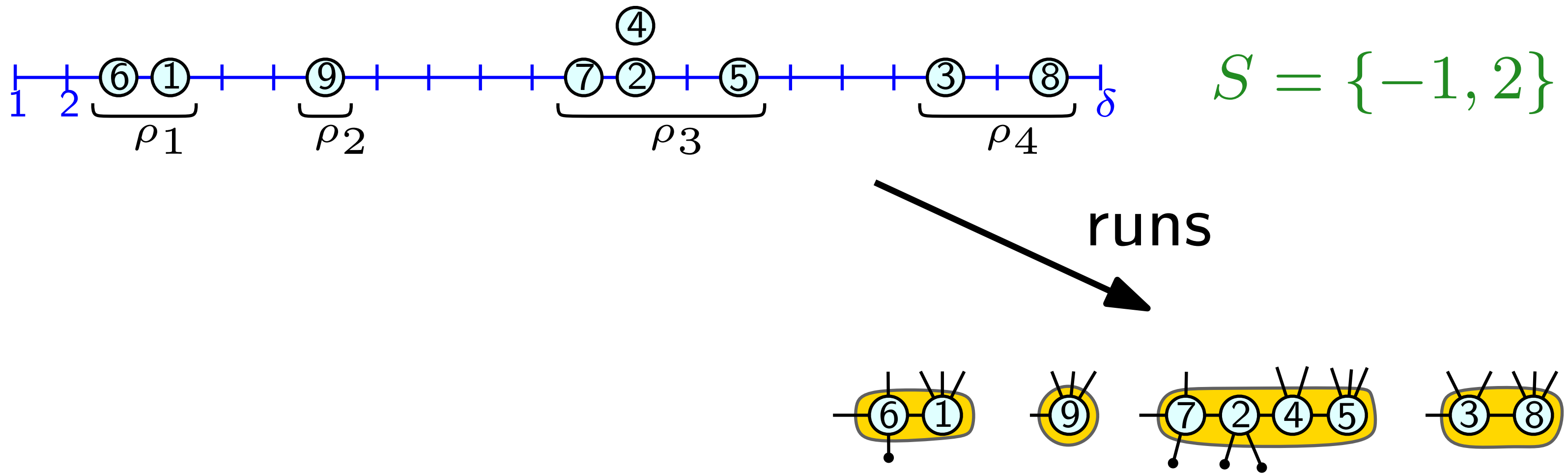

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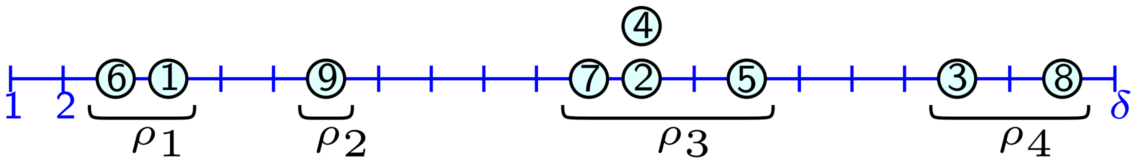

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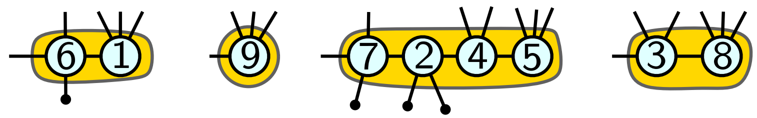
positions

$$\binom{\delta + m - \mathrm{width}(\rho 1) - \cdots - \mathrm{width}(\rho_r)}{r}$$

runs



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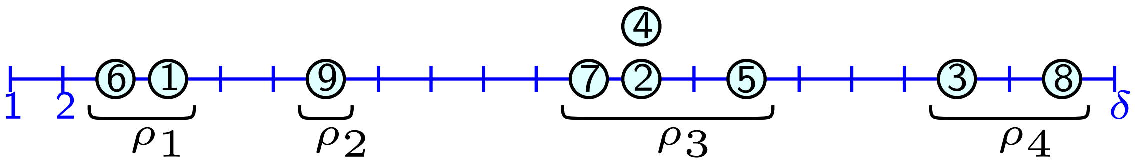

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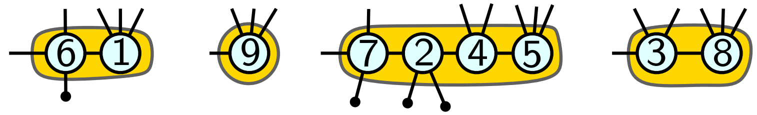
runs

$$\binom{\delta + m - \mathrm{width}(\rho 1) - \cdots - \mathrm{width}(\rho_r)}{r}$$

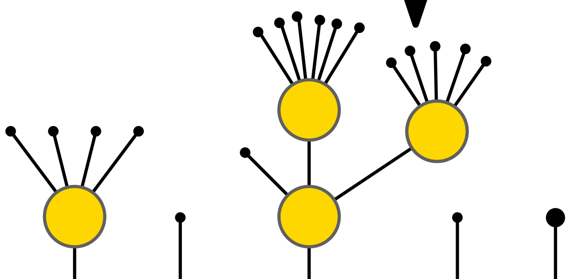


polynomial in δ

$\delta = -m-1-\gamma < 0$

$$(-1)^r \binom{\gamma + r + \mathrm{width}(\rho 1) + \cdots + \mathrm{width}(\rho_r)}{r}$$

$(-1)^r$

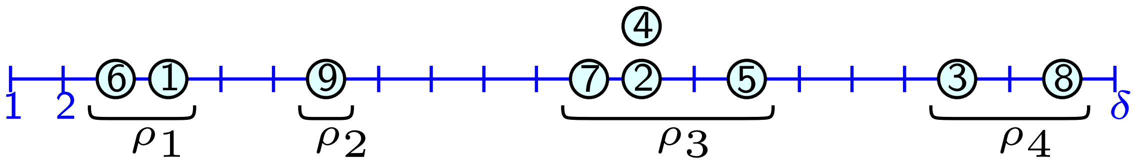


γ ordered trees, r nodes with $\mathrm{width}(\rho_1)+1, \ldots, \mathrm{width}(\rho_r)+1$ children

Def: $\mathcal{Z}_{S,\delta}(n) = \{(x_1, \dots, x_n) \in [\delta]^n \mid \forall i < j,\ x_i - x_j \notin S\}$.

Lemma 3: $Z_{S,\delta}(t) = U_S(t)^{-m-\delta-2}\, U_S^{\bullet}(t)$

Proof:



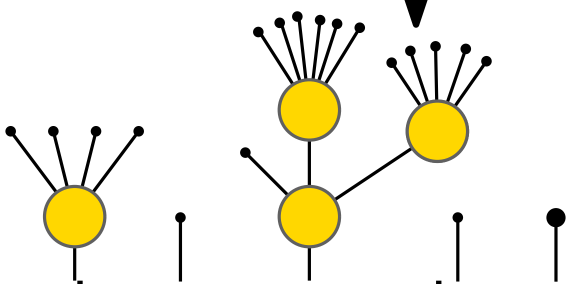
$S = \{-1, 2\}$

positions

$$\binom{\delta + m - \mathrm{width}(\rho 1) - \cdots - \mathrm{width}(\rho_r)}{r}$$

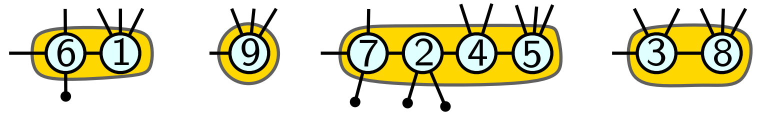
polynomial in δ $\quad \delta = -m-1-\gamma < 0$

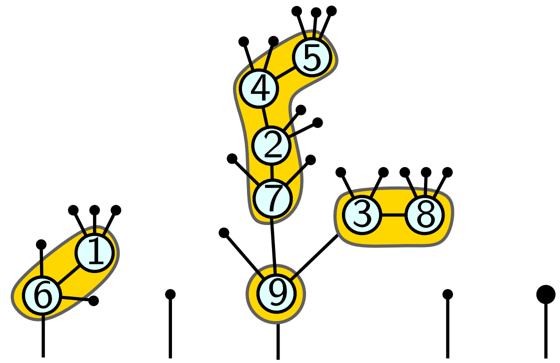
$$(-1)^r \binom{\gamma + r + \mathrm{width}(\rho 1) + \cdots + \mathrm{width}(\rho_r)}{r}$$

$(-1)^r$ 

γ ordered trees, r nodes with $\mathrm{width}(\rho_1) + 1, \dots, \mathrm{width}(\rho_r) + 1$ children

runs

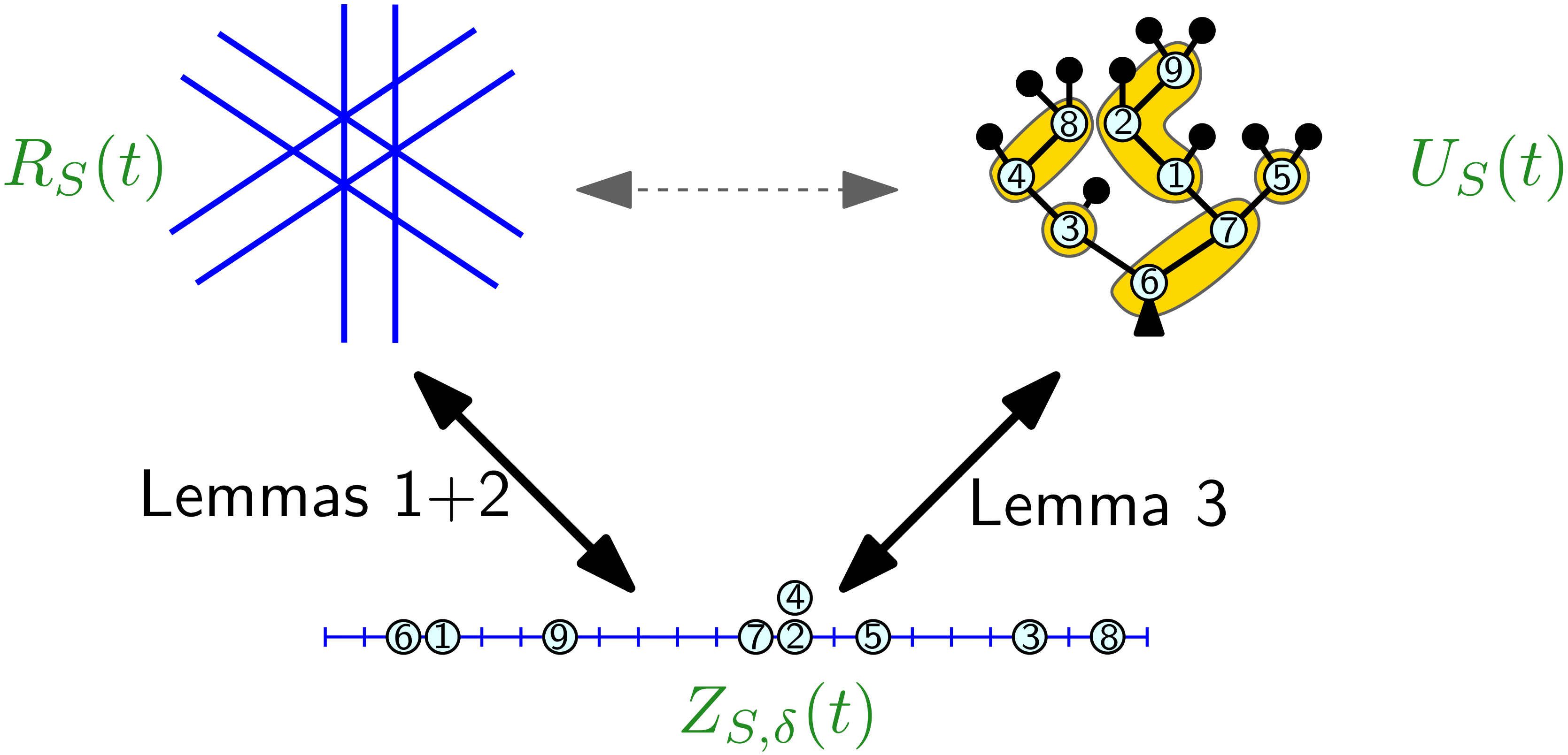




S-**boxed trees!**

□

Summary of the proof



$$
\begin{aligned}
\log\left(R_S(t)\right) &= \lim_{\delta\to\infty} -\frac{1}{\delta}\log(Z_{S,\delta}(-t)) \\
&= \lim_{\delta\to\infty} -\frac{1}{\delta}\log(U_S(-t)^{-\delta-m-2}U_S^{\bullet}(-t)) = \log\left(U_S(-t)\right)
\end{aligned}
$$

□

Extensions

Characteristic polynomial, **coboundary polynomial** of $\mathcal{A}_S(n)$:

$$\sum_{n=0}^{\infty} \chi_{\mathcal{A}_S(n)}(q) \frac{t^n}{n!} = R(0, -t)^{-q},$$

$$\sum_{n=0}^{\infty} P_{\mathcal{A}_S(n)}(q, y) \frac{t^n}{n!} = R(y, -t)^{-q},$$

where $R(y, t) = \sum_{T \ m\text{-boxed}} \frac{t^{|T|}}{|T|!} (-1)^{\#\text{boxes}} y^{\#S\text{-pairs}}$.

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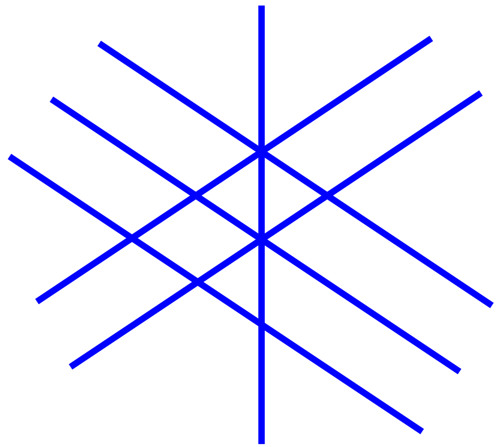
Bijection and counting results for more general arrangements:
$\mathcal{A}_{(S_{i,j})_{1\leq i<j\leq n}} \subset \mathbb{R}^n$ with hyperplanes $\{x_i - x_j \in S_{i,j}\}$.



Thanks.