Counting lattice paths by the number of crossings, major index, and descents

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A celebration of Ian Goulden and David Jackson

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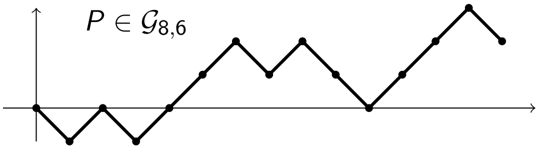
In this talk we will see that combining these statistics one gets surprisingly simple formulas.

I. Paths crossing a line

Lattice paths, descents and major index

Let $\mathcal{G}_{a,b}$ be the set of lattice paths in $\mathbb{Z}^2$ with a steps $U = (1, 1)$ and b steps $D = (1, -1)$, starting at the origin.



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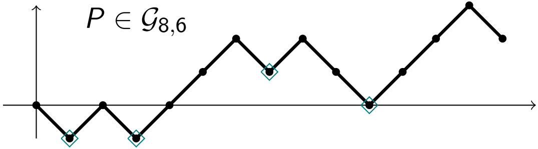
Definition

- A descent of P is a valley DU, let $\mathrm{des}(P) = \#$ descents.



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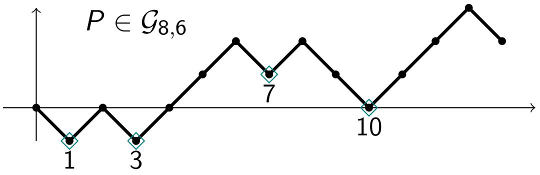
- The major index, $\text{maj}(P)$, is the sum of the x-coordinates of the descents.



$\text{des}(P) = 4 \qquad \text{maj}(P) = 1 + 3 + 7 + 10 = 21$

Lattice paths and major index

q-binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

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$$\sum_{P \in \mathcal{G}_{a,b}} q^{\mathrm{maj}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

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Example

$$\sum_{P\in\mathcal{G}_{3,2}} q^{\mathrm{maj}(P)} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q = 1+q+2q^2+2q^3+2q^4+q^5+q^6.$$

Refinement by the number of descents

Lemma (Fürlinger–Hofbauer '85)

$$\sum_{P\in\mathcal{G}_{a,b}} t^{\mathrm{des}(P)} q^{\mathrm{maj}(P)} = \sum_{n\geq 0} t^n q^{n^2} \begin{bmatrix} a \\ n \end{bmatrix}_q \begin{bmatrix} b \\ n \end{bmatrix}_q .$$

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Example

$$\sum_{P\in\mathcal{G}_{3,2}} t^{\mathrm{des}(P)} q^{\mathrm{maj}(P)} = 1+tq+2tq^2+2tq^3+(t+t^2)q^4+t^2q^5+t^2q^6.$$

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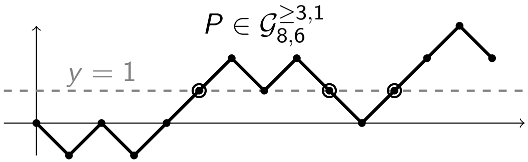
For $\ell \in \mathbb{Z}$ and $r \geq 0$, let $\mathcal{G}_{a,b}^{\geq r,\ell}$ be the set of paths in $\mathcal{G}_{a,b}$ that cross the line $y = \ell$ at least r times.



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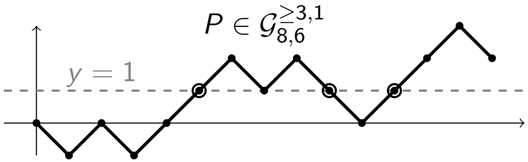

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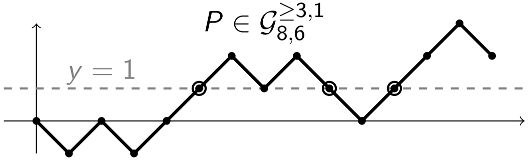

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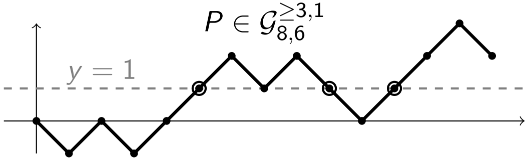

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For this talk, we focus on the specialization $G_{a,b}^{\geq r,\ell}(q) := G_{a,b}^{\geq r,\ell}(1,q)$.

Crossing the x-axis

First consider the case $\ell = 0$, which counts crossings of the x-axis.

Theorem

For any $a, b, r \geq 0$,

$$G_{a,b}^{\geq r,0}(q) = \begin{cases} q^{\binom{r+1}{2}} \begin{bmatrix} a+b \\ a+r \end{bmatrix}_q & \text{if } a > b, \\ (1+q^a)q^{\binom{r+1}{2}} \begin{bmatrix} 2a-1 \\ a+r \end{bmatrix}_q & \text{if } a = b, \\ q^{\binom{r}{2}} \begin{bmatrix} a+b \\ a-r \end{bmatrix}_q & \text{if } a < b. \end{cases}$$

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We give a bijective proof.

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- The specialization $t = q = 1$ (which ignores des and maj) is due to Engelberg '65 and Sen '65, and has later been rediscovered by other authors.

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- The theorem has applications to the enumeration of partitions λ with certain restrictions on the ranks $\lambda_i - \lambda'_i$, studied by Corteel–E.–Savage '22+.

Crossing an arbitrary horizontal line

Theorem

Let $a, b, m \geq 0$, *and let* $\ell \in \mathbb{Z} \setminus \{0\}$. *If* $0 < \ell < a - b$, *then*

$$G_{a,b}^{\geq 2m+1,\ell}(q) = G_{a,b}^{\geq 2m,\ell}(q) = q^{m(2m+1+\ell)} \begin{bmatrix} a+b \\ a+2m \end{bmatrix}_q .$$

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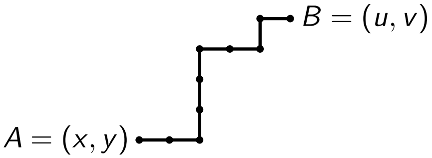
II. Pairs of paths crossing each other

Paths with north and east steps

For $A, B \in \mathbb{Z}^2$, let $\mathcal{P}_{A\to B}$ be the set of lattice paths from A to B with steps $N = (0, 1)$ and $E = (1, 0)$.



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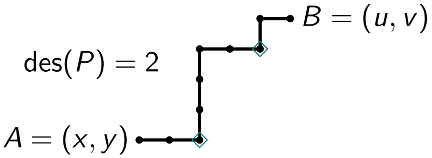

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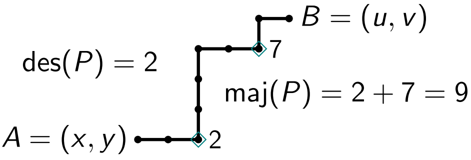

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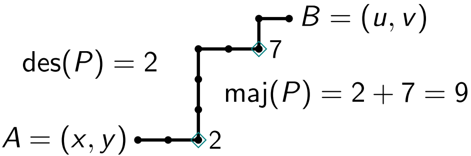


MacMahon's formula gives

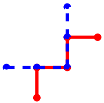
$$\sum_{P\in\mathcal{P}_{A\to B}} q^{\mathrm{maj}(P)} = \begin{bmatrix} u-x+v-y \\ u-x \end{bmatrix}_q .$$

Crossings of two paths

A crossing of two paths P and Q is a common vertex C such that:

- P and Q disagree in the step arriving at C;
- at the first step after C where P and Q disagree, each path has the same type of step (N or E) as it had when arriving at C.

crossings



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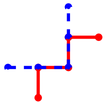
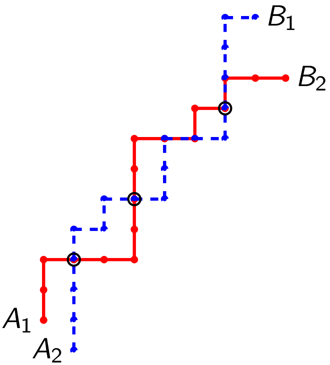


not a crossing

$$\mathcal{P}^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet} = \{(P,Q) : P \in \mathcal{P}_{A_1\to B_\circ},\, Q \in \mathcal{P}_{A_2\to B_\bullet},\ P \text{ and } Q \text{ have} \geq r \text{ crossings}\}.$$

Crossings of two paths

A pair in $\mathcal{P}^{\geq 3}_{A_1 \to B_2, A_2 \to B_1}$:



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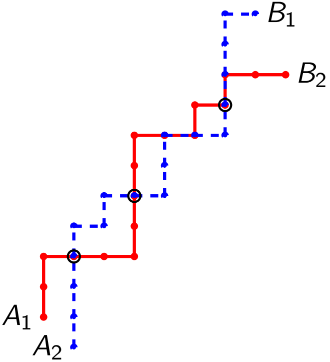

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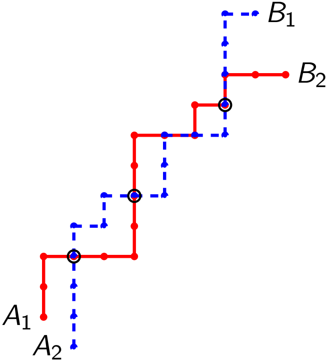

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For $r \geq 0$, define the polynomials

$$H^{\geq r}_{A_1 \to B_\circ, A_2 \to B_\bullet}(t, q) = \sum_{(P,Q) \in \mathcal{P}^{\geq r}_{A_1 \to B_\circ, A_2 \to B_\bullet}} t^{\operatorname{des}(P)+\operatorname{des}(Q)} q^{\operatorname{maj}(P)+\operatorname{maj}(Q)}$$

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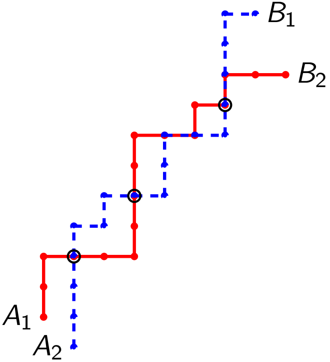

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and their specialization $H^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}(q) := H^{\geq r}_{A_1\to B_\circ, A_2\to B_\bullet}(1,q)$.

Easy cases and notation

Let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_1 = (u_1, v_1)$, $B_2 = (u_2, v_2)$.

For $r = 0$, we can choose the two paths independently, so

$$H^{\geq 0}_{A_1 \to B_\circ, A_2 \to B_\bullet}(q) = \begin{bmatrix} u_\circ - x_1 + v_\circ - y_1 \\ u_\circ - x_1 \end{bmatrix}_q \begin{bmatrix} u_\bullet - x_2 + v_\bullet - y_2 \\ u_\bullet - x_2 \end{bmatrix}_q .$$

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To give a general formula, first define

$$f_{r,A_1,A_2,B_2,B_1}(q) := q^{r(r+x_2-x_1)} \begin{bmatrix} u_2 - x_1 + v_2 - y_1 \\ u_2 - x_1 + r \end{bmatrix}_q \begin{bmatrix} u_1 - x_2 + v_1 - y_2 \\ u_1 - x_2 - r \end{bmatrix}_q .$$

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$$H^{\geq 0}_{A_1 \to B_\circ, A_2 \to B_\bullet}(q) = \begin{bmatrix} u_\circ - x_1 + v_\circ - y_1 \\ u_\circ - x_1 \end{bmatrix}_q \begin{bmatrix} u_\bullet - x_2 + v_\bullet - y_2 \\ u_\bullet - x_2 \end{bmatrix}_q .$$

To give a general formula, first define

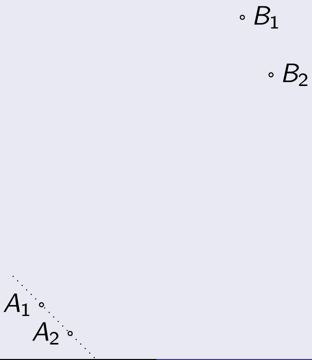
$$f_{r,A_1,A_2,B_2,B_1}(q) := q^{r(r+x_2-x_1)} \begin{bmatrix} u_2 - x_1 + v_2 - y_1 \\ u_2 - x_1 + r \end{bmatrix}_q \begin{bmatrix} u_1 - x_2 + v_1 - y_2 \\ u_1 - x_2 - r \end{bmatrix}_q .$$

Write $A_1 \prec A_2$ to mean that A_1 is strictly northwest of A_2.

Counting pairs of paths by crossings

Theorem

Let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $B_1 = (u_1, v_1)$, $B_2 = (u_2, v_2)$, *where* $A_1 \prec A_2$ *and* $B_1 \prec B_2$, *and* $x_1 + y_1 = x_2 + y_2$.



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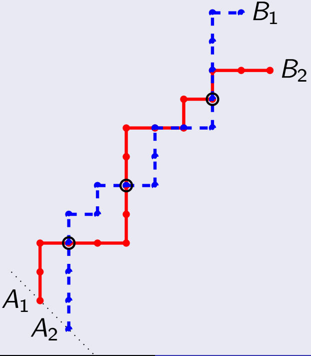
$$H^{\geq 2m+1}_{A_1 \to B_2, A_2 \to B_1}(q) = H^{\geq 2m}_{A_1 \to B_2, A_2 \to B_1}(q) = f_{2m, A_1, A_2, B_2, B_1}(q),$$



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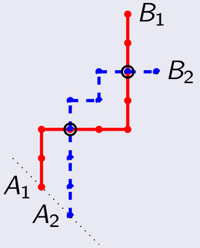

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Now let $A = (x, y)$ *and* $B = (u, v)$. *Then, for all* $r \geq 0$,

$$H^{\geq r}_{A \to B_1, A \to B_2}(q) = f_{r, A, A, B_2, B_1}(q),$$

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Now let $A=(x,y)$ *and* $B=(u,v)$. *Then, for all* $r\ge 0$,

$$H^{\ge r}_{A\to B_1, A\to B_2}(q)=f_{r,A,A,B_2,B_1}(q),$$

$$H^{\ge r}_{A_1\to B, A_2\to B}(q)=f_{r,A_1,A_2,B,B}(q),$$

$$H^{\ge r}_{A\to B, A\to B}(q)=\begin{cases} f_{0,A,A,B,B}(q) & \text{if } r=0,\\ 2\sum_{j\ge 1}(-1)^{j-1}f_{r+j,A,A,B,B}(q) & \text{if } r\ge 1.\end{cases}$$

Counting pairs of paths by crossings

With the specialization $t = q = 1$ (which ignores des and maj), the theorem still holds without the requirement $x_1 + y_1 = x_2 + y_2$.

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For the further refinement by des, our proof is inspired by Krattenthaler's '95 refinements of the LGV determinant by des and maj. It is still bijective but relies on certain two-rowed arrays.

III. Some bijections used in the proofs

The bijections $\bar{\tau}$ and $\bar{\sigma}$

Partition $\mathcal{P}_{A\to B} = \mathcal{P}^E_{A\to B} \cup \mathcal{P}^N_{A\to B}$ according to the last step of the path. Let $\mathbf{v} = (1, -1)$.

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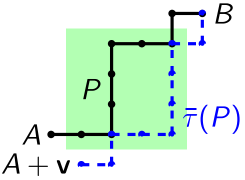
Define a bijection

$$\bar{\tau} : \mathcal{P}^E_{A\to B} \to \mathcal{P}^N_{A+\mathbf{v}\to B}$$

by placing the *NE* corners of $\bar{\tau}(P)$ at the coordinates of the *EN* corners of P:



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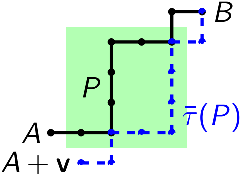


If $A = (x, y)$ and $B = (u, v)$, one can show that

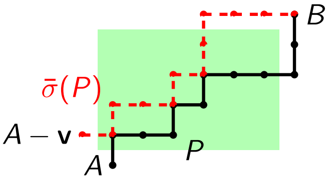
$$\text{maj}(\bar{\tau}(P)) = \text{maj}(P) + u - x - 1.$$

The bijections $\bar\tau$ and $\bar\sigma$

Its inverse is the bijection

$$\bar\sigma : \mathcal{P}^N_{A\to B} \to \mathcal{P}^E_{A-\mathbf{v}\to B}$$

obtained by placing the EN corners of $\bar\sigma(P)$ at the coordinates of the NE corners of P:



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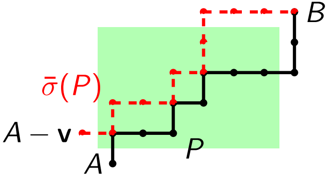
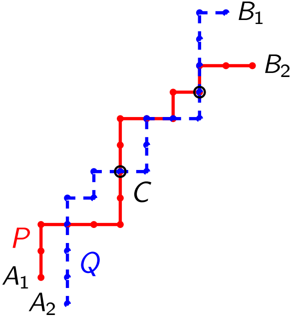


If $A = (x, y)$ and $B = (u, v)$, then

$$\text{maj}(\bar{\sigma}(P)) = \text{maj}(P) - u + x.$$

A bijection for pairs of paths

Given $(P, Q) \in \mathcal{P}^{\geq r}_{A_1 \to B_\circ, A_2 \to B_\bullet}$, let C be the rth crossing from the right. Suppose that P arrives to C with an N, and Q with an E.



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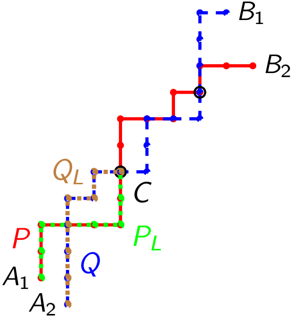
Splitting the paths at C, write $P = P_L P_R$ and $Q = Q_L Q_R$.



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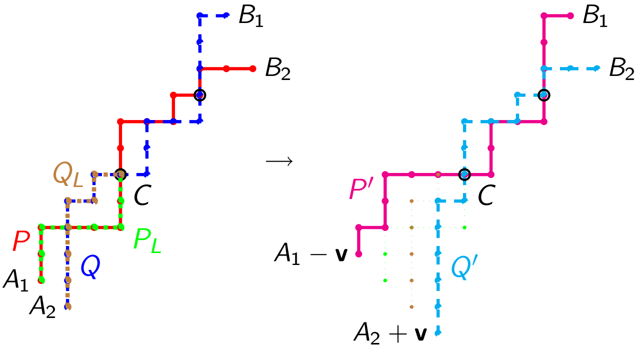
Splitting the paths at C, write $P = P_L P_R$ and $Q = Q_L Q_R$. Let

$$P' = \bar{\sigma}(P_L) Q_R \in \mathcal{P}_{A_1 - \mathbf{v} \to B_\bullet} \quad \text{and} \quad Q' = \bar{\tau}(Q_L) P_R \in \mathcal{P}_{A_2 + \mathbf{v} \to B_\circ}.$$



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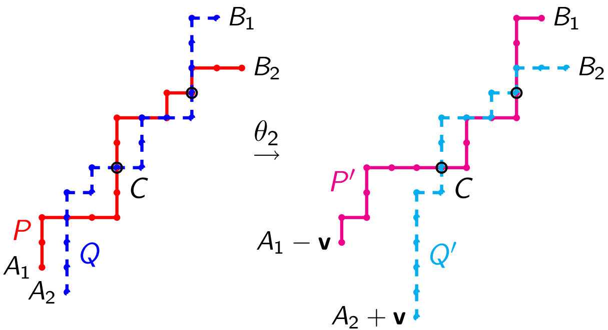

With the right setup, this map $(P, Q) \mapsto (P', Q')$ is a bijection, which we denote by θ_r.

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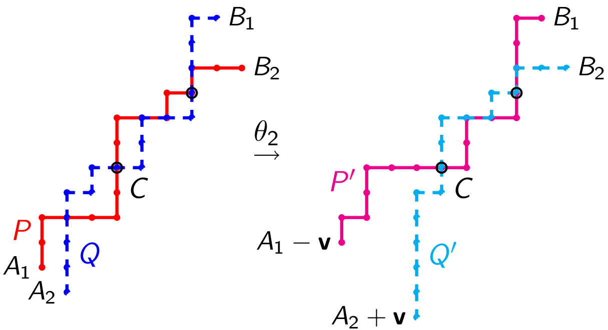

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If $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$, one can show that

$$\mathrm{maj}(P') + \mathrm{maj}(Q') = \mathrm{maj}(P) + \mathrm{maj}(Q) - (x_2 - x_1 + 1).$$

Composing bijections

To prove our theorem about pairs of paths, we use compositions such as $\theta_1 \circ \theta_2 \circ \cdots \circ \theta_r$, which decreases maj by $r(r + x_2 - x_1)$.

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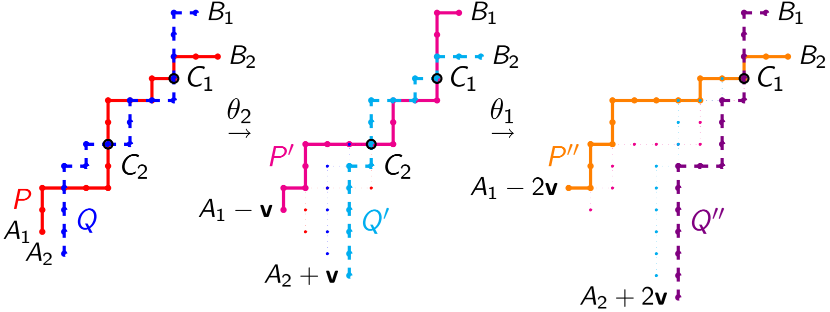

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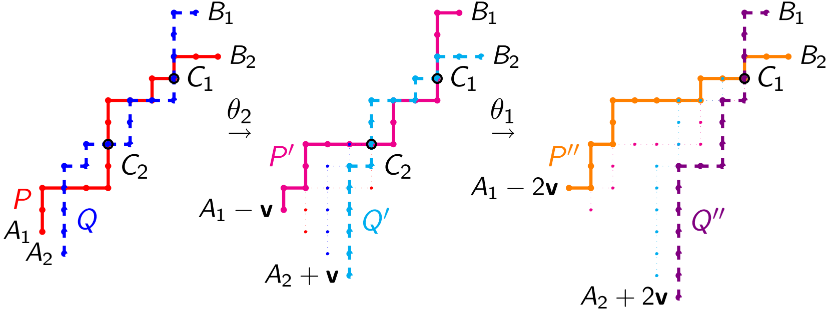

In this example, we have a bijection

$$\theta_1 \circ \theta_2 : \mathcal{P}^{\geq 2}_{A_1 \to B_2, A_2 \to B_1} \to \mathcal{P}^{\geq 0}_{A_1 - 2\mathbf{v} \to B_2, A_2 + 2\mathbf{v} \to B_1}.$$

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decreases maj by $2(2 + x_2 - x_1)$.

The pairs of paths in the image are easy to enumerate.

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The pairs of paths in the image are easy to enumerate. In this case, with the assumption $A_1 \prec A_2$ and $B_1 \prec B_2$, we obtain

$$H^{\geq 2}_{A_1 \to B_2, A_2 \to B_1}(q) = q^{2(2+x_2-x_1)} \begin{bmatrix} u_2 - x_1 + v_2 - y_1 \\ u_2 - x_1 + 2 \end{bmatrix}_q \begin{bmatrix} u_1 - x_2 + v_1 - y_2 \\ u_1 - x_2 - 2 \end{bmatrix}_q .$$

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As another application of these bijections, they can be used to give a simpler proof (directly in terms of paths) of Krattenthaler's refinement by maj of the Lindström–Gessel–Viennot determinantal formula for tuples of non-intersecting paths.

Bijections for paths crossing a horizontal line

For the problem of a single path crossing a horizontal line, we define similar bijections τ and σ. These apply to paths with U and D steps ending on the x-axis, and they fix the right endpoint.

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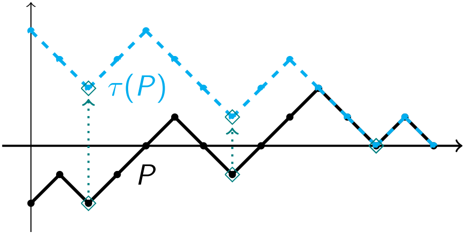
τ reflects the valleys along the x-axis:



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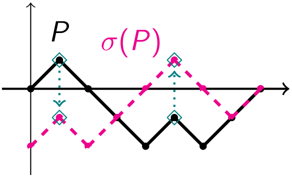

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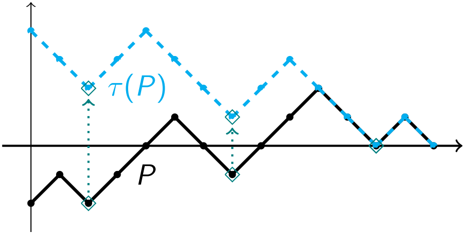

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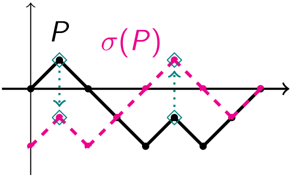
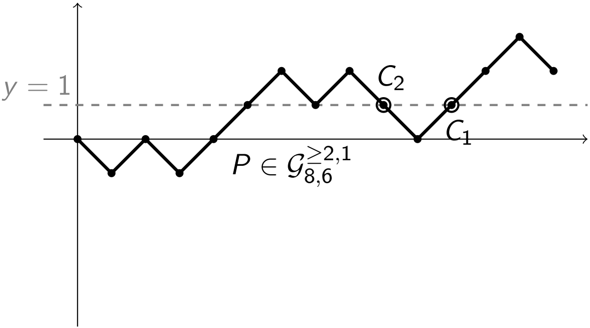


$\mathrm{maj}(\tau(P)) = \mathrm{maj}(P)$, $\mathrm{maj}(\sigma(P)) = \mathrm{maj}(P) + \#U - \#D - 1$

Composing bijections

To prove the theorems about paths crossing a line,



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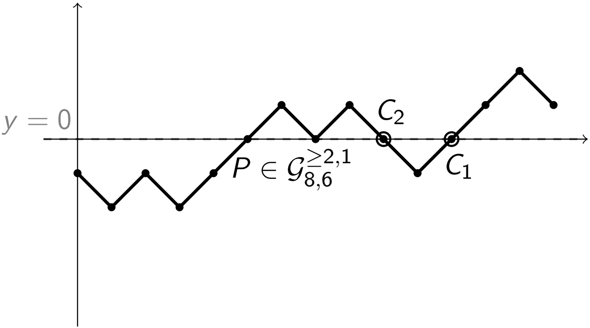

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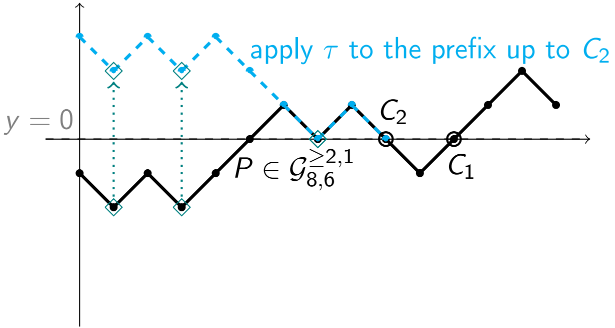

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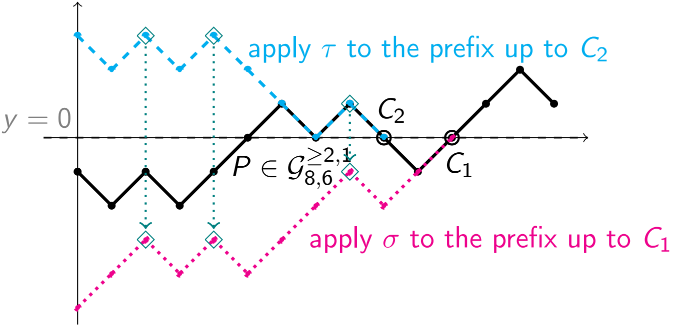

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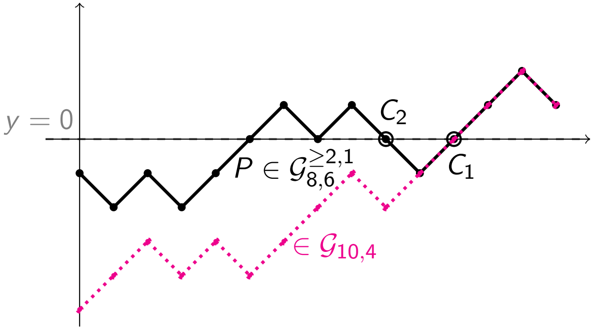

In this case, we get a bijection $\mathcal{G}_{a,b}^{\geq 2,\ell} \to \mathcal{G}_{a+2,b-2}$ that decreases maj by $\ell + 3$. The paths in the image are easy to count.

Refinement by the number of descents

Our theorems have refinements that also keep track of des (i.e., the number of DU or EN corners).

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Our theorems have refinements that also keep track of des (i.e., the number of *DU* or *EN* corners). Here are some sample formulas:

If $0 < \ell < a - b$, then

$$G_{a,b}^{\geq 2m,\ell}(t, q) = \sum_n t^n q^{n^2+m(m+\ell+1)} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} b \\ n+m \end{bmatrix}_q .$$

Refinement by the number of descents

Our theorems have refinements that also keep track of des (i.e., the number of *DU* or *EN* corners). Here are some sample formulas:

If $0 < \ell < a - b$, then

$$G_{a,b}^{\geq 2m,\ell}(t,q) = \sum_n t^n q^{n^2+m(m+\ell+1)} \begin{bmatrix} a \\ n-m \end{bmatrix}_q \begin{bmatrix} b \\ n+m \end{bmatrix}_q .$$

If $A_1 \prec A_2$, $B_1 \prec B_2$, and $x_1 + y_1 = x_2 + y_2$, then, for all $m \geq 0$,

$$H^{\geq 2m}_{A_1 \to B_2, A_2 \to B_1}(t,q)$$
$$= q^{2m(2m+x_2-x_1)} \cdot \left(\sum_n t^n q^{n(n+2m)} \begin{bmatrix} u_2 - x_1 \\ n \end{bmatrix}_q \begin{bmatrix} v_2 - y_1 \\ n+2m \end{bmatrix}_q \right)$$
$$\cdot \left(\sum_k t^n q^{n(n-2m)} \begin{bmatrix} u_1 - x_2 \\ n \end{bmatrix}_q \begin{bmatrix} v_1 - y_2 \\ n-2m \end{bmatrix}_q \right).$$

Refinement by the number of descents

Unfortunately, our bijections $\bar{\tau}$, $\bar{\sigma}$, σ do not behave well with respect to the number of descents.

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THANK YOU

References:

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