Generating trees for permutations avoiding generalized patterns

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Permutation Patterns 2006, Reykjavik

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- Definitions
  - Generalized patterns
  - Generating trees
  - Rightward generating trees
- Enumeration of permutations avoiding generalized patterns

Idea: Succession rule $\longrightarrow$ Functional equation $\longrightarrow$ Generating function

- Definitions
  - Generalized patterns
  - Generating trees
  - Rightward generating trees
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Idea: Succession rule $\longrightarrow$ Functional equation $\longrightarrow$ Generating function

- Generating trees with one label
  - $\{2\text{-}1\text{-}3, \bar{2}\text{-}31\}$-avoiding
  - $\{2\text{-}1\text{-}3, \overset{o}{\bar{2}}\text{-}31\}$-avoiding
  - $\{2\text{-}1\text{-}3, 2\text{-}3\text{-}41, 3\text{-}2\text{-}41\}$-avoiding
- Generating trees with two labels (Mireille Bousquet-Mélou)
  - $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding
  - $\{2\text{-}1\text{-}3, 32\text{-}1\}$-avoiding
  - $1\text{-}23$-avoiding
  - 123-avoiding
- Some unsolved cases

- Dashes can be inserted between entries in the pattern.
- Entries not separated by a dash have to be adjacent in an occurrence of the pattern in a permutation.

Examples:

$\pi = \underline{35}42\underline{7}1\underline{6}$ contains $\sigma = 12\text{-}4\text{-}3$

$\pi = 3542716$ avoids $12\text{-}43$ (it is $12\text{-}43$-avoiding)

Generating trees (usual kind)

- Nodes at each level are indexed by permutations of a given length.
- There is a rule that describes the children of each node.

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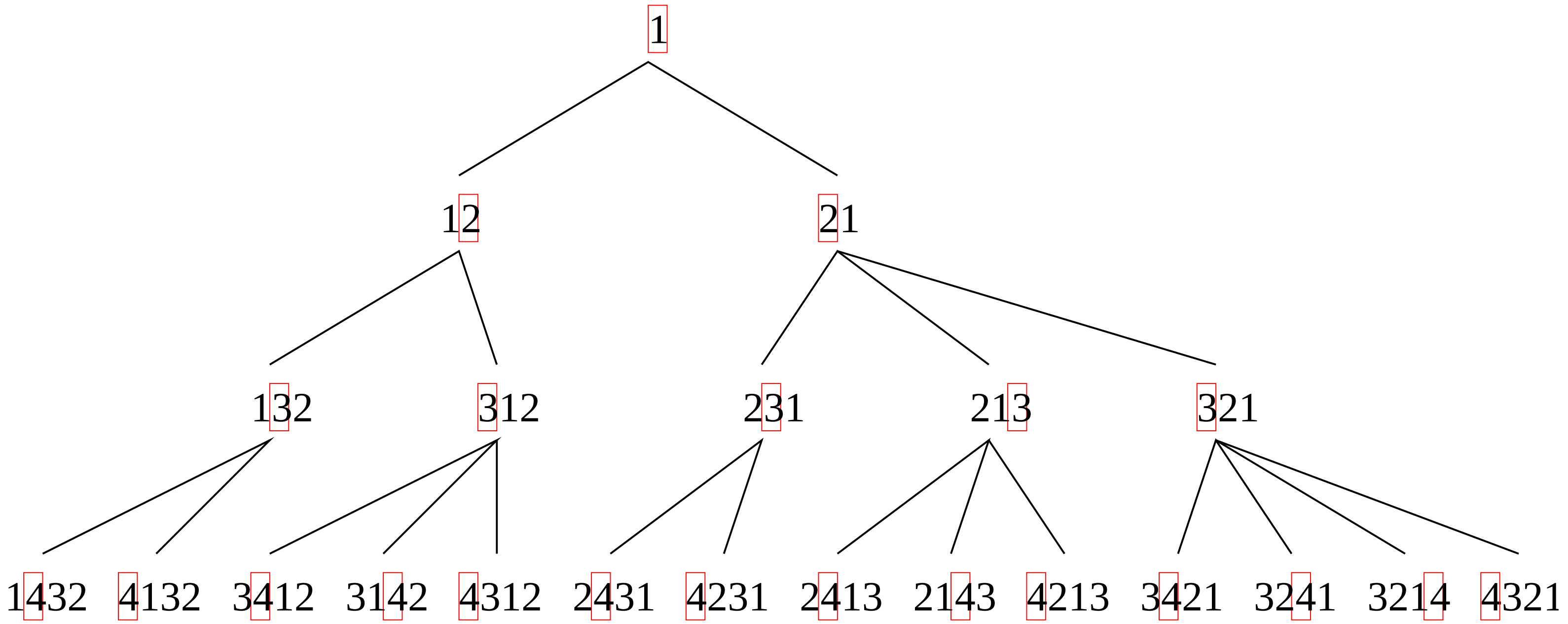
- There is a rule that describes the children of each node.

Usually, the children of a permutation are obtained by inserting the largest entry.

Example: Generating tree for 123-avoiding permutations:



Rightward generating trees (RGT)

To incorporate the adjacency condition in generalized patterns, it is more convenient to consider **rightward generating trees**.

To obtain a child of π:

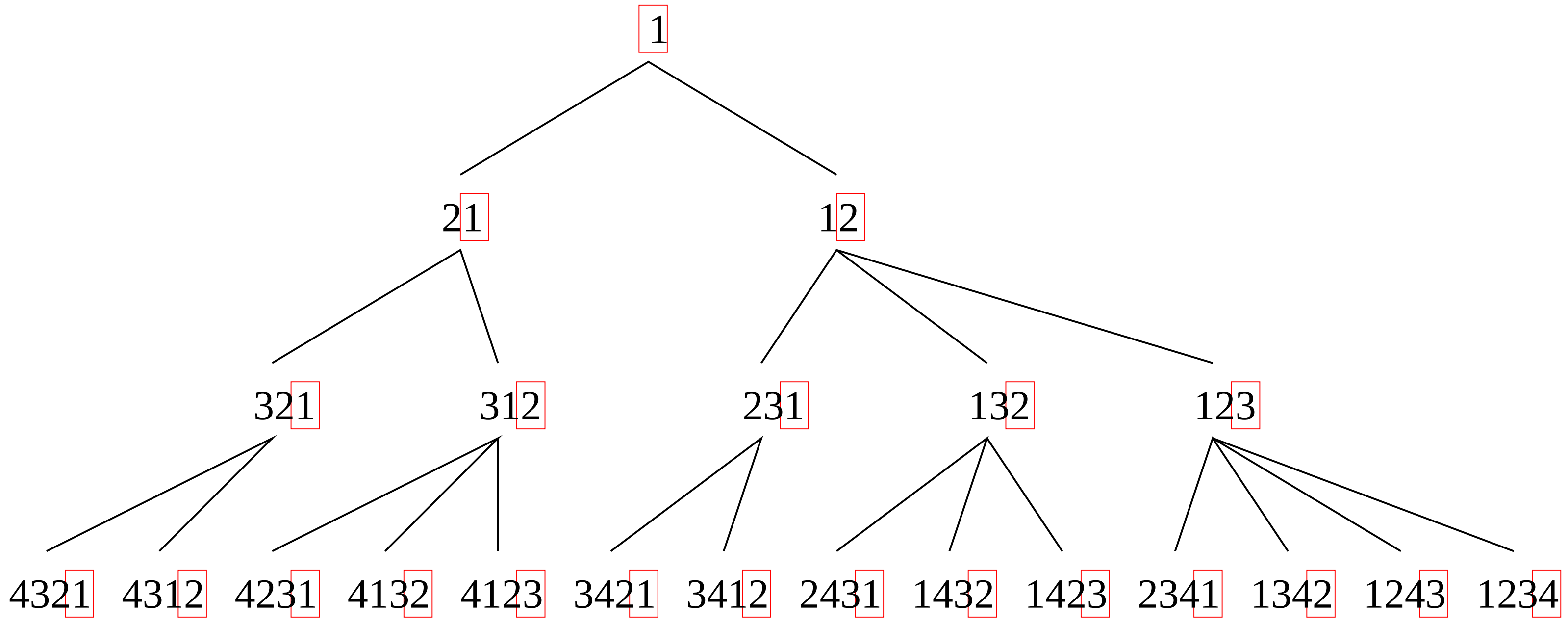
- append a new entry k to the right of π,
- shift up by one the entries of π that were $\geq k$.

Example:

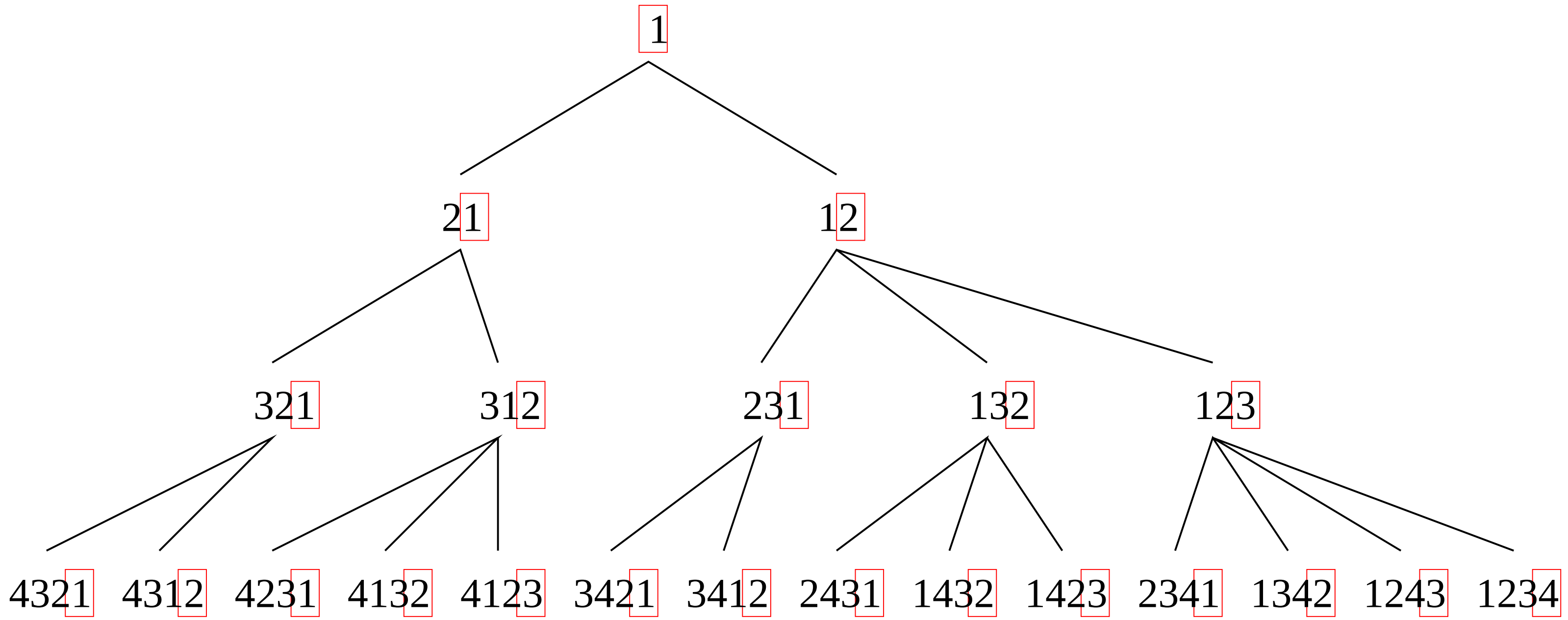
If we append 3 to the right of $\pi = 24135$, we obtain is the child 251463.

Example of RGT with one label

Generating tree for $2\text{-}13$-avoiding permutations:



Generating tree for $2\text{-}13$-avoiding permutations:



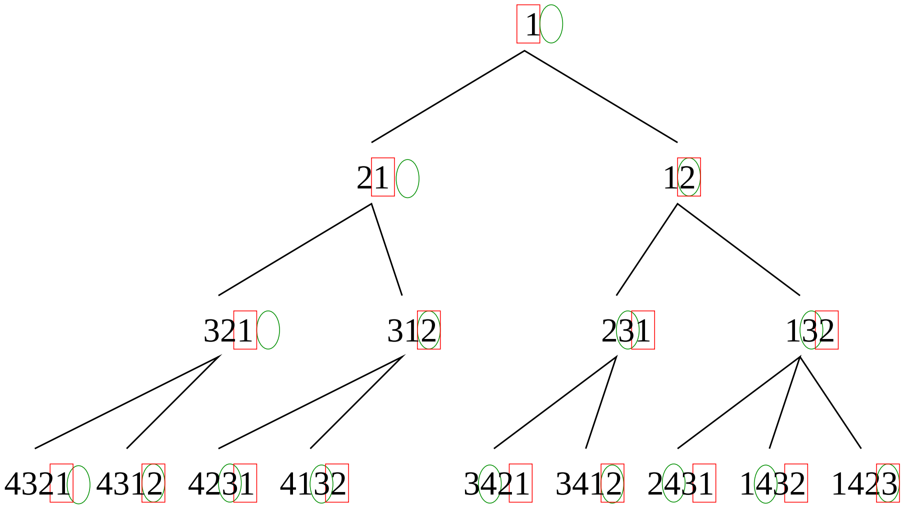
If $\pi \in \mathcal{S}_n$, let $r(\pi) = \pi_n$ be its rightmost entry.

This tree is described by the succession rule

$$(1)$$
$$(r) \longrightarrow (1)\ (2)\ \cdots\ (r)\ (r+1).$$

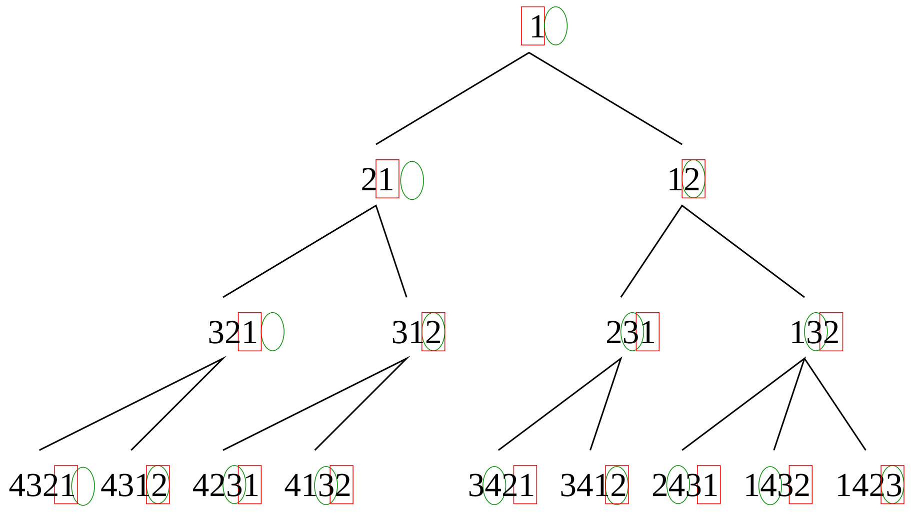
Example of RGT with two labels

Generating tree for $\{2\text{-}13, 12\text{-}3\}$-avoiding permutations:



Example of RGT with two labels

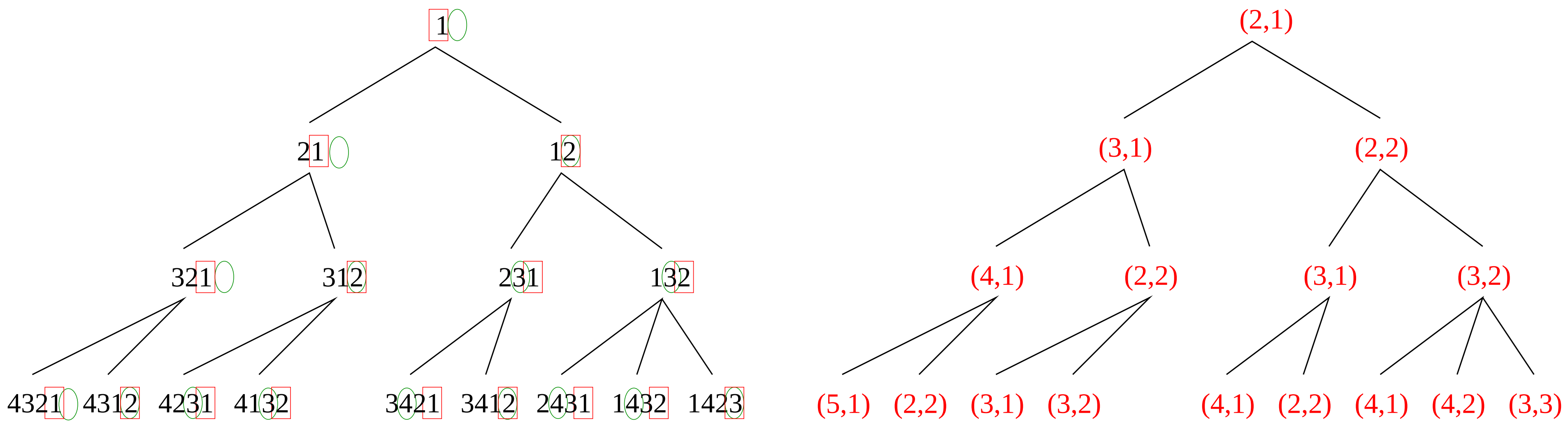
Generating tree for $\{2\text{-}13, 12\text{-}3\}$-avoiding permutations:



If $\pi \in \mathcal{S}_n$, let $l(\pi) = \begin{cases} n+1 & \text{if } \pi = n(n-1)\cdots 21, \\ \min\{\pi_i : i > 1,\ \pi_{i-1} < \pi_i\} & \text{otherwise.} \end{cases}$

Example of RGT with two labels

Generating tree for $\{2\text{-}13, 12\text{-}3\}$-avoiding permutations:



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This tree is described by the succession rule

$$(2,1)$$

$$(l,r) \longrightarrow \begin{cases} (l+1,1)\ (l+1,2)\ \cdots\ (l+1,l) & \text{if } l = r, \\ (l+1,1)\ (l+1,2)\ \cdots\ (l+1,r)\ (r+1,r+1) & \text{if } l > r. \end{cases}$$

RGT with one label: $\{2$**-**1**-**$3, \overline{2}$**-**$31\}$***-avoiding permutations (1)***

π avoids $\overline{2}$-31 if every occurrence of 31 in π is part of an occurrence of 2-31

Example: $\pi = 4623751$ avoids $\overline{2}$-31

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Proposition. *The number of* $\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$*-avoiding permutations of size* n *is the* n*-th Motzkin number* M_n*.*

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Proposition. *The number of $\{2\text{-}1\text{-}3, \bar{2}\text{-}31\}$-avoiding permutations of size n is the n-th Motzkin number M_n.*

Proof:

The RGT for this class is described by the succession rule

$$\begin{array}{l} (1) \\ (r) \longrightarrow (1)\ (2)\ \cdots\ (r-1)\ (r+1). \end{array}$$

Let $D(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,\bar{2}\text{-}31)} u^{r(\pi)} t^n = \sum_{r\geq 1} D_r(t) u^r$.

The succession rule translates into

$$\begin{aligned} D(t,u) &= tu + t\sum_{r\geq 1} D_r(t)(u + u^2 + \cdots + u^{r-1} + u^{r+1}) \\ &= tu + t\sum_{r\geq 1}\left[\frac{D_r(t)(u^r - u)}{u-1} + D_r(t)u^{r+1}\right] = tu + \frac{t}{u-1}[D(t,u) - uD(t,1)] + tuD(t,u) \end{aligned}$$

RGT with one label: $\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$*-avoiding permutations (2)*

$$\left(1 - \frac{t}{u-1} - tu\right) D(t,u) = tu - \frac{tu}{u-1} D(t,1)$$

Kernel method:

$$1 - \frac{t}{u_0(t) - 1} - t\, u_0(t) = 0 \implies u_0(t) = \frac{1 + t - \sqrt{1 - 2t - 3t^2}}{2t}$$

Substitute $u = u_0(t)$ to cancel the left hand side:

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Substitute $u = u_0(t)$ to cancel the left hand side:

$$D(t,1) = u_0(t) - 1 = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t},$$

which is the generating function for the Motzkin numbers. □

RGT with one label: $\{2\text{-}1\text{-}3, \overset{o}{\overline{2}}\text{-}31\}$***-avoiding permutations (1)***

π avoids $\overset{o}{\overline{2}}$-$31$ if every occurrence of 31 in π is part of an odd number of occurrences of 2-31

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Proposition. *The number of* $\{2\text{-}1\text{-}3, \overset{o}{\overline{2}}\text{-}31\}$*-avoiding permutations of size* n *is*

$$\begin{cases} \frac{1}{2k+1}\binom{3k}{k} & \text{if } n = 2k, \\ \frac{1}{2k+1}\binom{3k+1}{k+1} & \text{if } n = 2k+1. \end{cases}$$

RGT with one label: $\{2\text{-}1\text{-}3, \overset{o}{\overline{2}}\text{-}31\}$***-avoiding permutations (1)***

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Proof sketch:

The RGT for this class is described by the succession rule

$$\begin{aligned} &(1) \\ &(r) \longrightarrow (r+1)\ (r-1)\ (r-3) \cdots \end{aligned}$$

RGT with one label: $\{2\text{-}1\text{-}3, \overset{o}{\overline{2}}\text{-}31\}$-avoiding permutations (2)

Let $D(t,u) = \sum_{n \geq 1} \sum_{\pi \in \mathcal{S}_n(2\text{-}1\text{-}3, \overset{o}{\overline{2}}\text{-}31)} u^{r(\pi)} t^n$,

RGT with one label: $\{2\text{-}1\text{-}3, \overset{o}{\overline{2}}\text{-}31\}$*-avoiding permutations (2)*

Let $\eth(t,u) = \sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(2\text{-}1\text{-}3, \overset{o}{\overline{2}}\text{-}31)} u^{r(\pi)} t^n$,

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and $\eth^e(t,u) =$ terms in $\eth(t,u)$ with even exponent in u.

The succession rule translates into

$$\left(1 - \frac{tu^3}{u^2-1}\right)\eth(t,u) = tu - \frac{tu^2}{u^2-1}\eth(t,1) + \frac{tu(u-1)}{u^2-1}\eth^e(t,1)$$

Using two different roots $u_1(t)$ and $u_2(t)$ of the Kernel, we get two equations relating $\eth(t,1)$ and $\eth^e(t,1)$.
Solve for $\eth(t,1)$. □

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Solve for $\eth(t,1)$. $\square$

A similar argument gives the number of $\{2\text{-}1\text{-}3, \overset{e}{\overline{2}}\text{-}31\}$-avoiding permutations.

(π avoids $\overset{e}{\overline{2}}\text{-}31$ if every occurrence of 31 in π is part of an even number of occurrences of $2\text{-}31$)

RGT with one label: $\{2\text{-}1\text{-}3, 2\text{-}3\text{-}41, 3\text{-}2\text{-}41\}$***-avoiding perms.***

Let $K(t,u) = \sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(2\text{-}1\text{-}3, 2\text{-}3\text{-}41, 3\text{-}2\text{-}41)} u^{r(\pi)} t^n = \sum_{r \geq 1} K_r(t) u^r$.

Proposition.

$$K(t,u) = \frac{1 - t - 2tu - \sqrt{1 - 2t - 3t^2}}{2t(\frac{1}{u} + 1 + u) - 2}.$$

RGT with one label: $\{2\text{-}1\text{-}3, 2\text{-}3\text{-}41, 3\text{-}2\text{-}41\}$***-avoiding perms.***

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Proof sketch:

Succession rule:

$$(1)$$
$$(r) \longrightarrow \begin{cases} (r-1)\ (r)\ (r+1) & \text{if } r>1, \\ (r)\ (r+1) & \text{if } r=1. \end{cases}$$

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Functional equation:

$$\left[1-t\left(\frac{1}{u}+1+u\right)\right]K(t,u) = tu - tK_1(t).$$

Apply Kernel method to find $K_1(t)$, and then find $K(t,u)$. $\square$

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Functional equation:

$$\left[1 - t\left(\frac{1}{u} + 1 + u\right)\right] K(t,u) = tu - tK_1(t).$$

Apply Kernel method to find $K_1(t)$, and then find $K(t,u)$. □

Known (Mansour): $K(t,1)$ also enumerates $\{1\text{-}3\text{-}2, 123\text{-}4\}$-avoiding perms.

Open: Bijective proof of $|\mathcal{S}_n(2\text{-}1\text{-}3, 2\text{-}3\text{-}41, 3\text{-}2\text{-}41)| = |\mathcal{S}_n(1\text{-}3\text{-}2, 123\text{-}4)|$?

RGT with two labels: $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations (1)

Known (Claesson): $|\mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3)| = M_n$.

RGT with two labels: $\{2\text{-}1\text{-}3, 12\text{-}3\}$***-avoiding permutations (1)***

Known (Claesson): $|\mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3)| = M_n$.

The RGT for $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations is described by

$$(2,1)$$

$$(l,r) \longrightarrow \begin{cases} (l+1,1)\ (l+1,2)\ \cdots\ (l+1,l) & \text{if } l = r, \\ (l+1,1)\ (l+1,2)\ \cdots\ (l+1,r)\ (r+1,r+1) & \text{if } l > r, \end{cases}$$

where $l(\pi)$ is the smallest value of the top of a rise in π.

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\end{aligned}
$$

where $l(\pi)$ is the smallest value of the top of a rise in π.
Let

$$M(t,u,v) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3)} u^{l(\pi)} v^{r(\pi)}\ t^n = \sum_{l,r} M_{l,r}(t) u^l v^r$$

Proposition.

$$M(t,u,v) = \frac{[(1-u)v + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 - ((1-u)v + tu + t^2u^2v)\sqrt{1-2t-3t^2})]u^2 v}{2(1-u-tu(1-u)+t^2u^2)(1-uv+tuv+t^2u^2v^2)},$$

where $c_1 = 2-u-v-uv+2u^2v,\quad c_2 = u(-1+(2-u)v+2(u-1)v^2),$
$c_3 = u^2v(-3+2v-2uv),\quad c_4 = -2u^3v^2.$

RGT with two labels: $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations (2)

Proof sketch: The succession rule

$$(2,1)$$
$$(l,r) \longrightarrow \begin{cases} (l+1,1)\ (l+1,2)\ \cdots\ (l+1,l) & \text{if } l=r, \\ (l+1,1)\ (l+1,2)\ \cdots\ (l+1,r)\ (r+1,r+1) & \text{if } l>r, \end{cases}$$

translates into

$$M(t,u,v) = tu^2v + t\sum_{l} M_{l,l}(t)u^{l+1}(v+v^2+\cdots+v^l) + t\sum_{l>r} M_{l,r}(t)[u^{l+1}(v+v^2+\cdots+v^r)+u^{r+1}v^{r+1}]$$

RGT with two labels: $\{2\text{-}1\text{-}3, 12\text{-}3\}$***-avoiding permutations (2)***

Proof sketch: The succession rule

$$(2,1)$$

$$(l,r) \longrightarrow \begin{cases} (l+1,1)\ (l+1,2)\ \cdots\ (l+1,l) & \text{if } l = r, \\ (l+1,1)\ (l+1,2)\ \cdots\ (l+1,r)\ (r+1,r+1) & \text{if } l > r, \end{cases}$$

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Let $M_>(t,u,v) =$ terms in $M(t,u,v)$ where exponent of $u >$ exponent of v.

$$M_>(t,u,v) = tu^2v + \frac{tuv}{v-1}\left[tuv\ M_>(t,1,uv) - tu\ M_>(t,1,u) + M_>(t,u,v) - M_>(t,u,1)\right].$$

RGT with two labels: $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations (2)

Proof sketch:

$$M(t,u,v) = tu^2v + t\sum_{l} M_{l,l}(t)u^{l+1}(v+v^2+\cdots+v^l) + t\sum_{l>r} M_{l,r}(t)[u^{l+1}(v+v^2+\cdots+v^r)+u^{r+1}v^{r+1}]$$

Let $M_>(t,u,v) =$ terms in $M(t,u,v)$ where exponent of $u >$ exponent of v.

$$M_>(t,u,v) = tu^2v + \frac{tuv}{v-1}\left[tuv\, M_>(t,1,uv) - tu\, M_>(t,1,u) + M_>(t,u,v) - M_>(t,u,1)\right].$$

For $u=1$,

$$\left(1 - \frac{t^2v^2}{v-1} - \frac{tv}{v-1}\right) M_>(t,1,v) = tv - \frac{t(t+1)v}{v-1} M_>(t,1,1).$$

Apply the Kernel method to find $M_>(t,1,1)$ and $M_>(t,1,v)$.

RGT with two labels: $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations (2)

Proof sketch:

$$M(t,u,v) = tu^2v + t\sum_{l} M_{l,l}(t)u^{l+1}(v+v^2+\cdots+v^l) + t\sum_{l>r} M_{l,r}(t)[u^{l+1}(v+v^2+\cdots+v^r)+u^{r+1}v^{r+1}]$$

Let $M_>(t,u,v) =$ terms in $M(t,u,v)$ where exponent of $u >$ exponent of v.

$$M_>(t,u,v) = tu^2v + \frac{tuv}{v-1}\left[tuv\, M_>(t,1,uv) - tu\, M_>(t,1,u) + M_>(t,u,v) - M_>(t,u,1)\right].$$

For $u = 1$,

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Apply the Kernel method to find $M_>(t,1,1)$ and $M_>(t,1,v)$.

Now apply the Kernel method to

$$\left(1 - \frac{tuv}{v-1}\right) M_>(t,u,v) = tu^2v + \frac{tuv}{v-1}\left[tuv\, M_>(t,1,uv) - tu\, M_>(t,1,u) - M_>(t,u,1)\right]$$

to find $M_>(t,u,1)$ and $M_>(t,u,v)$.

Finally, $M(t,u,v) = M_>(t,u,v) + tuv\, M_>(t,1,uv)$. □

RGT with two labels: $\{2\text{-}1\text{-}3, 32\text{-}1\}$-avoiding permutations

Known (Claesson): $|\mathcal{S}_n(2\text{-}1\text{-}3, 32\text{-}1)| = 2^{n-1}$.

RGT with two labels: $\{2\text{-}1\text{-}3, 32\text{-}1\}$*-avoiding permutations*

Known (Claesson): $|\mathcal{S}_n(2\text{-}1\text{-}3, 32\text{-}1)| = 2^{n-1}$.

Let

$$h(\pi) = \begin{cases} 0 & \text{if } \pi = 12\cdots n, \\ \max\{\pi_i : i > 1,\ \pi_{i-1} > \pi_i\} & \text{otherwise.} \end{cases}$$

The RGT for $\{2\text{-}1\text{-}3, 32\text{-}1\}$-avoiding permutations is described by

$$(0, 1)$$
$$(h, r) \longrightarrow (h+1, h+1)\ (h+1, h+2)\ \cdots\ (r, r)\ (h, r+1).$$

RGT with two labels: $\{2\text{-}1\text{-}3, 32\text{-}1\}$***-avoiding permutations***

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Let $N(t, u, v) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,32\text{-}1)} u^{h(\pi)} v^{r(\pi)}\, t^n$.

From the succession rule,

$$N(t, u, v) = tv + tvN(t, u, v) + \frac{tuv[N(t, 1, uv) - N(t, uv, 1)]}{uv - 1}.$$

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Solving this functional equation we get

$$N(t,u,v) = \frac{tv(1-t+tu-tuv)}{(1-tv)(1-t-tuv)}.$$

RGT with two labels: $1\text{-}23$-avoiding permutations

Known (Claesson): $|\mathcal{S}_n(1\text{-}23)| = B_n$, the n-th Bell number.

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$$(r,n) \longrightarrow \begin{cases} (1,n+1)\ (2,n+1)\ \cdots\ (n+1,n+1) & \text{if } r=1, \\ (1,n+1)\ (2,n+1)\ \cdots\ (r,n+1) & \text{if } r>1. \end{cases}$$

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Let $G(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(1\text{-}23)} u^{r(\pi)}\, t^n$. From the succession rule,

$$
\left(1-\frac{tu}{u-1}\right)G(t,u) = tu + t^2u^2 + \frac{tu}{u-1}\left[tu^2G(tu,1) - (1+tu)G(t,1)\right]
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$$G(t,1) = \frac{t}{1-t}\left(1+G(\frac{t}{1-t},1)\right)$$

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$$G(t,1) = \frac{t}{1-t}\left(1+\frac{t}{1-2t}\left(1+\frac{t}{1-3t}\,(1+\cdots)\right)\right) = \sum_{k\geq 1}\frac{t^k}{(1-t)(1-2t)\cdots(1-kt)}$$

We can also get a formula for $G(t,u)$.

RGT with two labels: 123*-avoiding permutations (1)*

Known (E, Noy): The exponential GF for 123-avoiding permutations is

$$\frac{\sqrt{3}}{2}\frac{e^{t/2}}{\cos(\frac{\sqrt{3}}{2}t+\frac{\pi}{6})}.$$

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Let $C(t,u) = \sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(123)} u^{r(\pi)}\, t^n = A(t,u) + B(t,u)$, where A (resp. B) are the terms with a label of the form $(\ ,\ )$ (resp. $(\ ,\ )'$).

RGT with two labels: 123-avoiding permutations (2)

The succession rule translates into

$$\begin{aligned} A(t,u) &= tu + \frac{tu}{u-1}[C(t,u) - C(t,1)] \\ B(t,u) &= \frac{tu}{u-1}[uA(tu,1) - A(t,u)] \end{aligned}$$

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Solved by Bousquet-Mélou:

$$C(t,1) = \frac{3+i\sqrt{3}}{2(3t-i\sqrt{3})}\, C\left(\frac{t}{1+i\sqrt{3}t}, 1\right) - \frac{3(2t+1-i\sqrt{3})t}{(2t-1-i\sqrt{3})(3t-i\sqrt{3})}.$$

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From this, one can obtain a recurrence for the coefficients of $C(t,1)$, and derive their exponential generating function.

Unsolved RGT with three labels: 1-2-34*-avoiding perms. (1)*

If $\pi \in \mathcal{S}_n$, let $m(\pi) = \begin{cases} n+1 & \text{if } \pi = n(n-1)\cdots 21, \\ \min\{\pi_i : \exists j < i \text{ with } \pi_j < \pi_i\} & \text{otherwise.} \end{cases}$

Unsolved RGT with three labels: 1*-*2*-*34*-avoiding perms. (1)*

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The RGT for 1-2-34-avoiding permutations is described by:

$$(2,1,1)$$

$$(m,r,n) \longrightarrow \begin{cases} (m+1,1,n+1)\ (2,2,n+1)\ (3,3,n+1)\ \cdots\ (m,m,n+1) \\ \qquad (m,m+1,n+1)\ \cdots\ (m,n+1,n+1) & \text{if } r=1, \\ (m+1,1,n+1)\ (2,2,n+1)\ (3,3,n+1)\ \cdots\ (m,m,n+1) \\ \qquad (m,m+1,n+1)\ \cdots\ (m,n+1,n+1) & \text{if } m=r, \\ (m+1,1,n+1)\ (2,2,n+1)\ (3,3,n+1)\ \cdots\ (m,m,n+1) \\ \qquad (m,m+1,n+1)\ \cdots\ (m,r,n+1) & \text{if } m<r. \end{cases}$$

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Let $G(t,u,v) = \sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(1\text{-}2\text{-}34)} u^{m(\pi)} v^{r(\pi)}\, t^n$.

Unsolved RGT with three labels: 1-2-34*-avoiding perms. (2)*

Functional equation:

$$\left(1-\frac{tv}{v-1}\right)G(t,u,v)=\left(tuv-\frac{t^2uv^2}{v-1}\right)G(t,u,1)-\frac{t(u-1)v+t^2uv^2}{(v-1)(uv-1)}G(t,uv,1)$$
$$+\left(\frac{t^2u^2v^3}{(v-1)(uv-1)}-\frac{tu^2v^2}{uv-1}\right)G(t,1,1)+\frac{t^2u^2v^3}{(u-1)(v-1)}G(tv,u,1)$$
$$-\frac{t^2u^2v^3}{(u-1)(v-1)}G(tv,1,1)+tu^2v+tu^2v^2$$

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Functional equation:

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$$+ \left(\frac{t^2u^2v^3}{(v-1)(uv-1)} - \frac{tu^2v^2}{uv-1}\right) G(t,1,1) + \frac{t^2u^2v^3}{(u-1)(v-1)} G(tv,u,1)$$
$$- \frac{t^2u^2v^3}{(u-1)(v-1)} G(tv,1,1) + tu^2v + tu^2v^2$$

Don't know how to solve it...

Unsolved RGT with three labels: 12*-*34*-avoiding perms.*

The RGT for 12-34-avoiding permutations is described by:

$$(2,1,1)$$

$$(l,r,n) \longrightarrow \begin{cases} (l+1,1,n+1)\,(l+1,2,n+1)\ \cdots\ (l+1,r,n+1)\,(r+1,r+1,n+1) & \\ \qquad (r+2,r+2,n+1)\ \cdots (l,l,n+1)\,(l,l+1,n+1)\ \cdots\ (l,n+1,n+1) & \text{if } l \geq r, \\ (l+1,1,n+1)\,(l+1,2,n+1)\ \cdots\ (l+1,l,n+1) & \\ \qquad (l,l+1,n+1)\,(l,l+2,n+1)\ \cdots\ (l,r,n+1) & \text{if } l < r. \end{cases}$$

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Let $H(t,u,v) = \sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(12\text{-}34)} u^{l(\pi)} v^{r(\pi)}\ t^n$,
$J(t,u,v) =$ terms of $H(t,u,v)$ with $l \geq r$.

Functional equations:

$$\left(1 - \frac{tv}{v-1}\right) H(t,u,v) = -\frac{tv}{v-1} H(t,uv,1) + \left(1 - \frac{tv}{v-1}\right) J(t,u,v) + \frac{tv^2}{v-1} J(tv,u,1)$$

$$\left(1 - \frac{tuv}{v-1}\right) J(t,u,v) = \frac{tuv}{v-1} H(t,uv,1) - \frac{tuv}{v-1} H(t,u,1) + tuv\left(\frac{1}{uv-1} - \frac{1}{v-1}\right) J(t,uv,1) - \frac{tuv}{uv-1} J(t,1,uv) + tu^2 v$$

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Don’t know how to solve either...

TAKK