Central Twisted Transformation Groups and Group C^*-algebras of Central Group Extensions

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ABSTRACT. We examine the structure of central twisted transformation group C^*-algebras $C_0(X) \rtimes_{\mathrm{id},u} G$, and apply our results to the group C^*-algebras of central group extensions. Our methods require that we study Moore's cohomology group $H^2(G, C(X, \mathbf{T}))$, and, in particular, we prove an inflation result for pointwise trivial cocyles which may be of use elsewhere.

Twisted transformation group C^*-algebras and Moore's cohomology group $H^2(G, C(X, \mathbf{T}))$ arise naturally in the study of the group C^*-algebras of group extensions and have far reaching applications in operator algebras. Packer's survey articles [13], and especially [12], give an excellent historical context as well as providing additional references and the basic definitions for what follows.

In this paper we examine the structure of central twisted transformation group C^*-algebras $C_0(X) \rtimes_{\mathrm{id},u} G$, where G is a second countable locally compact group acting trivially on a second countable locally compact space X, and $u \in Z^2(G, C(X, \mathbf{T}))$ is a 2-cocycle on G taking values in the center $C(X, \mathbf{T})$ of the multiplier algebra of $C_0(X)$. A primary motivation is the study of the group C^*-algebra $C^*(L)$ of a central group extension

$$(*) \qquad 1 \longrightarrow N \longrightarrow L \underset{p}{\longrightarrow} G \longrightarrow 1.$$

It is well-known that $C^*(L)$ is isomorphic to the central twisted transformation group $C_0(\widehat{N}) \rtimes_{\mathrm{id},\eta} G$, where $\eta \in Z^2(G, N)$ is a 2-cocycle associated to the extension $(*)$ viewed as taking values in $C(\widehat{N}, \mathbf{T})$.

Our results apply, in particular, to groups G which are smooth in the sense of Moore [8], in that there is a central group extension

$$1 \longrightarrow Z \longrightarrow H \longrightarrow G \longrightarrow 1$$

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(called a representation group for G) such that the transgression map $\mathrm{tg}:\widehat{Z}\to H^2(G,\mathbf{T})$ is an isomorphism of topological groups. Thus our results apply in particular to groups G that are either discrete, compact, compactly generated abelian or connected, simply connected Lie groups. Our results substantially generalize results in [23, 24, 20], where central twisted group algebras have been studied for abelian groups and pointwise trivial cocycles $u\in Z^2(G,C(X,\mathbf{T}))$, and results in [25], where Smith considers central twisted group algebras for groups with $H^2(G,\mathbf{T})$ discrete (which, for example, is the case whenever G is compact).

The structure of $C_0(X)\rtimes_{\mathrm{id},u}G$ is easiest to describe when u is pointwise trivial; that is, $u(x)$ is trivial in $H^2(G,\mathbf{T})$ for each $x\in X$, where $u(x)(s,t):=u(s,t)(x)$. As was shown in [23, 20] for abelian G, each pointwise trivial cocycle is naturally associated to a locally compact $\widehat{G}_{\mathrm{ab}}$-bundle $p:\mathcal{E}_u\to X$, where $G_{\mathrm{ab}}:=G/\overline{[G,G]}$ is the *abelianization* of G (Definition 1.2). The bundles $\mathcal{E}_u$ and $\mathcal{E}_v$ are isomorphic if and only if u and v are equivalent in $H^2(G,C(X,\mathbf{T}))$. Furthermore, the map $u\mapsto\mathcal{E}_u$ is multiplicative in that $\mathcal{E}_{uv}\cong\mathcal{E}_u*\mathcal{E}_v$, where the $\mathcal{E}_u*\mathcal{E}_v$ is the usual product of $\widehat{G}_{\mathrm{ab}}$-bundles. In the case of abelian G, the bundles $\mathcal{E}_u$ appeared in [24] under the name of *characteristic bundles*, the isomorphism classes of which form an (often proper) subset of the set of all isomorphism classes of free and proper $\widehat{G}$-bundles over X. Moreover, in this case, the algebra $C_0(X)\rtimes_{\mathrm{id},u}G$ is isomorphic to $C_0(\mathcal{E}_u)$. We shall give a short proof of this well-known isomorphism below.

When G is not abelian, $C_0(X)\rtimes_{\mathrm{id},u}G$ is still a $C_0(X)$-algebra, and therefore can be thought of as the (semi-continuous) sections of a C^*-bundle over X. If A is a $C_0(X)$-algebra admitting a $C_0(X)$-linear $\widehat{G}_{\mathrm{ab}}$-action, then we can form the C^*-analogue of the bundle product above, and obtain a C^*-algebra $\mathcal{E}_u*A$ ([4, Definition 3.3]). One benefit of this construction is that, if u is pointwise trivial and G is smooth, then we can show that

$$C_0(X)\rtimes_{\mathrm{id},u}G\sim_{ME}\mathcal{E}_u*C_0(X,C^*(G))) \tag{†}$$

(Theorem 2.11). A crucial ingredient in the proof of (†), which may be of independent interest, is showing that u is equivalent to a cocycle inflated from a pointwise trivial cocycle on G_{ab} (Proposition 1.7). More generally, we show that

$$H^2(G,C(X,\mathbf{T}))\cong C(X,H^2(G,\mathbf{T}))\oplus H^2_{\mathrm{pt}}(G_{\mathrm{ab}},C(X,\mathbf{T})) \tag{‡}$$

(Proposition 1.10). If $u\in Z^2(G,C(X,\mathbf{T}))$, then the map $x\mapsto[u(x)]$ gives us an element $\varphi\in C(X,H^2(G,\mathbf{T}))$ and we can use (‡) to write $u=v\cdot\bar{u}_\varphi$ with v pointwise trivial. Since H is a representation group for G, $C^*(H)$ is a $C_0(\widehat{Z})$-algebra. Since we can view φ as a continuous map of X into $\widehat{Z}\cong H^2(G,\mathbf{T})$, we can form the pull-back $\varphi^*(C^*(H))$. Our main result (Theorem 2.9) implies that

$$C_0(X)\rtimes_{\mathrm{id},u}G\sim_{ME}\mathcal{E}_v*\varphi^*(C^*(H)).$$

This result is new even in the special situation where $H^2(G,\mathbf{T})$ is discrete as is the case in [25] (see Theorem 2.11 below).

In the case of a central group extension $C^*(L) \cong C_0(\widehat{N}) \rtimes_{\mathrm{id},\eta} G$, our results take a more elegant form as our auxiliary constructions can be formulated in group theoretic terms. For example, if L is pointwise trivial—that is, the associated cocycle η is pointwise trivial—then $\mathrm{res} : \widehat{L}_{\mathrm{ab}} \to \widehat{N}$ is a $\widehat{G}_{\mathrm{ab}}$-bundle which is isomorphic to $\mathcal{E}_\eta$, and we have

$$C^*(L) \cong \widehat{L}_{\mathrm{ab}} * C_0(\widehat{N}, C^*(G))$$

(Corollary 3.3). In general, we prove (Theorem 3.5) that

$$C^*(L) \cong \widehat{L'}_{\mathrm{ab}} * \varphi^*(C^*(H)),$$

where L' is the quotient of $\{(\ell,h) \in L \times H \mid p(\ell) = q(h)\}$ by the subgroup $\Delta(Z) := \{(\hat{\varphi}(z),z) \mid z \in Z\}$, where $\hat{\varphi} : Z \to N$ is the dual to $\varphi : \widehat{N} \to H^2(G,\mathbf{T}) \cong \widehat{Z}$.

The paper is organized as follows. In Section 1, we make a careful study of pointwise trivial cocycles and prove our results on inflation of pointwise trivial cocycles, and on the decomposition of $H^2(G,C(X,\mathbf{T}))$. In Section 2, we prove our structure theorems for general central twisted transformation groups. In Section 3, we show how to apply our results to the group C^*-algebras of central group extensions.

We should mention that this paper is very much related to the papers [5, 4], where we consider more general systems. However, in the special situations considered here, the results are much easier to describe, and allow more general statements.

1. POINTWISE TRIVIAL COCYCLES

If X is a second countable locally compact space, then the set of continuous functions $C(X,\mathbf{T})$ from X into the circle group $\mathbf{T}$ is a Polish group when equipped with pointwise multiplication and the compact-open topology. In this section, we look carefully at the Moore cohomology group $H^2(G,C(X,\mathbf{T}))$ for a second countable locally compact group G acting trivially on $C(X,\mathbf{T})$.

The definition and basic properties of Moore's cohomology groups $H^n(G,A)$ for an arbitrary polish G-module A are laid out in Moore's original paper [9]. (A summary with additional references can be found in Section 7.4 of [21].) An important facet of the theory is that these groups can be computed by using two different complexes: one can either take the complex

$$\cdots \underset{\partial}{\longrightarrow} C^n(G,A) \underset{\partial}{\longrightarrow} C^{n+1}(G,A) \underset{\partial}{\longrightarrow} \cdots$$

where $C^n(G,A)$ denotes the group of A-valued Borel functions on G^n and ∂ denotes the usual group coboundary, or one can work with the complex

$$\cdots \xrightarrow[\underline{\partial}]{} \underline{C}^n(G,A) \xrightarrow[\underline{\partial}]{} \underline{C}^{n+1}(G,A) \xrightarrow[\underline{\partial}]{} \cdots$$

where $\underline{C}^n(G,A)$ is the quotient of $C^n(G,A)$ obtained by identifying Borel functions on G^n which coincide almost everywhere and $\underline{\partial}$ is the induced map. Moore shows in [9, Theorem 5] that the canonical maps $C^n(G,A) \to \underline{C}^n(G,A)$ induce isomorphisms of $H^n(G,A)$ with $\underline{H}^n(G,A)$ for all $n \geq 0$. One advantage of working with $\underline{C}^n(G,A)$ is that $\underline{C}^n(G,A)$ is a Polish group when equipped with the topology of convergence in measure (after replacing Haar measure with an equivalent finite measure). Thus we have a topology on $\underline{H}^n(G,A)$ (and therefore on $H^n(G,A)$), although this topology can be non-Hausdorff in general. On the other hand, elements in $\underline{C}^n(G,A)$ are not defined everywhere, and this can often be a nuisance; thus it is useful to work with both definitions simultaneously, and we shall do so below.

Definition 1.1. Let $u \in Z^2(G,C(X,\mathbf{T}))$ and define $u(x) \in Z^2(G,\mathbf{T})$ by evaluation at x: $u(x)(s,t) := u(s,t)(x)$. We say that u is *pointwise trivial* if $u(x)$ is trivial (i.e., $u(x) \in B^2(G,\mathbf{T})$) for all $x \in X$. We say that u is *locally trivial* if each $x \in X$ has an open neighborhood V such that the restriction $u_V \in Z^2(G,C(V,\mathbf{T}))$ of u to V is trivial (i.e., $u_V \in B^2(G,C(V,\mathbf{T}))$).

We denote by $Z^2_{\mathrm{pt}}(G,C(X,\mathbf{T}))$ and $Z^2_{\mathrm{loc}}(G,C(X,\mathbf{T}))$, respectively, the subsets of pointwise trivial cocycles and locally trivial cocycles in $Z^2(G,C(X,\mathbf{T}))$. Similarly, we let $H^2_{\mathrm{pt}}(G,C(X,\mathbf{T}))$ and $H^2_{\mathrm{loc}}(G,C(X,\mathbf{T}))$, respectively, be the images of $Z^2_{\mathrm{pt}}(G,C(X,\mathbf{T}))$ and $Z^2_{\mathrm{loc}}(G,C(X,\mathbf{T}))$ in $H^2(G,C(X,\mathbf{T}))$.

Pointwise trivial cocycles have been studied extensively in the literature. A particularly important example—having applications in the study of C^*-dynamical systems—is Rosenberg's [22, Theorem 2.1], which shows that

$$Z^2_{\mathrm{pt}}(G,C(X,\mathbf{T})) = Z^2_{\mathrm{loc}}(G,C(X,\mathbf{T})),$$

whenever $H^2(G,\mathbf{T})$ is Hausdorff and the abelianization G_{ab} is compactly generated. However, Rosenberg's Theorem can fail without the assumptions on G_{ab} and $H^2(G,\mathbf{T})$ (see [5] and Example 1.8 below).

We shall study pointwise trivial cocycles $u \in Z^2_{\mathrm{pt}}(G,C(X,\mathbf{T}))$ via a canonical $\widehat{G}_{\mathrm{ab}}$-space $\mathcal{E}_u$. Our construction of $\mathcal{E}_u$ is identical to that in [20] where the group G was assumed to be abelian. It follows from [9, Theorem 3] that $H^1(G,\mathbf{T})$, and therefore $\underline{H}^1(G,\mathbf{T})$, can be identified with $\widehat{G}_{\mathrm{ab}}$. Since there are natural maps of $\widehat{G}_{\mathrm{ab}}$ into $C^1(G,\mathbf{T})$ and $\underline{C}^1(G,\mathbf{T})$, we always have *algebraic* short exact sequences of groups

$$1 \longrightarrow \widehat{G}_{\mathrm{ab}} \longrightarrow C^1(G,\mathbf{T}) \underset{\partial}{\longrightarrow} B^2(G,\mathbf{T}) \longrightarrow 1$$

and

$$1 \longrightarrow \widehat{G}_{\mathrm{ab}} \longrightarrow \underline{C}^1(G,\mathbf{T}) \underset{\underline{\partial}}{\longrightarrow} \underline{B}^2(G,\mathbf{T}) \longrightarrow 1$$

which are related to each other via the inclusions $C^n \to \underline{C}^n$. Although the homomorphisms in the second sequence are always continuous, the second sequence may fail to be a short exact sequence of topological groups as the associated quotient map $\underline{C}^1(G,\mathbf{T})/\widehat{G}_{\mathrm{ab}} \to \underline{B}^2(G,\mathbf{T})$ may fail to be a homeomorphism. In fact, this quotient map is a topological isomorphism if and only if $\underline{B}^2(G,\mathbf{T})$ is a Polish subgroup of $\underline{C}^2(G,\mathbf{T})$. This happens exactly when $\underline{B}^2(G,\mathbf{T})$ is closed, which is equivalent to $\underline{H}^2(G,\mathbf{T})$ being Hausdorff (cf. e.g. [9, Section 5]).

Definition 1.2. Let $u \in Z^2_{\mathrm{pt}}(G, C(X,\mathbf{T}))$. Then we define

$$\mathcal{E}_u = \{(f,x) \in C^1(G,\mathbf{T}) \times X \mid \partial(f) = u(x)\}.$$

Similarly, if $\underline{u}$ denotes the image of u in $\underline{Z}^2(G, C(X,\mathbf{T}))$, we define

$$\mathcal{E}_{\underline{u}} = \{(\underline{f},x) \in \underline{C}^1(G,\mathbf{T}) \times X \mid \underline{\partial}(\underline{f}) = \underline{u}(x)\}.$$

It follows from the short exact sequences mentioned above that both, $\mathcal{E}_u$ and $\mathcal{E}_{\underline{u}}$ are free $\widehat{G}_{\mathrm{ab}}$-spaces, and that $\mathcal{E}_{\underline{u}}$ becomes a topological $\widehat{G}_{\mathrm{ab}}$-space when equipped with the relative topology from $\underline{C}^1(G,\mathbf{T}) \times X$. Moreover, the canonical projections

$$p : \mathcal{E}_u \to X \quad \text{and} \quad \underline{p} : \mathcal{E}_{\underline{u}} \to X$$

induce bijections between the quotient spaces $\mathcal{E}_u/\widehat{G}_{\mathrm{ab}}$ and $\mathcal{E}_{\underline{u}}/\widehat{G}_{\mathrm{ab}}$ and X, respectively. However, there are cases where $\underline{p} : \mathcal{E}_{\underline{u}} \to X$ fails to be open (e.g. Example 1.8 below).

Proposition 1.3. *Let $u \in Z^2_{\mathrm{pt}}(G, C(X,\mathbf{T}))$. Then the following are true:*

(a) *The map $(f,x) \mapsto (\underline{f},x)$ is a bijection $\varphi : \mathcal{E}_u \to \mathcal{E}_{\underline{u}}$.*
(b) *If we topologize $\mathcal{E}_u$ via the identification with $\mathcal{E}_{\underline{u}}$ of* (a)*, then the topology on $\mathcal{E}_u \subset C^1(G,\mathbf{T}) \times X$ is given by pointwise convergence in the first variable, and the given topology on X.*
(c) *If $[u] = [v] \in H^2_{\mathrm{pt}}(G, C(X,\mathbf{T}))$, then $\mathcal{E}_u \cong \mathcal{E}_v$ as* (*topological*) *$\widehat{G}_{\mathrm{ab}}$-spaces.*
(d) *$\mathcal{E}_u$ is isomorphic to the trivial $\widehat{G}_{\mathrm{ab}}$-bundle $\widehat{G}_{\mathrm{ab}} \times X$* (*as a $\widehat{G}_{\mathrm{ab}}$-space*) *if and only if u is trivial.*

Proof. For the proof of (a), it is enough to show that the given map induces bijections between the $\widehat{G}_{\mathrm{ab}}$-orbits in $\mathcal{E}_u$ and $\mathcal{E}_{\underline{u}}$. But, by the above discussion, the orbits are just the sets $p^{-1}(\{x\})$ and $\underline{p}^{-1}(\{x\})$, respectively, and it is clear that $p^{-1}(\{x\})$ is mapped into $\underline{p}^{-1}(\{x\})$. The result then follows from the freeness of the $\widehat{G}_{\mathrm{ab}}$-actions.

For (b), we have to show that a sequence $\{(f_n, x_n)\}$ converges to (f, x) in $\mathcal{E}_u$ if and only if $f_n \to f$ pointwise on G and $x_n \to x$. Since pointwise convergence of $\{f_n\}$ in $C^1(G, \mathbf{T})$ implies convergence of $\{\underline{f_n}\}$ in $\underline{C}^1(G, \mathbf{T})$, it is enough to show that convergence of $\{(\underline{f_n}, x_n)\}$ in $\mathcal{E}_{\underline{u}}$ implies pointwise convergence of $\{f_n\}$. But this follows from part (a) and [20, Lemma 3.6].

For (c), let $g \in C^1(G, C(X, \mathbf{T}))$ be a Borel cochain such that $v = \partial(g)u$. Then $\mathcal{E}_u \to \mathcal{E}_v : (f, x) \mapsto (g(x)f, x)$ is an isomorphism which is bicontinuous by part (b).

The last assertion follows from the proof of [20, Proposition 3.4], which did not make use of the assumption that G was abelian. □

Remark 1.4. It follows from part (d) of the proposition that $\mathcal{E}_u$ is a locally trivial $\widehat{G}_{\mathrm{ab}}$-bundle if and only if u is locally trivial. In particular, when u is locally trivial, $\mathcal{E}_u$ is a free and proper locally compact $\widehat{G}_{\mathrm{ab}}$-bundle.

If G is abelian, then [20, Proposition 3.1] implies that $\mathcal{E}_u$ is locally compact, and $p : \mathcal{E}_u \to X$ is a free and proper $\widehat{G}$-bundle. A careful look at the proof reveals that the hypothesis that G be abelian was only used to guarantee that the coboundary map $\underline{\partial} : \underline{C}^1(G, \mathbf{T}) \to \underline{B}^2(G, \mathbf{T})$ is open.[1] Thus we get the following result.

Proposition 1.5. *Assume that $H^2(G, \mathbf{T})$ is Hausdorff and*

$$u \in Z^2_{\mathrm{pt}}(G, C(X, \mathbf{T})).$$

Then $\mathcal{E}_u$ is locally compact and $p : \mathcal{E}_u \to X$ is a free and proper $\widehat{G}_{\mathrm{ab}}$-bundle.

We shall also need the following lemma, which can be proved along the lines of the final part of the proof of [20, Proposition 3.4].

Lemma 1.6. *Let Y be a second countable topological space and let $g : Y \to C(X, \mathbf{T})$ be a map such that for each $x \in X$ the map $g(x) : Y \to \mathbf{T} : g(x)(y) = g(y, x)$ is Borel. Then $g : Y \to C(X, \mathbf{T})$ is Borel.*

We are now ready for our lifting result for pointwise trivial cocycles.

Proposition 1.7. *Assume that $u \in Z^2_{\mathrm{pt}}(G, C(X, \mathbf{T}))$. Then the following are equivalent:*

[1] If G is abelian, then $H^2(G, \mathbf{T})$ is Hausdorff [10, Theorem 7], and this is equivalent to the openness of $\underline{\partial}$.

(a) *There exists a cocycle* $\tilde{u} \in Z^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X,\mathbf{T}))$ *such that* u *is cohomologous to the inflation* $\inf \tilde{u}$ *of* $\tilde{u}$ *to* G.
(b) *The projection* $p : \mathcal{E}_u \to X$ *is open.*
(c) $\mathcal{E}_u$ *is locally compact and* $p : \mathcal{E}_u \to X$ *is a free and proper* $\widehat{G}_{\mathrm{ab}}$*-bundle.*

Moreover, if (a) *holds, then* $\mathcal{E}_u$ *is isomorphic to* $\mathcal{E}_{\tilde{u}}$ *as a* $\widehat{G}_{\mathrm{ab}}$*-space.*

Proof. We will first show the last statement: indeed, if $u \sim \inf \tilde{u}$, we can use part (c) of Proposition 1.3 to assume without loss of generality that $u = \inf \tilde{u}$. It is then easy to check that the map from $\mathcal{E}_{\tilde{u}} \to \mathcal{E}_u$ given by

$$(f, x) \mapsto (\inf f, x)$$

is a $\widehat{G}_{\mathrm{ab}}$-equivariant bijection, which is bicontinuous by part (b) of Proposition 1.3.

We now show (a)⇒(c)⇒(b)⇒(a). In fact (a)⇒(c) follows from the isomorphism $\mathcal{E}_u \cong \mathcal{E}_{\tilde{u}}$ together with [20, Proposition 3.1] (see the discussion preceding Proposition 1.5 above), and (c)⇒(b) follows by definition.

We now check (b)⇒(a). Choose a Borel section $c : G_{\mathrm{ab}} \to G$ with $c(\dot{e}) = e$ and define a map $\mu : G_{\mathrm{ab}} \times G_{\mathrm{ab}} \times \mathcal{E}_u \to \mathbf{T}$ by

$$\mu(\dot{s}, \dot{t}, (f, x)) = \partial_{G_{\mathrm{ab}}}(f \circ c)(\dot{s}, \dot{t}).$$

This map is continuous in (f, x) by Proposition 1.3(b), and since

$$\partial_{G_{\mathrm{ab}}}(\gamma \cdot (f \circ c)) = \partial_{G_{\mathrm{ab}}}(f \circ c) \quad \text{for all } \gamma \in \widehat{G}_{\mathrm{ab}},$$

it follows that μ induces a map $\tilde{u} : G_{\mathrm{ab}} \times G_{\mathrm{ab}} \times X \to \mathbf{T}$ which is continuous on X (since part (b) implies that $p : \mathcal{E}_u \to X$ induces a homeomorphism $\mathcal{E}_u/\widehat{G}_{\mathrm{ab}} \cong X$). Thus we may view $\tilde{u}$ as a map from $G_{\mathrm{ab}} \times G_{\mathrm{ab}}$ to $C(X, \mathbf{T})$. Since for each $x \in X$, $\tilde{u}(x) = \partial_{G_{\mathrm{ab}}}(f \circ c)$ for any $(f, x) \in \mathcal{E}_u$, it follows that all evaluations $\tilde{u}(x)$ are Borel. Therefore, $\tilde{u}$ is Borel by Lemma 1.6.

It remains to check that $u \sim \inf \tilde{u}$. For this we define a map $v : G \times \mathcal{E}_u \to \mathbf{T}$ by

$$v(s, (f, x)) = \overline{f(s)} \cdot (f \circ c)(\dot{s}).$$

Then v is continuous on $\mathcal{E}_u$, and since $v(s, (\gamma \cdot f, x)) = v(s, (f, x))$ for all $\gamma \in \widehat{G}_{\mathrm{ab}}$, it follows that v factors through a map $g : G \times X \to \mathbf{T}$ which is continuous on $X \cong Z_u/\widehat{G}_{\mathrm{ab}}$. Lemma 1.6 implies that g, viewed as a map from G to $C(X, \mathbf{T})$, is Borel. Moreover, if $(f, x) \in \mathcal{E}_u$, then

$$\partial g(s, t)(x) = \partial \bar{f} \cdot \partial(f \circ c) = \overline{u(x)} \cdot \inf \tilde{u}(x).$$

This completes the proof. □

Example 1.8 (cf. [5, Example 7.3]). There exist groups G with cocycles $u \in Z^2_{\mathrm{pt}}(G, C(X, \mathbf{T}))$ such that $p : \mathcal{E}_u \to X$ fails to be open; it follows that such u are not lifted from pointwise trivial cocycles on G_{ab}. Let θ be an irrational number and let $G = \mathbf{R}^2 \times \mathbf{T}^2$ with multiplication given by the formula

$$(s_1, t_1, z_1, w_1)(s_2, t_2, z_2, w_2) = (s_1 + s_2, t_1 + t_2, e^{is_1t_2} z_1 z_2, e^{i\theta s_1 t_2} w_1 w_2).$$

This is the example of a group with non-Hausdorff $H^2(G, \mathbf{T})$ presented by Moore in [8, p. 85] (see also [5, Example 7.2]).

Let $X = \{\frac{1}{n} \mid n \in \mathbf{N}\} \cup \{0\}$ and choose a continuous map $\lambda : X \to \mathbf{Z} + \theta\mathbf{Z} \subseteq \mathbf{R}$ such that $\lambda_0 = 0$ and such that $\lambda_{1/n} \neq 0$ for all $n \in \mathbf{N}$. Define a cocycle $v \in Z^2(\mathbf{R}^2, C(X, \mathbf{T}))$ by

$$v((s_1, t_1), (s_2, t_2))(x) = e^{i\lambda(x) s_1 t_2}, \quad s, t \in G,\ x \in X.$$

Since the $v(x)$ are non-trivial cocycles on $\mathbf{R}^2 = G_{\mathrm{ab}}$ if $x \neq 0$, it follows that v is not pointwise trivial. However, the inflation $\inf v \in Z^2(G, C(X, \mathbf{T}))$ *is* pointwise trivial. Indeed, a short computation shows that if we define $\lambda(x) = \ell(x) + \theta m(x)$ where $\ell, m : X \to \mathbf{Z}$, and if we define $f_x \in C^1(G, \mathbf{T})$ by $f_x(s, t, z, w) = z^{\ell(x)} w^{m(x)}$, then $\inf v(x) = \partial f_x$.

We now show that the projection $p : \mathcal{E}_{\inf v} \to X$ is not open. If it were, then Proposition 1.7 would imply that $\mathcal{E}_{\inf v}$ would be a locally compact free and proper $\widehat{G}_{\mathrm{ab}} \cong \mathbf{R}^2$-space. Since every free and proper $\mathbf{R}^2$-bundle is locally trivial by Palais's Slice Theorem [16, Theorem 4.1], and therefore trivial since $\check{H}^1(X, \mathbf{R}^2) = \{0\}$, this would imply that $\mathcal{E}_{\inf v}$ would be a trivial bundle. In that case, there exists a continuous section $\varphi : X \to \mathcal{E}_{\inf v}$. Then we could find elements $\gamma_x \in \widehat{\mathbf{R}}^2$ such that

$$\varphi(x) = (\gamma_x \cdot f_x, x), \quad \text{with} \quad \gamma_0 = 1_G,$$

and where f_x is defined as above. This would imply that $\gamma_{1/n} \cdot f_{1/n}$ converges pointwise to 1_G, and hence, that $f_{1/n}|_{\mathbf{T}^2}$ converges pointwise to $1_{\mathbf{T}^2}$. But this is impossible since for all $x \neq 0$, $f_x|_{\mathbf{T}^2}$ is a non-trivial character of $\mathbf{T}^2$, $\widehat{\mathbf{T}}^2 \cong \mathbf{Z}^2$ is discrete, and pointwise convergence of characters implies convergence [9, Theorem 8].

Corollary 1.9. *Assume that $H^2(G, \mathbf{T})$ is Hausdorff. Then the inflation map*

$$\inf : H^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X, \mathbf{T})) \to H^2_{\mathrm{pt}}(G, C(X, \mathbf{T}))$$

sending $[v] \mapsto [\inf v]$ is an isomorphism of abelian groups.

Similarly, if G is any second countable locally compact group, then the inflation map $\inf : H^2_{\mathrm{loc}}(G_{\mathrm{ab}}, C(X, \mathbf{T})) \to H^2_{\mathrm{loc}}(G, C(X, \mathbf{T}))$ *is an isomorphism.*

Proof. It follows from Propositions 1.5 and 1.7 that

$$\inf : H^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X,\mathbf{T})) \to H^2_{\mathrm{pt}}(G, C(X,\mathbf{T}))$$

is surjective. Injectivity follows since $[\inf v] = [1]$ in $H^2_{\mathrm{pt}}(G, C(X,\mathbf{T}))$ if and only if $\mathcal{E}_{\inf v}$ is the trivial bundle (Proposition 1.3(d)). But $\mathcal{E}_{\inf v} \cong \mathcal{E}_v$ by Proposition 1.7, and this implies $[v] = 1$. The second statement is proved similarly using Remark 1.4 and Proposition 1.7. □

The above results can be used to give a description of $H^2(G, C(X,\mathbf{T}))$ along the lines of [5, Section 5]. We restrict our attention to groups which are *smooth* in the sense of Moore (see [8]—an extensive discussion of smooth groups is also given in [5, Section 4]). Recall that if $1 \to Z \to H \to G \to 1$ is a central group extension, then the *transgression map* $\mathrm{tg} : \widehat{Z} \to H^2(G,\mathbf{T})$ is defined by composing the characters of Z with a cocycle $\eta \in Z^2(G, Z)$ corresponding to the extension (recall that such an η is given by $\eta = \partial c$ for any Borel section $c : G \to H$). A group G is called smooth if there exists a central extension as above such that $\mathrm{tg} : \widehat{Z} \to H^2(G,\mathbf{T})$ is bijective, which automatically implies that it is an isomorphism of topological groups. The extension H is then called a *representation group* for G. In particular, if G is smooth, then $H^2(G,\mathbf{T})$ is Hausdorff. The list of smooth groups is quite large; it contains all discrete groups, all compact groups, all compactly generated abelian groups, and all simply connected and connected Lie groups (see [9, 5]).

Suppose now that G is smooth and that $1 \to Z \to H \to G \to 1$ is a representation group of G. Let $\eta \in Z^2(G, Z)$ be a corresponding cocycle. Then any continuous map $\varphi : X \to H^2(G,\mathbf{T}) \cong \widehat{Z}$ determines a cocycle $u_\varphi \in Z^2(G, C(X,\mathbf{T}))$ by defining $u_\varphi(s,t)(x) = \varphi(x) \circ \eta(s,t)$. It is easy to check (and it follows from the proof of [5, Theorem 5.4]) that $\varphi \mapsto [u_\varphi]$ determines a well defined group homomorphism $\Psi_H : C(X, H^2(G,\mathbf{T})) \to H^2(G, C(X,\mathbf{T}))$, which depends only on the particular choice of the representation group, but not on the particular choice of $\eta \in Z^2(G, Z)$ corresponding to this extension.

Proposition 1.10. *Suppose that G is smooth with representation group H. Then the map*

$$\inf \oplus \Psi_H : H^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X,\mathbf{T})) \oplus C(X, H^2(G,\mathbf{T})) \to H^2(G, C(X,\mathbf{T})),$$

sending $(v,\varphi) \mapsto \inf v \cdot u_\varphi$*, is an isomorphism of groups.*

Proof. Let

$$\Phi : H^2(G, C(X,\mathbf{T})) \to C(X, H^2(G,\mathbf{T}))$$

be the evaluation map given by $\Phi([u])(x) := [u(x)]$. Then Φ is a group homomorphism and $\ker\Phi = H^2_{\mathrm{pt}}(G, C(X,\mathbf{T}))$. Since $H^2(G,\mathbf{T})$ is Hausdorff, we can apply Corollary 1.9 to see that $\inf : H^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X,\mathbf{T})) \to H^2_{\mathrm{pt}}(G, C(X,\mathbf{T})) = \ker\Phi$

is an isomorphism. The result then follows from the fact that $\Psi_H : C(X, H^2(G, \mathbf{T})) \to H^2(G, C(X, \mathbf{T}))$ is a splitting homomorphism for Φ (see the proof of [5, Theorem 5.4] for more details). □

Remark 1.11.

(1) If $H^2(G, \mathbf{T})$ is discrete, then every continuous map $\varphi : X \to H^2(G, \mathbf{T})$ is locally constant. Thus, together with the classification of characteristic bundles given in [24], the above result subsumes [25, Corollary 9]—without having to assume that G is abelian as in [25]!

(2) Since every countable discrete group G has a representation group by [8, Theorem 3.1] (see also [15, Corollary 1.3]), the above decomposition applies to such groups. If, in addition, G_{ab} is free abelian, then it follows from the classification of characteristic bundles in [24, Lemma 3] (but see also [13]), that $H^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X, \mathbf{T})) = \{0\}$. Thus Proposition 1.10 implies that

$$H^2(G, C(X, \mathbf{T})) \cong C(X, H^2(G, \mathbf{T})) \tag{1.1}$$

in these cases. Notice that if G is a non-abelian free group, then $H^2(G, \mathbf{T}) = \{0\}$. Then (1.1) implies the well know result that $H^2(G, C(X, \mathbf{T})) = \{0\}$.

2. CENTRAL TWISTED TRANSFORMATION GROUP ALGEBRAS

In this section, we want to describe the central twisted transformation group algebra $C_0(X) \rtimes_{\mathrm{id},u} G$ corresponding to a cocycle $u \in Z^2(G, C(X, \mathbf{T}))$. The basic properties of twisted crossed products are given in [14]. We recall some of the fundamentals here. A (Busby-Smith) *twisted action* (α, u) of G on a C^*-algebra A consists of a strongly measurable map $\alpha : G \to \operatorname{Aut} A$ together with a strictly measurable map $u : G \times G \to UM(A)$ such that

$$\alpha_s \alpha_t = \operatorname{Ad} u(s,t) \circ \alpha_{st} \quad \text{and} \quad \alpha_r(u(s,t))u(r,st) = u(r,s)u(rs,t),$$

for all s, t, $r \in G$. We also require that $\alpha_e = \mathrm{id}$ and $u(e,s) = u(s,e) = 1$ for all $s \in G$. The *twisted crossed product* $A \rtimes_{\alpha,u} G$ is a completion of $L^1(G, A)$ with convolution defined by

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))u(t, t^{-1}s)\, ds.$$

The *covariant representations* of the twisted system (A, G, α, u) are the pairs (π, U) in which $\pi : A \to B(\mathcal{H})$ is a nondegenerate $*$-representation of A and $U : G \to U(\mathcal{H})$ is a measurable map such that

$$U_e = 1, \quad \pi(\alpha_s(a)) = U_s \pi(a) U_s^*, \quad \text{and} \quad U_s U_t = \pi(u(s,t))U_{st}.$$

There is a natural one-to-one correspondence between covariant representations (π, U) of (A, G, α, u) and nondegenerate $*$-representations of $A \rtimes_{\alpha,u} G$ associating (π, U) to its *integrated form*

$$\pi \rtimes U(f) = \int_G \pi(f(s))u_s\,ds, \quad f \in L^1(G, A).$$

The *dual action* $(\alpha, u)^\wedge$ of $\widehat{G}_{\mathrm{ab}}$ on $A \rtimes_{\alpha,u} G$ is defined on the L^1-functions via

$$(\alpha, u)^\wedge_\gamma(f)(s) = \overline{\gamma}(s)f(s).$$

(Just as with the Fourier transform, there is a choice to be made when defining the dual action, and it is a matter of convenience whether one multiplies with $\gamma(s)$ or $\overline{\gamma(s)}$ in the formula above. It should be noted that our convention here is the opposite of that in [4]). There are canonical embeddings $i_A : A \to M(A \rtimes_{\alpha,u} G)$ and $i_G : G \to UM(A \rtimes_{\alpha,u} G)$ given on $f \in L^1(G, A)$ by

$$(i_A(a)f)(t) = a \cdot f(t) \quad \text{and} \quad (i_G(s)f)(t) = \alpha_s(f(s^{-1}t))u(s, s^{-1}t).$$

Then (i_A, i_G) is a covariant homomorphism of (A, G, α, u) into $M(A \rtimes_{\alpha,u} G)$, and for $f \in L^1(G, A)$, $i_A \times i_G(f)$ is the image of f in $A \rtimes_{\alpha,u} G$ under the embedding of $L^1(G, A)$ into its completion $A \rtimes_{\alpha,u} G$.

If $u \in Z^2(G, C(X, \mathbf{T}))$, with $C(X, \mathbf{T})$ regarded as a trivial G-module, then (id, u) is a twisted action of G on $C_0(X)$, and the central twisted transformation group algebras are precisely the crossed products $C_0(X) \rtimes_{\mathrm{id},u} G$ which arise in this way. For a good survey article on twisted transformation group algebras we refer to [12]. Since we usually have $\alpha = \mathrm{id}$ in this section, we shall often write $\hat{u}$ for the dual action $(\mathrm{id}, u)^\wedge$ of $\widehat{G}_{\mathrm{ab}}$.

To further reduce the notational overhead, we won't distinguish between $v \in Z^2(G_{\mathrm{ab}}, C(X, \mathbf{T}))$ and its inflation, $\inf v$ in $Z^2(G, C(X, \mathbf{T}))$.

If $u \in Z^2(G, C(X, \mathbf{T}))$ is a product $u = v \cdot \sigma$ for some $v \in Z^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X, \mathbf{T}))$ and $\sigma \in Z^2(G, C(X, \mathbf{T}))$, we want to obtain a description of $C_0(X) \rtimes_{\mathrm{id},u} G$ in terms of the algebras $C_0(X) \rtimes_{\mathrm{id},v} G_{\mathrm{ab}}$ and $C_0(X) \rtimes_{\mathrm{id},\sigma} G$. (Of course this is motivated by Proposition 1.10.)

We start by giving a description of $C_0(X) \rtimes_{\mathrm{id},v} G_{\mathrm{ab}}$ in terms of the bundle $p : \mathcal{E}_v \to X$. The following result is well known. It follows from the work of Smith [23] and the discussion given in [20, Remark 3.11]. However, we feel it worthwhile to include the following direct (and much shorter) proof.

Lemma 2.1. *Suppose that G is abelian and $v \in Z^2_{\mathrm{pt}}(G, C(X, \mathbf{T}))$. Then the dual system $(C_0(X) \rtimes_{\mathrm{id},v} G, \widehat{G}, \hat{v})$ is isomorphic to $(C_0(\mathcal{E}_v), \widehat{G}, \tau)$, where $\tau : \widehat{G} \to \mathrm{Aut}\, C_0(\mathcal{E}_v)$ is given by $\tau_\gamma(\psi)(f, x) = \psi(\overline{\gamma} \cdot f, x)$.*

Proof. Since v is pointwise trivial, it is symmetric, i.e., $v(s, t) = v(t, s)$ for every s, $t \in G$. Using this it is easy to check that convolution on $L^1(G, C(X))$ is commutative. Thus, $C_0(X) \rtimes_{\mathrm{id},v} G$ is commutative, and isomorphic to $C_0((C_0(X) \rtimes_{\mathrm{id},v} G)^\wedge)$ via the Gelfand-Naimark theorem.

Thus we have to show that $\mathcal{E}_v$ is $\hat{G}$-equivariantly homeomorphic to

$$(C_0(X) \rtimes_{\mathrm{id},u} G)^\wedge.$$

It is straightforward to check that the one-dimensional covariant representations are precisely the pairs (ε_x, f) with $(f, x) \in \mathcal{E}_v$, where $\varepsilon_x : C_0(X) \to \mathbf{C}$ denotes evaluation at x. Thus we get a canonical bijection $\Phi : \mathcal{E}_v \to (C_0(X) \rtimes_{\mathrm{id},u} G)^\wedge$, given by

$$(f, x) \mapsto \varepsilon_x \rtimes f.$$

Since the action of a character $\gamma \in \hat{G}$ on a covariant representations (π, U) is given by $(\pi, \gamma \cdot U)$, it follows that Φ is $\hat{G}$-equivariant.

So it only remains to check that Φ is continuous and open. For continuity, let (f_n, x_n) converge to (f, x) in $\mathcal{E}_u$. Then $\varepsilon_{x_n} \rtimes f_n(h)$ converges to $\varepsilon_x \rtimes f(h)$ for all $h \in C_c(G \times X)$ by Lebesgue's dominated convergence theorem. Since $C_c(G \times X)$ is dense in $C_0(X) \rtimes_{\mathrm{id},u} G$ this implies that $\varepsilon_{x_n} \rtimes f_n$ converges to $\varepsilon_x \rtimes f$ in the weak-$*$ topology. This proves continuity.

To prove openness, we could appeal to some deep results of Olesen and Raeburn such as [11, Corollary 2.3]. Instead, we give a more elementary argument. We suppose that $\varepsilon_{x_n} \rtimes f_n$ converges to $\varepsilon_x \rtimes f$ in $(C_0(X) \rtimes_{\mathrm{id},u} G)^\wedge$. In view of Proposition 1.3, it will suffice to show that $\{(\underline{f}_n, x_n)\}$ converges to $(\underline{f}, x)$ in $\mathcal{E}_{\underline{v}}$. Let $h \in C_c(G \times X)$ be such that $\varepsilon_x \rtimes f(h) = 1$. Then if $\psi \in C_c(X)$, we note that

$$\varepsilon_{x_n} \rtimes f_n(i_{C_0(X)}(\psi)h) = \psi(x_n)(\varepsilon_{x_n} \rtimes f_n(h)).$$

Since $\varepsilon_{x_n} \rtimes f_n(i_{C_0(X)}(\psi)h)$ converges to $\varepsilon_x \rtimes f(i_{C_0(X)}(\psi)h)$, we must have $\psi(x_n) \to \psi(x)$ for all $\psi \in C_c(X)$. Thus $x_n \to x$. Thus we may assume there is a $\psi \in C_c(X)$ such that $\psi(x_n) = 1$ for all n. If $\varphi \in L^1(G)$ and we define $h \in L^1(G, C_0(X))$ by $h(s) = \varphi(s)\psi$, then for all $\varphi \in L^1(G)$ we have

$$\int_G \varphi(s) f_n(s)\, d\mu(s) = \varepsilon_{x_n} \rtimes f_n(h) \to \varepsilon_x \rtimes f(h) = \int_G \varphi(s) f(s)\, d\mu(s). \tag{2.1}$$

We claim (2.1) implies that $\underline{f}_n \to \underline{f}$ in $\underline{C}^1(G, \mathbf{T})$. In view of [9, Proposition 6], it will suffice to show that

$$\int_K |f_n(s) - f(s)|\, d\mu(s) \to 0 \tag{2.2}$$

for each compact set $K \subset G$. Since $|f(s) f_n(s)| = 1$, we have

$$|f_n(s) - f(s)|^2 = |1 - \overline{f(s)} f_n(s)|^2 \le 2 - 2\,\mathrm{Re}(\overline{f(s)} f_n(s)).$$

But then by Hölder's inequality

$$(2.3)\qquad \Big(\int_K |f_n(s) - f(s)|\,d\mu(s)\Big)^2 \\ \le \mu(K)\int_K |f_n(s) - f(s)|^2\,d\mu(s) \\ \le 2\mu(K)\,\mathrm{Re}\Big(\int_K 1 - \overline{f(s)}f_n(s)\,d\mu(s)\Big) \\ = 2\mu(K)\,\mathrm{Re}\Big(\int_G \mathbb{1}_K(s) - \mathbb{1}_K(s)\overline{f(s)}f_n(s)\,d\mu(s)\Big).$$

Since $\mathbb{1}_K \cdot \overline{f} \in L^1(G)$, (2.1) implies that (2.3) goes to 0, and the result follows. □

To state our main result in this section, we want to recall some constructions from [4, Section 3]. A $C_0(X)$*-algebra* is a C^*-algebra A equipped with a fixed nondegenerate $*$-homomorphism Φ from $C_0(X)$ into the center $ZM(A)$ of the multiplier algebra $M(A)$ of A. This allows us to view A as a $C_0(X)$-module, and we shall usually write $\varphi \cdot a$ in place of $\Phi(\varphi)a$. A twisted action (α, u) of G on A is called $C_0(X)$*-linear* if $\alpha_s(f \cdot a) = f \cdot \alpha_s(a)$ for all $s \in G$, $f \in C_0(X)$ and $a \in A$. A $C_0(X)$-algebra A should be thought of as an algebra of (semi-continuous) sections of a bundle of C^*-algebras A_x, $x \in X$, where $A_x \cong A/(C_0(X \setminus \{x\}) \cdot A)$: the image of $a \in A$ in A_x is denoted by $a(x)$. Then $C_0(X)$-linearity means that α induces compatible twisted actions (α^x, u^x) on the fibres A_x. The crossed product $A \rtimes_{\alpha,u} G$ of a $C_0(X)$-linear twisted action is a $C_0(X)$-algebra with respect to the composition $i_A \circ \Phi : C_0(X) \to M(A \rtimes_{\alpha,u} G)$, and the fibres are just the crossed products $A_x \rtimes_{\alpha^x,u^x} G$. In particular, the dual action of $\widehat{G}_{\mathrm{ab}}$ on $A \rtimes_{\alpha,u} G$ is again $C_0(X)$-linear and restricts to the respective dual actions on the fibres $A_x \rtimes_{\alpha^x,u^x} G$. It is worth mentioning that the dual $\hat{A}$ (respectively, the primitive ideal space $\operatorname{Prim} A$) of a $C_0(X)$-algebra A has an induced bundle structure $q : \hat{A} \to X$ (resp. $q : \operatorname{Prim} A \to X$) with fibres $\hat{A}_x$ (resp. $\operatorname{Prim} A_x$).

Suppose that L is an *abelian* group, that $\alpha : L \to \operatorname{Aut} A$ is a $C_0(X)$-linear (untwisted) action and that $p : Z \to X$ is a locally compact free and proper L-bundle. Since L is abelian, α^{-1} is an action and we can form the *induced algebra* $\operatorname{Ind}_L^Z(A, \alpha^{-1})$ which is the set of bounded continuous functions $F : Z \to A$ satisfying

$$\alpha_\ell(F(z)) = F(\ell^{-1} \cdot z), \quad \text{for all } s \in G \text{ and } z \in Z, \quad \text{and}$$

such that $z \mapsto \|F(z)\|$ vanishes at infinity on $X = Z/L$. As in [4], we'll write $Z \times_L A$ in place of $\operatorname{Ind}_L^Z(A, \alpha^{-1})$ to stress the analogy with classical topological bundle constructions (see [4, Definition 3.1(s)]). Note that $Z \times_L A$ is a C^*-algebra when equipped with the pointwise operations and the supremum norm. Moreover, $Z \times_L A$ carries a strongly continuous action $\operatorname{Ind}\alpha$ of L given by

$$(\operatorname{Ind}\alpha)_\ell(F)(z) = F(\ell^{-1} \cdot z) = \alpha_\ell(F(z)).$$

If (A, L, α) is a $C_0(X)$-system, then there is a $C_0(X \times X)$-action on $Z \times_L A$ given by

$$(h \cdot F)(z)(x) = h(p(z), x)F(z)(x), \quad h \in C_0(X \times X),$$

and the *L-fibre product* $Z * A$ is defined as the restriction of $Z \times_L A$ to the diagonal $\Delta(X) = \{(x, x) \mid x \in X\}$. Identifying X with $\Delta(X)$ gives $Z * A$ the structure of a $C_0(X)$-algebra and $\operatorname{Ind}\alpha$ restricts to a $C_0(X)$-linear action $Z * \alpha$ of L on $Z * A$. Further details on this construction and those in the previous two paragraphs can be found in [4, Section 3].

Note that the construction of $Z * A$ is the C^*-algebraic analogue of the usual construction of L-fibre products of topological L-bundles given by

$$Z * Y = \{(z, y) \in Z \times Y \mid p(z) = q(y)\}/L,$$

where $q : Y \to X$ is assumed to be a topological bundle over X equipped with a compatible L-action on the fibres Y_x, and where the quotient space is taken by the anti-diagonal action $\ell \cdot (z, y) = (\ell z, \ell^{-1} y)$. In particular, we always have $(Z * A)^\wedge \cong Z * \hat{A}$ and $\operatorname{Prim}(Z * A) \cong Z * \operatorname{Prim} A$ with respect to the bundle structures of $\hat{A}$ and $\operatorname{Prim} A$ (see [4, Proposition 3.5]).

If A and B are $C_0(X)$-algebras, and if $\alpha : L \to \operatorname{Aut} A$ and $\beta : L \to \operatorname{Aut} B$ are $C_0(X)$-linear actions, then (A, L, α) and (B, L, β) are $C_0(X)$-*Morita equivalent* if there exists an $A - B$ imprimitivity bimodule X satisfying[2] $\varphi \cdot \xi = \xi \cdot \varphi$ for all $\varphi \in C_0(X)$ and $\xi \in \mathsf{X}$, and such that X carries a $C_0(X)$-linear action $\delta : L \to \operatorname{Aut} \mathsf{X}$ such that

$${}_A\langle \delta_\ell(\xi) \,, \delta_\ell(\eta) \rangle = \alpha_\ell({}_A\langle \xi \,, \eta \rangle) \quad \text{and} \quad \langle \delta_\ell(\xi) \,, \delta_\ell(\eta) \rangle_B = \beta_\ell(\langle \xi \,, \eta \rangle_B)$$

for all ξ, $\eta \in \mathsf{X}$ and $\ell \in L$. Note that $C_0(X)$-Morita equivalence implies equivalence of the topological L-bundles $\hat{A}$ and $\hat{B}$ (resp. $\operatorname{Prim} A$ and $\operatorname{Prim} B$)—see [21, Proposition 5.7].

We can now state the main result of this section.

Theorem 2.2 (cf. [4, Theorem 5.3])**.** *Suppose that u has the form $v \cdot \sigma$ with $v \in Z^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X, \mathbf{T}))$ and $\sigma \in Z^2(G, C(X, \mathbf{T}))$. Let $p : Z_v \to X$ be the free and proper $\widehat{G}_{\mathrm{ab}}$-bundle associated to v as in Definition* 1.2*. Then the systems*

$$(C_0(X) \rtimes_{\mathrm{id}, u} G,\ \widehat{G}_{\mathrm{ab}},\ \hat{u}) \quad \text{and} \quad (Z_v * (C_0(X) \rtimes_{\mathrm{id}, \sigma} G),\ \widehat{G}_{\mathrm{ab}},\ \mathcal{E}_v * \hat{\sigma})$$

are $C_0(X)$-Morita equivalent systems.

[2] The left and right actions of $C_0(X)$ on X are obtained from extending the left and right actions of A and B to $M(A)$ and $M(B)$, respectively.

Remark 2.3. If v is actually locally trivial, then a stronger result holds. It follows from [4, Theorem 5.3] that there exists a $\widehat{G}_{\mathrm{ab}}$-equivariant and $C_0(X)$-linear *isomorphism* between $C_0(X) \rtimes_{\mathrm{id},u} G$ and $Z_v * (C_0(X) \rtimes_{\mathrm{id},\sigma} G)$. The action δ appearing in that theorem, and which is used to compare (id, u) with (id, σ), is the locally unitary action $\delta : G \to \mathrm{Aut}\, C_0(X, \mathcal{K})$ corresponding to $\inf v \in Z^2_{\mathrm{loc}}(G, C(X, \mathbf{T}))$ as constructed in [6, Proposition 3.1]. However, the proof of [4, Theorem 5.3] requires localizations of both systems and breaks down if v is only assumed to be pointwise trivial. We do not know whether the stronger result of $C_0(X)$-linear isomorphism also holds in the situation of Theorem 2.2 above.

In what follows, we denote by $A \otimes_{C_0(X)} B$ the *maximal* balanced tensor product of $C_0(X)$-algebras A and B (see [2] and [4, Section 2]). It is obtained by restriction of the $C_0(X \times X)$-algebra $A \otimes_{\max} B$ to the diagonal $\Delta(X)$ and therefore carries a canonical structure as $C_0(X)$-algebra. If (α, u) and (β, v) are $C_0(X)$-linear twisted actions on A and B, respectively, then the diagonal twisted action $(\alpha \otimes \beta, u \otimes v)$ on $A \otimes_{\max} B$ restricts to a $C_0(X)$-linear twisted action $(\alpha \otimes_X \beta, u \otimes_X v)$ on $A \otimes_{C_0(X)} B$ (see [4, Section 4] for more details).

Suppose now that $\alpha : L \to \mathrm{Aut}\, A$ is a $C_0(X)$-linear (untwisted) action of the abelian group L, and let $p : Z \to X$ be a free and proper L-bundle over X. Let τ denote the action of L on $C_0(Z)$ given by $\tau_\ell(\psi)(z) = \psi(\ell^{-1} \cdot z)$. Then it follows that the crossed product $(C_0(Z) \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} L$ is a $C_0(X \times X)$-algebra, and the restriction to the diagonal $\Delta(X)$ is isomorphic to $(C_0(Z) \otimes_{C_0(X)} A) \rtimes_{\tau \otimes_X \alpha^{-1}} L$. We define a $C_0(X \times X)$-linear action $\underline{\alpha}$ of L on $(C_0(Z) \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} L$ by the formula

$$\underline{\alpha}_\ell(f)(s) = \mathrm{id} \otimes \alpha_\ell(f(s)) \quad \text{for } f \in L^1(L, C_0(Z) \otimes_{C_0(X)} A) \text{ and } \ell, s \in L.$$

Note that this extends to an automorphism of the crossed product since $\mathrm{id} \otimes_X \alpha_\ell$ commutes with $\tau_h \otimes \alpha_{h^{-1}}$ for all $\ell, h \in L$. Since $\underline{\alpha}$ is $C_0(X \times X)$-linear, it restricts to a $C_0(X)$-linear action $\underline{\alpha}^X$ of L on $(C_0(Z) \otimes_{C_0(X)} A) \rtimes_{\tau \otimes_X \alpha^{-1}} L$. The proof of Theorem 2.2 depends heavily on the following result.

Proposition 2.4. *In the situation above, the systems*

$$((C_0(Z) \otimes A) \rtimes_{\tau \otimes \alpha^{-1}} L,\ L,\ \underline{\alpha}) \quad \text{and} \quad (Z \times_L A,\ L,\ \mathrm{Ind}\, \alpha)$$

are $C_0(X \times X)$-Morita equivalent, and the systems

$$((C_0(Z) \otimes_{C_0(X)} A) \rtimes_{\tau \otimes_X \alpha^{-1}} L,\ L,\ \underline{\alpha}^X) \quad \text{and} \quad (Z * A,\ L,\ Z * \alpha)$$

are $C_0(X)$-Morita equivalent.

Proof. Note that it is enough to prove the first Morita equivalence, since the second will follow from the first by restricting to the diagonal $\Delta(X)$.

The proof of the first equivalence is based on the proof of [19, Theorem 2.2]. Note that our algebra $Z \times_L A$ is equal to the algebra $GC(Z, A)^{\tau \otimes \alpha^{-1}}$ in the language of [19]. The proof of [19, Theorem 2.2] shows that we obtain a $(C_0(Z)\otimes A)\rtimes_{\tau\otimes\alpha^{-1}} L - Z\times_L A$ imprimitivity bimodule X by taking the completion of $C_c(Z,A)$ with respect to the left and right $(C_0(Z)\otimes A)\rtimes L$- and $Z\times_L A$-valued inner products and left and right actions of $C_c(L, C_0(Z)\otimes A)\subset (C_0(Z)\otimes A)\rtimes L$ and $Z\times_L A$ on X given by the formulas

$$\begin{aligned} {}_{C_c(L,C_0(Z)\otimes A)}\langle \xi\,,\eta\rangle(\ell,z) &= \xi(z)\alpha_{\ell^{-1}}(\eta(\ell^{-1}z)^*),\\ \langle \xi\,,\eta\rangle_{Z\times_L A} &= \int_L \alpha_\ell(\xi(\ell z)^*\eta(\ell z))\,d\mu(\ell),\\ f\cdot\xi(z) &= \int_L f(\ell,z)\alpha_\ell(\xi(\ell z))\,d\mu(\ell),\\ \xi\cdot F(z) &= \xi(z)F(z), \end{aligned}$$

where $F\in Z\times_L A$, $f\in C_c(L, C_0(Z)\otimes A)$ and $\xi,\eta\in C_c(Z,A)$. To see that these formulas are equivalent to those given in [19], note that our action α^{-1} plays the role of the action β in [19], and that we may replace ℓ by ℓ^{-1} in all integrals above since L is abelian and therefore unimodular.

It is now easy to check that the $C_0(X\times X)$-actions on both sides of X are given by the formula

$$\varphi\cdot\xi(z)(x) = \varphi(p(z),x)\xi(z)(x),\quad \varphi\in C_0(X\times X) \text{ and } \xi\in C_c(Z,A).$$

Moreover, if we define

$$\delta_\ell(\xi)(z) = \alpha_\ell(\xi(z)),\quad \text{for } \xi\in C_c(Z,A),\ \ell\in L, \text{ and } z\in Z,$$

then it is straightforward to check that δ extends to a $C_0(X\times X)$-linear action on the completion X of $C_c(Z,A)$ which is compatible with the actions $\underline{\alpha}$ and $\operatorname{Ind}\alpha$. □

Suppose that (α,u) is a twisted action of G on A and that N is a closed normal subgroup of G. Then, depending on a choice of a Borel section $c: G/N\to G$, Packer and Raeburn [14, Theorem 4.1] showed that there is a twisted action (β,w) of G/N on the crossed product $A\rtimes_{\alpha,u} N$ such that

$$A\rtimes_{\alpha,u} G\cong (A\rtimes_{\alpha,u} N)\rtimes_{\beta,w} G/N.$$

We want to apply their result in the very special case where G is the direct product $L\times N$. In this case, we get a particularly nice description of the twisted action (β,w) and the isomorphism Φ in the following proposition.

Proposition 2.5. *Suppose that $G = L\times N$ is the direct product of two second countable locally compact groups L and N and let (α,u) be a twisted action of G on*

A. Then there is a twisted action (β, w) of L on $A \rtimes_{\alpha,u} N$ given by the formulas

$$\begin{aligned} \beta_\ell(f)(n) &= \alpha_\ell(f(n))u((\ell,e),(e,n))u((e,n),(\ell,e))^*, & f \in L^1(N,A), \text{ and} \\ w(\ell,h) &= i_A(u((\ell,e),(h,e))), & \ell, h \in L. \end{aligned}$$

With this action, there is an isomorphism between $A \rtimes_{\alpha,u} G$ and $(A \rtimes_{\alpha,u} N) \rtimes_{\beta,w} L$ which restricts to a homomorphism of L^1-algebras $\Phi : L^1(L \times N, A) \to L^1(L, L^1(N,A))$ given by the formula

$$\Phi(f)(\ell)(n) = f(\ell,n)u((e,n),(\ell,e))^*.$$

Proof. The proof is basically a consequence of [14, Theorem 4.1]—in particular, the formula for the action (β, w) directly follows from the formulas as given in [14] with respect to the cross section $L \to L \times N$ defined by $\ell \mapsto (\ell, e)$. We only have to check that the isomorphism is given on the L^1-algebras by the above formula. For this let (i_A, i_N) denote the canonical embeddings of (A, N) into $M(A \rtimes_{\alpha,u} N)$ and, similarly, let $(j_{A\rtimes N}, j_L)$ denote the embeddings of $(A \rtimes_{\alpha,u} N, L)$ into $M((A \rtimes_{\alpha,u} N) \rtimes_{\beta,w} L)$. Then it is shown on [14, p. 307] that the pair (k_A, k_G) defined by

$$k_A = j_{A\rtimes N} \circ i_A \quad \text{and} \quad k_G((\ell,n)) = j_{A\rtimes N}(i_A(u((e,n),(\ell,e))^*)i_N(n)j_L(\ell)$$

is a covariant homomorphism of (A, G, α, u) into $M((A \rtimes_{\alpha,u} N) \rtimes_{\beta,w} L)$ such that the integrated form $k_A \rtimes k_G$ is the desired isomorphism. Thus for $f \in L^1(L \times N, A)$ we get

$$\begin{aligned} &k_A \rtimes k_G(f) \\ &\quad = \int_G k_A(f(\ell,n))k_G((\ell,n))\, d\mu_L(\ell,n) \\ &\quad = \int_L \int_N j_{A\rtimes N}(i_A(f((\ell,n))u((e,n),(\ell,e))^*)i_N(n))j_L(\ell)\, d\mu_N(n)\, d\mu_L(\ell) \\ &\quad = \int_L j_{A\rtimes N}(\Phi(f)(\ell))j_L(\ell)\, d\mu_L(\ell) = j_{A\rtimes N} \rtimes j_L(\Phi(f)). \end{aligned}$$

This completes the proof. ◻

Remark 2.6. Note that it follows directly from the formula of the isomorphism $A \rtimes_{\alpha,u} G \cong (A \rtimes_{\alpha,u} N) \rtimes_{\beta,w} L$ given in the proposition that this isomorphism is $\widehat{L}_{\mathrm{ab}} \times \widehat{N}_{\mathrm{ab}}$-equivariant. Moreover, if A is a $C_0(X)$-algebra and (α, u) is $C_0(X)$-linear, then it follows from the definition of (β, w), that it is $C_0(X)$-linear and the formula for the above isomorphism shows that it is $C_0(X)$-linear, too.

We shall also need the following result.

Proposition 2.7. *Suppose that* (α, u) *and* (β, v) *are* $C_0(X)$*-linear twisted actions of* L *and* N *on the* $C_0(X)$*-algebras* A *and* B*, respectively. Define the twisted action* $(\alpha \otimes_X \beta, u \otimes_X v)$ *of* $L \times N$ *on* $A \otimes_{C_0(X)} B$ *in the obvious way. Then there exists a* $C_0(X)$*-linear and* $\widehat{L}_{\mathrm{ab}} \times \widehat{N}_{\mathrm{ab}}$*-equivariant isomorphism between*

$$(A \otimes_{C_0(X)} B) \rtimes_{\alpha \otimes_X \beta, u \otimes_X v} L \times N \quad \text{and} \quad (A \rtimes_{\alpha,u} L) \otimes_{C_0(X)} (B \rtimes_{\beta,v} N).$$

Proof. Restricting the twisted action $(\alpha \otimes_X \beta, u \otimes_X v)$ to the subgroup $N \cong \{e\} \times N$ of $L \times N$ gives the action $(\mathrm{id} \otimes_X \beta, 1 \otimes_X v)$ of N on $A \otimes_{C_0(X)} B$. It follows from [4, Proposition 4.3] that $(A \otimes_{C_0(X)} B) \rtimes_{\mathrm{id} \otimes_X \beta, 1 \otimes_X v} N$ is $C_0(X)$-linearly and $\widehat{N}_{\mathrm{ab}}$-equivariantly isomorphic to $A \otimes_{C_0(X)} (B \rtimes_{\beta,v} N)$. Using the formula for this isomorphism as given [4, Propsition 4.3], it follows from Proposition 2.5 that the decomposition action of L on $(A \otimes_{C_0(X)} B) \rtimes_{\mathrm{id} \otimes_X \beta, 1 \otimes_X v} N$ corresponds to the twisted action $(\alpha \otimes_X \mathrm{id}, u \otimes_X 1)$ of L on $A \otimes_{C_0(X)} (B \rtimes_{\beta,v} N)$. The result then follows from another application of [4, Proposition 4.3]. □

We are now ready for the proof of Theorem 2.2. The proof relies on the above decomposition results, and the Takesaki-Takai duality for twisted actions of abelian groups.

Proof of Theorem 2.2. We consider the diagonal twisted action $(\mathrm{id} \otimes_X \mathrm{id}, v \otimes_X \sigma)$ of $G_{\mathrm{ab}} \times G$ on $C_0(X) \otimes_{C_0(X)} C_0(X) \cong C_0(X)$. If we restrict this action to the diagonal $\Delta(G) = \{(\dot{s}, s) \mid s \in G\} \subset G_{\mathrm{ab}} \times G$ and identify G with $\Delta(G)$ via $s \mapsto (\dot{s}, s)$, then it follows that the isomorphism $C_0(X) \otimes_{C_0(X)} C_0(X) \to C_0(X)$, given on elementary tensors by $\varphi \otimes \psi \to \varphi \cdot \psi$, carries $(\mathrm{id}_{\widehat{G}_{\mathrm{ab}}} \otimes_X \mathrm{id}_G, v \otimes_X \sigma)$ to the twisted action $(\mathrm{id}_G, v \cdot \sigma) = (\mathrm{id}_G, u)$. Thus we get a natural $C_0(X)$-linear isomorphism

$$C_0(X) \rtimes_{\mathrm{id},u} G \cong (C_0(X) \otimes_{C_0(X)} C_0(X)) \rtimes_{\mathrm{id} \otimes_X \mathrm{id}, v \otimes_X \sigma} \Delta(G),$$

which transforms the dual action of $\widehat{G}_{\mathrm{ab}}$ to the dual action of $\Delta(G)^{\wedge}_{\mathrm{ab}}$.

For the crossed product by the full group $G_{\mathrm{ab}} \times G$, it follows from Proposition 2.5 that we have a $C_0(X)$-linear and $\widehat{G}_{\mathrm{ab}} \times \widehat{G}_{\mathrm{ab}}$-equivariant isomorphism

$$\begin{aligned}(C_0(X) \otimes_{C_0(X)} C_0(X)) &\rtimes_{\mathrm{id} \otimes_X \mathrm{id}, v \otimes_X \sigma} G_{\mathrm{ab}} \times G \\ &\cong (C_0(X) \rtimes_{\mathrm{id},v} G_{\mathrm{ab}}) \otimes_{C_0(X)} (C_0(X) \rtimes_{\mathrm{id},\sigma} G).\end{aligned}$$

By Lemma 2.1, the algebra $C_0(X) \rtimes_{\mathrm{id},v} G_{\mathrm{ab}}$ is $\widehat{G}_{\mathrm{ab}}$-equivariantly isomorphic to $C_0(\mathcal{E}_v)$, and this isomorphism is clearly $C_0(X)$-linear. Thus we obtain a $C_0(X)$-linear and $\widehat{G}_{\mathrm{ab}} \times \widehat{G}_{\mathrm{ab}}$-equivariant isomorphism between

$$\begin{gathered}(C_0(X) \otimes_{C_0(X)} C_0(X)) \rtimes_{\mathrm{id} \otimes_X \mathrm{id}, v \otimes_X \sigma} G_{\mathrm{ab}} \times G \quad \text{and} \\ C_0(\mathcal{E}_v) \otimes_{C_0(X)} (C_0(X) \rtimes_{\mathrm{id},\sigma} G).\end{gathered}$$

We now split $G_{\rm ab} \times G$ as the product $G_{\rm ab} \times \Delta(G)$ via the isomorphism $(\dot{s}, t) \mapsto (\dot{t}^{-1}\dot{s}, (\dot{t}, t))$. Iterating the crossed product with respect to this decomposition of $G_{\rm ab} \times G$ now provides $C_0(X)$-linear isomorphisms

$$\begin{aligned} C_0(\mathcal{E}_v) &\otimes_{C_0(X)} (C_0(X) \rtimes_{\mathrm{id},\sigma} G) \\ &\cong (C_0(X) \otimes_{C_0(X)} C_0(X)) \rtimes_{\mathrm{id}\otimes_X \mathrm{id}, v\otimes_X \sigma} G_{\rm ab} \times G \\ &\cong ((C_0(X) \otimes_{C_0(X)} C_0(X)) \rtimes_{\mathrm{id}\otimes_X \mathrm{id}, v\otimes_X \sigma} \Delta(G)) \rtimes_{\beta,w} G_{\rm ab} \\ &\cong (C_0(X) \rtimes_{\mathrm{id},u} G) \rtimes_{\beta,w} G_{\rm ab}. \end{aligned}$$

We need to compare the natural $\widehat{G}_{\rm ab} \times \widehat{G}_{\rm ab}$-action on $(C_0(X) \rtimes_u G) \rtimes_{\beta,w} G_{\rm ab}$ with the $\widehat{G}_{\rm ab} \times \widehat{G}_{\rm ab}$-action on $C_0(\mathcal{E}_v) \otimes_{C_0(X)} (C_0(X) \rtimes_{\mathrm{id},\sigma} G)$ under the above isomorphism. Indeed, if we identify $\widehat{G}_{\rm ab}$ with $\Delta(G)^\wedge_{\rm ab}$ via $\chi(\dot{t}, \dot{t}) = \chi(\dot{t})$ (as we do in the last isomorphism above), we see that our given isomorphism $\psi : G_{\rm ab} \times G \to G_{\rm ab} \times \Delta(G)$ induces the isomorphism $\hat{\psi} : \widehat{G}_{\rm ab} \times \Delta(G)^\wedge_{\rm ab} \to \widehat{G}_{\rm ab} \times \widehat{G}_{\rm ab}$ given by

$$\hat{\psi}(\gamma, \chi)(\dot{s}, \dot{t}) = \gamma(\dot{t}^{-1}\dot{s})\chi(\dot{t}).$$

It follows from this that the dual action $\hat{u}$ of $\widehat{G}_{\rm ab}$ on $C_0(X) \rtimes_{\mathrm{id},u} G$ (and then extended to $(C_0(X) \rtimes_u G) \rtimes_{\beta,w} G_{\rm ab}$) corresponds to the action $\mathrm{id} \otimes_X \hat{\sigma}$ of $\widehat{G}_{\rm ab}$ on $C_0(\mathcal{E}_v) \otimes_{C_0(X)} (C_0(X) \rtimes_{\mathrm{id},\sigma} G)$, while the dual action $(\beta, w)^\wedge$ of $\widehat{G}_{\rm ab}$ on $(C_0(X) \rtimes_u G) \rtimes_{\beta,w} G_{\rm ab}$ corresponds to the action $\tau \otimes_X \hat{\sigma}^{-1}$ of $G_{\rm ab}$ on $C_0(\mathcal{E}_v) \otimes_{C_0(X)} (C_0(X) \rtimes_{\mathrm{id},\sigma} G)$. Thus it follows from Proposition 2.4 that we get a $C_0(X)$-Morita equivalence between the systems

$$\begin{aligned} (((C_0(X) \rtimes_{\mathrm{id},u} G) \rtimes_{\beta,w} G_{\rm ab}) \rtimes_{(\beta,w)^\wedge} \widehat{G}_{\rm ab},\ \widehat{G}_{\rm ab},\ \underline{\hat{u}}) \\ \cong ((C_0(\mathcal{E}_v) \otimes_{C_0(X)} (C_0(X) \rtimes_{\mathrm{id},\sigma} G)) \rtimes_{\tau\otimes\hat{\sigma}^{-1}} \widehat{G}_{\rm ab},\ \widehat{G}_{\rm ab},\ \underline{\hat{\sigma}}^X) \end{aligned}$$

and

$$(\mathcal{E}_v * (C_0(X) \rtimes_{\mathrm{id},\sigma} G),\ \widehat{G}_{\rm ab},\ \mathcal{E}_v * \hat{\sigma}),$$

where $\underline{\hat{u}}$ denotes the canonical action induced by $\hat{u}$ on $((C_0(X) \rtimes_u G) \rtimes_{\beta,w} G_{\rm ab}) \rtimes_{(\beta,w)^\wedge} \widehat{G}_{\rm ab}$. Now the Takesaki-Takai theorem for twisted actions (see [17, Theorem 3.1]) implies that

$$((C_0(X) \rtimes_{\mathrm{id},u} G) \rtimes_{\beta,w} G_{\rm ab}) \rtimes_{(\beta,w)^\wedge} \widehat{G}_{\rm ab} \cong (C_0(X) \rtimes_{\mathrm{id},u} G) \otimes \mathcal{K}(L^2(G_{\rm ab})),$$

and this isomorphism carries the action $\underline{\hat{u}}$ to $\hat{u} \otimes \mathrm{id}_{\mathcal{K}}$. Since the systems

$$((C_0(X) \rtimes_{\mathrm{id},u} G) \otimes \mathcal{K}(L^2(G_{\rm ab})),\ \widehat{G}_{\rm ab},\ \hat{u} \otimes \mathrm{id}_{\mathcal{K}}) \quad \text{and}$$
$$(C_0(X) \rtimes_{\mathrm{id},u} G,\ \widehat{G}_{\rm ab},\ \hat{u})$$

are clearly $C_0(X)$-Morita equivalent, the result follows. □

We are now going to use Theorem 2.2 to give a bundle theoretic description of $C_0(X) \rtimes_{\mathrm{id},u} G$ when G is smooth. Then Proposition 1.10 implies that we obtain a factorization $u = v \cdot u_\varphi$, where $v \in Z^2_{\mathrm{pt}}(G_{\mathrm{ab}}, C(X,\mathbf{T}))$ and u_φ is obtained by pulling back a given cocycle $\eta \in Z^2(G,Z)$ corresponding to a representation group $1 \to Z \to H \to G \to 1$ for G via the continuous map $\varphi : X \to H^2(G,\mathbf{T}) \cong \widehat{Z}$ defined by $x \mapsto [u(x)]$. Recall from [19, 4] that if A is a $C_0(Y)$-algebra and if $\varphi : X \to Y$ is a continuous map, then the pull-back $\varphi^*(A)$ is defined as the balanced tensor product $C_0(X) \otimes_{C_0(Y)} A$, where $C_0(X)$ is viewed as a $C_0(Y)$-algebra via $\varphi : X \to Y$. Note that $\varphi^*(A)$ becomes a $C_0(X)$-algebra via the canonical embedding $C_0(X) \to M(C_0(X) \otimes_{C_0(Y)} A)$. Moreover, if $\alpha : L \to \operatorname{Aut} A$ is a $C_0(Y)$-linear action on A, then $\varphi^*(\alpha) = \mathrm{id} \otimes_Y \alpha$ is a $C_0(X)$-linear action on $\varphi^*(A)$. The following description of $C_0(X) \rtimes_{\mathrm{id},u_\varphi} G$ follows from [4, Lemmas 6.3 and 6.5].

Proposition 2.8. *Let $1 \to Z \to H \to G \to 1$ be a representation group for G. Let $C^*(H)$ be viewed as a $C_0(\widehat{Z})$-algebra via the canonical embedding $C_0(\widehat{Z}) \cong C^*(Z) \to ZM(C^*(H))$ given by convolution. Let $\varphi : X \to H^2(G,\mathbf{T}) \cong \widehat{Z}$ be a continuous map, and let $u_\varphi \in Z^2(G, C(X,\mathbf{T}))$ be the cocycle defined in Proposition 1.10 (with respect to* any *cocycle $\eta \in Z^2(G,Z)$ corresponding to H). Further, let δ denote the dual action of $\widehat{G}_{\mathrm{ab}} = \widehat{H}_{\mathrm{ab}}$ on $C^*(H)$. Then the systems*

$$(C_0(X) \rtimes_{\mathrm{id},u_\varphi} G,\ \widehat{G}_{\mathrm{ab}},\ \hat{u}_\varphi) \quad \text{and} \quad (\varphi^*(C^*(H)),\ \widehat{G}_{\mathrm{ab}},\ \varphi^*(\delta))$$

are $C_0(X)$-isomorphic.

We can now gather our results to obtain a general description of the bundle structure of $C_0(X) \rtimes_{\mathrm{id},u} G$ in terms of a given representation group H for G.

Theorem 2.9. *Suppose that G is smooth with representation group $1 \to Z \to H \to G \to 1$, and that $u \in Z^2(G, C(X,\mathbf{T}))$. Let $u_\varphi \in Z^2(G, C(X,\mathbf{T}))$ be as above with $\varphi(x) = [u(x)]$ for all $x \in X$. If $v := u \cdot \overline{u}_\varphi$, then $v \in Z^2_{\mathrm{pt}}(G, C(X,\mathbf{T}))$ and there exists a $C_0(X)$-Morita equivalence between the systems*

$$(C_0(X) \rtimes_{\mathrm{id},u} G,\ \widehat{G}_{\mathrm{ab}},\ \hat{u}) \quad \text{and} \quad (\mathcal{E}_v * \varphi^*(C^*(H)),\ \widehat{G}_{\mathrm{ab}},\ \mathcal{E}_v * (\varphi^*(\delta))).$$

Proof. The proof is now a direct consequence of Propositions 1.10 and 2.8, and Theorem 2.2. ◻

Remark 2.10.

(1) If the cocycle $v = u \cdot \overline{u}_\varphi$ in the above theorem is actually *locally trivial*, then it follows from Remark 2.3 that the $C_0(X)$-Morita equivalence in the theorem can be replaced by $C_0(X)$-isomorphism. By [22, Theorem 2.1], this is automatically the case whenever $\widehat{G}_{\mathrm{ab}}$ is compactly generated.

(2) If A is a $C\mathcal{R}(X)$-algebra in the sense of [5, 4] (e.g. if A is unital and X is the complete regularization of $\operatorname{Prim} A$ as in [5, Definition 2.5]), then any inner action

of G on A determines a unique class $[u] \in H^2(G, C(X, \mathbf{T}))$ (see [18, Section 0] and [5, Section 2]). It is shown in [4, Corollary 4.7] that the crossed product $A \rtimes_\alpha G$ is then $C_0(X)$-linearly and $\widehat{G}_{\mathrm{ab}}$-equivariantly isomorphic to $(C_0(X) \rtimes_{\mathrm{id},u} G) \otimes_{C_0(X)} A$, so Theorem 2.9 also gives new insights into the structure of crossed products by inner actions.

We end this section by giving a description of $C_0(X) \rtimes_{\mathrm{id},u} G$, where u is a cocycle in $Z^2(G, C(X, \mathbf{T}))$ with constant evaluation map $[u(x)] = [\omega] \in H^2(G, \mathbf{T})$. Note that if $H^2(G, \mathbf{T})$ is discrete (as considered by Smith in [25]), then every cocycle $u \in Z^2(G, C(X, \mathbf{T}))$ has a direct sum decomposition into cocycles $u_i \in Z^2(G, C(X_i, \mathbf{T}))$ such that X is a disjoint union of the open subsets $X_i \subseteq X$, and each u_i has constant evaluation map. It is then easy to see that we get a decomposition

$$C_0(X) \rtimes_{\mathrm{id},u} G \cong \bigoplus_i C_0(X_i) \rtimes_{\mathrm{id},u_i} G.$$

As is standard, $C^*(G, \omega)$ will denote the twisted group algebra $\mathbf{C} \rtimes_{\mathrm{id},\omega} G$ of G with respect to $\omega \in Z^2(G, \mathbf{T})$. Of course, if ω is trivial, then $C^*(G, \omega)$ is the full group C^*-algebra $C^*(G)$ of G.

Theorem 2.11. *Assume that* $u \in Z^2(G, C(X, \mathbf{T}))$ *has constant evaluation map* $x \mapsto [u(x)] := [\omega] \in H^2(G, \mathbf{T})$. *Suppose further that* $v = u \cdot \overline{\omega} \in Z^2_{\mathrm{pt}}(G, C(X, \mathbf{T}))$ *satisfies one of the equivalent conditions of Proposition* 1.7 (*which is automatic if* $H^2(G, \mathbf{T})$ *is Hausdorff*). *Then the system* $(C_0(X) \rtimes_{\mathrm{id},u} G,\ \widehat{G}_{\mathrm{ab}},\ \hat{v})$ *is* $C_0(X)$*-Morita equivalent to*

$$(\mathcal{E}_v \times_{\widehat{G}_{\mathrm{ab}}} C^*(G, \omega), \widehat{G}_{\mathrm{ab}}, \mathrm{Ind}\, \delta),$$

where $\delta : G_{\mathrm{ab}} \to \mathrm{Aut}(C^*(G, \omega))$ *denotes the dual action, and the* $C_0(X)$*-structure of* $\mathcal{E}_v \times_{\widehat{G}_{\mathrm{ab}}} C^*(G, \omega)$ *is given by* $(\psi \cdot F)(z) = \psi(p(z))F(z)$.

Proof. If we apply Theorem 2.2 to the factorization $u = v \cdot \omega$, we obtain a $\widehat{G}_{\mathrm{ab}}$-equivariant $C_0(X)$-Morita equivalence between $C_0(X) \rtimes_{\mathrm{id},u} G$ and

$$\mathcal{E}_v * (C_0(X) \rtimes_{\mathrm{id},\omega} G) \cong \mathcal{E}_v * (C_0(X) \otimes C^*(G, \omega)) \cong \mathcal{E}_v \times_{\widehat{G}_{\mathrm{ab}}} C^*(G, \omega)$$

(with respect to the obvious identifications), where the last isomorphism follows from [4, Remark 3.4(c)]. □

If $v \in Z^2_{\mathrm{loc}}(G, C(X, \mathbf{T}))$, then the conditions of Proposition 1.7 are automatically satisfied (Remark 1.4), and Remark 2.3 implies that for such u we may replace $C_0(X)$-Morita equivalence by $C_0(X)$-isomorphism in the statement of Theorem 2.11.

3. THE GROUP C^*-ALGEBRAS OF CENTRAL GROUP EXTENSIONS

In this section we use our methods to study the group C^*-algebra $C^*(L)$ of a central extension $1 \to N \to L \to G \to 1$. Of course, the study of such algebras and their dual spaces is one of the main motivations for studying central twisted transformation group algebras.

To each central extension as above, we can associate a cocycle $\eta \in Z^2(G,N)$ of the form $\eta = \partial c$ for a Borel cross-section $c : G \to L$ satisfying $c(e) = e$. Viewing N as the dual of $\widehat{N}$, η can be viewed as a cocycle in $Z^2(G, C(\widehat{N}, \mathbf{T}))$, and it follows from [14, Theorem 4.1] (but see also [4, Lemma 6.3(a)]), that $C^*(L)$ is $C_0(\widehat{N})$-linearly and $\widehat{G}_{\mathrm{ab}}$-equivariantly isomorphic to $C_0(\widehat{N}) \rtimes_{\mathrm{id},\eta} G$, where the $C_0(\widehat{N})$-structure on $C^*(L)$ is given by the canonical inclusion $C_0(\widehat{N}) \cong C^*(N) \to ZM(C^*(L))$ given by convolution (compare with the discussion preceding Proposition 2.8). Thus the results of the preceding sections are directly applicable to the study of $C^*(L)$. However, for central twisted transformation group algebras associated to central group extensions, many of the abstract constructions in the preceding sections, like the bundle $\mathcal{E}_v$, can be realized quite naturally on the group level (e.g. Remark 3.2(5)).

Definition 3.1. A central extension $1 \to N \to L \to G \to 1$ of G by N is called *pointwise trivial* if every character $\chi \in \widehat{N}$ can be extended to a character of L; that is, if the restriction map $\mathrm{res} : H^1(L,\mathbf{T}) \to H^1(N,\mathbf{T}) = \widehat{N}$ is surjective. We denote by $Z^2_{\mathrm{pt}}(G,N)$ the cocycles, and by $H^2_{\mathrm{pt}}(G,N)$ the classes in $H^2(G,N)$, corresponding to pointwise trivial extensions.

Remark 3.2. We collect some straightforward observations on pointwise trivial extensions, which are no doubt well known to the experts.

(1) A central extension $1 \to N \to L \to G \to 1$ is pointwise trivial if and only if any corresponding cocycle η, viewed as an element of $Z^2(G, C(\widehat{N}, \mathbf{T}))$ is pointwise trivial in the sense of Definition 1.1. This follows directly from the Hochschild-Serre exact sequence

$$1 \longrightarrow H^1(G,\mathbf{T}) \underset{\mathrm{inf}}{\longrightarrow} H^1(L,\mathbf{T}) \underset{\mathrm{res}}{\longrightarrow} H^1(N,\mathbf{T}) \underset{\mathrm{tg}}{\longrightarrow} H^2(G,\mathbf{T})$$

(see [7, Chap. I Section 5]), but can easily be computed directly.

(2) If G is abelian, then the pointwise unitary extensions of G by N are precisely the abelian extensions. Indeed, if $1 \to N \to L \to G \to 1$ is a pointwise trivial extension with G abelian, then one easily checks that the characters of L separate the points of L. It follows that $[L,L]$ is trivial and L is abelian. Thus if G is abelian, then $H^2_{\mathrm{pt}}(G,N) = H^2_{\mathrm{ab}}(G,N)$, where $H^2_{\mathrm{ab}}(G,N)$ denotes the set of cohomology classes corresponding to the abelian extensions.

(3) If $1 \to N \to L \to G \to 1$ is a pointwise trivial extension, then the quotient map $L \mapsto L_{\mathrm{ab}}$ is injective on N (since the characters of L separate the points of

N). Thus we obtain an abelian exact sequence $1 \to N \to L_{\mathrm{ab}} \to G_{\mathrm{ab}} \to 1$, and the extension $1 \to N \to L \to G \to 1$ is actually inflated from this abelian extension. Recall that if $1 \to N \to M \xrightarrow{p} G_{\mathrm{ab}} \to 1$ is any extension of G_{ab}, then the inflation of this extension is the extension $1 \to N \to \inf(M) \to G \to 1$ obtained as follows. We set $\inf(M) = \{(m,s) \in M \times G \mid p(m) = q(s)\}$, where $q : G \to G_{\mathrm{ab}}$ is the quotient map. The inclusion $N \to \inf M$ sends $n \mapsto (n,e)$, and the quotient map $\inf M \to G$ sends $(m,s) \mapsto s$. The isomorphism $L \cong \inf(L_{\mathrm{ab}})$ is given by $\ell \mapsto (p(\ell), q(\ell))$, where $p : L \to L_{\mathrm{ab}}$ and $q : L \to G$ are the quotient maps.

(4) Of course, inflation of extensions in the above sense corresponds to the inflation of the corresponding group cocycles. Indeed, if $1 \to N \to M \to G_{\mathrm{ab}} \to 1$ is as above and if $c : G_{\mathrm{ab}} \to M$ is a Borel section, then we obtain a Borel section $d : G \to \inf(M)$ by defining $d(s) = (c(q(s)), s)$. Of course we then get $\partial_G d = \inf \partial_{G_{\mathrm{ab}}} c$. Combining this with (3), we see that inflation determines an isomorphism $\inf : H^2_{\mathrm{ab}}(G_{\mathrm{ab}}, N) \to H^2_{\mathrm{pt}}(G, N)$.

(5) If $\eta \in Z^2_{\mathrm{pt}}(G, C(\widehat{N}, \mathbf{T}))$ is a cocycle corresponding to a pointwise trivial extension $1 \to N \to L \to G \to 1$, then the $\widehat{G}_{\mathrm{ab}}$-bundle $p : \mathcal{E}_\eta \to \widehat{N}$ is isomorphic to the bundle $\mathrm{res} : \widehat{L}_{\mathrm{ab}} = H^1(L, \mathbf{T}) \to \widehat{N}$. Indeed, if $\eta = \partial c$ for some cross section $c : G \to L$, then we define a map $\Psi : \mathcal{E}_\eta \to \widehat{L}_{\mathrm{ab}}$ by

$$\Psi(f, \chi)(c(s)n) = f(s)\chi(n).$$

This is well defined, since

$$\begin{aligned}\partial_L(\Psi(f,\chi))(c(s)n, c(t)m) &= f(s)\chi(n)f(t)\chi(m)\overline{f(st)\chi(c(st)^{-1}c(s)c(t)nm)} \\ &= \partial_G(f)(s,t)\overline{\eta(s,t)(\chi)} = 1,\end{aligned}$$

so $\Psi(f,\chi) \in \widehat{L}_{\mathrm{ab}}$. Since pointwise convergence of characters implies uniform convergence on compact sets (see [9, Theorem 8]), the map Φ is continuous, and it is certainly $\widehat{G}_{\mathrm{ab}}$-equivariant. The assertion follows from the fact that $\widehat{L}_{\mathrm{ab}} \to \mathcal{E}_\eta$ defined by $\mu \mapsto (\mu \circ c, \mu|_N)$ is a continuous inverse for Ψ.

Thus, as a direct corollary of item (5) of the above remark and Theorem 2.11 we obtain the following result.

Corollary 3.3. *Suppose that* $1 \to N \to L \to G \to 1$ *is a pointwise trivial central group extension and let* $\delta : \widehat{G}_{\mathrm{ab}} \to \operatorname{Aut} C^*(G)$ *denote the dual action. Then* $C^*(L)$ *is* $C_0(\widehat{N})$*-Morita equivalent to* $\widehat{L}_{\mathrm{ab}} \times_{\widehat{G}_{\mathrm{ab}}} C^*(G) \cong \operatorname{Ind}_{\widehat{G}_{\mathrm{ab}}}^{\widehat{L}_{\mathrm{ab}}}(C^*(G), \delta^{-1})$. (*In fact, the corresponding* $\widehat{G}_{\mathrm{ab}}$*-systems are* $C_0(\widehat{N})$*-Morita equivalent.*)

Again, if the bundle $\mathrm{res} : \widehat{L}_{\mathrm{ab}} \to \widehat{N}$ is locally trivial (which is automatic if G_{ab} is compactly generated), then we can replace $C_0(\widehat{N})$-Morita equivalence by $C_0(\widehat{N})$-isomorphism.

We are now going to discuss the group algebra of general central extensions of a smooth group G with a fixed representation group $1 \to Z \to H \to G \to 1$. Let $\mu \in Z^2(G, Z)$ be a corresponding cocycle. Then, identifying $H^2(G, \mathbf{T})$ with $\widehat{Z}$, the transgression map for $1 \to N \to L \to G \to 1$ (which is just the evaluation map $\chi \mapsto [\eta(\chi)] \in H^2(G, \mathbf{T})$, if $\eta \in Z^2(G, C(\widehat{N}, \mathbf{T}))$ is a cocycle corresponding to the given extension) determines a homomorphism $\varphi : \widehat{N} \to \widehat{Z}$. Let $\widehat{\varphi} : Z \to N$ denote the dual homomorphism defined by $\widehat{\varphi}(z)(\chi) = z(\varphi(\chi))$ (where we identify Z with the dual of $\widehat{Z}$ and N with the dual of $\widehat{N}$ via Pontryagin duality). Then we obtain a cocycle $\widehat{\varphi}_*(\mu) \in Z^2(G, N)$ by defining $\widehat{\varphi}_*(\mu)(s,t) = \widehat{\varphi}(\mu(s,t))$. A short computation shows that this cocycle, viewed as a cocycle in $Z^2(G, C(\widehat{N}, \mathbf{T}))$, is precisely the one we obtain from the evaluation map for η via the process described in Proposition 1.10. It follows in particular that $\eta \cdot \widehat{\varphi}_*(\mu)^{-1} \in Z^2_{\mathrm{pt}}(G, N)$. Thus, a small variation on the proof of Proposition 1.10 gives us the following result.

Proposition 3.4. *Let* $1 \to Z \to H \to G \to 1$ *be a representation group for G and let* $\mu \in Z^2(G, Z)$ *be a corresponding cocycle. Then, for any locally compact abelian group N, viewed as a trivial G-module, we get an isomorphism*

$$H^2_{\mathrm{ab}}(G_{\mathrm{ab}}, N) \times \mathrm{Hom}(Z, N) \cong H^2(G, N),$$

defined by $([\eta], \psi) \mapsto [\inf \eta \cdot \psi_*(\mu)]$.

As a consequence of the above discussion and Theorem 2.9 we obtain the next theorem.

Theorem 3.5. *Let* $1 \to N \to L \to G \to 1$, $\eta \in Z^2(G, N)$, $\varphi : \widehat{N} \to \widehat{Z}$ *and* $\eta \cdot \widehat{\varphi}_*(\mu) \in Z^2_{\mathrm{pt}}(G, N)$ *be as in the discussion preceding Proposition 3.4, and let* $1 \to N \to L' \to G \to 1$ *be the central extension corresponding to the pointwise trivial cocycle* $\eta \cdot \widehat{\varphi}_*(\mu)^{-1}$. *Then* $C^*(L)$ *is* $C_0(\widehat{N})$*-Morita equivalent to* $\widehat{L'}_{\mathrm{ab}} * \varphi^*(C^*(H))$. *(Again, if we consider the* $\widehat{G}_{\mathrm{ab}}$*-actions, the Morita equivalence passes to the dynamical systems.)*

It is actually easy to give a direct construction of the pointwise trivial extension $1 \to N \to L' \to G \to 1$ corresponding to the cocycle $\eta \cdot \widehat{\varphi}_*(\mu)^{-1}$ without even mentioning the cocycles. For this let $1 \to N \to L \xrightarrow{p} G \to 1$ be the original extension corresponding to η. Let $q : H \to G$ denote the quotient map for the representation group H. Define

$$L' = \{(\ell, h) \in L \times H \mid p(\ell) = q(h)\}/\Delta(Z),$$

where $\Delta(Z) = \{(\widehat{\varphi}(z), z) \mid z \in Z\}$. Then we obtain a central extension

$$1 \longrightarrow N \xrightarrow{n \mapsto [n,e]} L' \xrightarrow{[\ell,h] \mapsto p(\ell)} G \longrightarrow 1.$$

We claim that this extension corresponds to the cocycle $\eta \cdot \widehat{\varphi}_*(\eta)$ of the theorem. Indeed, if we choose Borel sections $c : G \to L$ and $d : G \to H$ such that $\eta = \partial c$

and $\mu = \partial d$, then we get a Borel section $c \times d : G \to L'$ by defining $(c \times d)(s) = [c(s), d(s)]$. We then compute

$$\begin{aligned}\partial(c \times d)(s,t) &= [c(s), d(s)][c(t), d(t)][c(st), d(st)]^{-1} \\ &= [\eta(s,t), \mu(s,t)] \\ &= [\eta(s,t)\hat{\varphi}(\mu(s,t))^{-1}, e],\end{aligned}$$

which clearly proves the claim.

We finish with some straightforward examples which illustrate our results.

Example 3.6. Let $G = \mathbf{Z}^2$. Recall that the discrete Heisenberg group H_d is the set $\mathbf{Z}^3$ with multiplication given by

$$(n_1, m_1, \ell_1)(n_2, m_2, \ell_2) = (n_1 + n_2, m_1 + m_2, \ell_1 + \ell_2 + n_1 m_2).$$

It is easy to check that

$$1 \longrightarrow \mathbf{Z} \longrightarrow H_d \longrightarrow \mathbf{Z}^2 \longrightarrow 1$$

is a representation group for G and that $\hat{\mathbf{Z}} = \mathbf{T} \cong H^2(\mathbf{Z}^2, \mathbf{T})$ via $z \mapsto [\omega_z]$, where

$$\omega_z((n_1, m_1), (n_2, m_2)) = z^{n_1 m_2}.$$

Since every abelian extension of $\mathbf{Z}^2$ by some group N splits, it follows from Proposition 3.4 that $H^2(\mathbf{Z}^2, N) \cong \mathrm{Hom}(\mathbf{Z}, N) = N$. Thus each $n \in N$ determines a central extension $1 \longrightarrow N \longrightarrow L \longrightarrow \mathbf{Z}^2 \longrightarrow 1$, and Theorem 3.5 gives an isomorphism between $C^*(L)$ and the pull-back $n^*(C^*(H_d))$ (where we identify n with the character of $\hat{N}$ given by evaluation). Recall from [1] that $C^*(H_d)$ is a continuous bundle over $\mathbf{T}$ with fibers A_z given by the rotation algebras A_θ where $z = e^{2\pi i\theta}$ for some $\theta \in [0,1]$. Hence $C^*(L)$ is a continuous bundle over $\hat{N}$ with fibre $A_{\chi(n)}$ at the base point $\chi \in \hat{N}$, whose global structure is completely determined by the global structure of $C^*(H_d)$ as a bundle over $\hat{\mathbf{Z}} = \mathbf{T}$.

In [5] we gave explicit constructions for the representation groups for $\mathbf{Z}^n$ and $\mathbf{R}^n$. Using these, we can also apply our results to these groups. In all these cases, the abelian extensions vanish, so that we have $H^2(G, N) \cong \mathrm{Hom}(Z, N)$, where Z denotes the center of the corresponding representation group H. Therefore, the group algebras of central extensions of $\mathbf{Z}^n$ and $\mathbf{R}^n$ by N are isomorphic to the pull-backs of $C^*(H)$ via the corresponding dual maps $\varphi : \hat{N} \to \hat{Z}$.

Example 3.7. Let $G = \mathbf{Z} \times \mathbf{Z}_2$, where $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. Then an easy application of [4, Proposition 4.5] shows that $H = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ with multiplication given by

$$(n, [i], [j]) \cdot (m, [\ell], [k]) = (n+m, [i+\ell], [j+k+n \cdot \ell]), \quad n, m, i, j, k, \ell \in \mathbf{Z},$$

is a representation group for G with center $Z = \mathbf{Z}_2$. In particular, we have $H^2(G,\mathbf{T}) \cong \widehat{\mathbf{Z}_2} = \mathbf{Z}_2$. As there are many non-trivial abelian extensions of G by locally compact abelian groups N, we have, in general, a nontrivial decomposition $H^2(G,N) = H^2_{\mathrm{ab}}(G,N) \oplus \mathrm{Hom}(Z,N)$. It is then an straightforward exercise to apply our results to the group algebras of the corresponding central extensions of G by N.

It is also interesting to revisit Example 1.8 to illustrate some differences between the general situation of central twisted crossed products compared to central group extensions for possibly non-smooth groups.

Example 3.8. Let $G = \mathbf{R}^2 \times \mathbf{T}^2$ with multiplication given by

$$(s_1,t_1,z_1,w_1)(s_2,t_2,z_2,w_2) = (s_1+s_2, t_1+t_2, e^{is_1t_2}z_1z_2, e^{i\theta s_1t_2}w_1w_2),$$

where θ is any fixed irrational real number. In Example 1.8 we constructed a pointwise unitary cocycle $u \in Z^2(G, C(X,\mathbf{T}))$, with $X = \{\frac{1}{n} \mid n \in \mathbf{N}\} \cup \{0\}$, which is not inflated from $G_{\mathrm{ab}} = \mathbf{R}^2$. In particular, $H^2_{\mathrm{pt}}(G, C(X,\mathbf{T})) \neq H^2_{\mathrm{pt}}(\mathbf{R}^2, C(X,\mathbf{T})) = \{0\}$. On the other hand, it follows from part (4) of Remark 3.2 that for every abelian locally compact group N we do have an inflation isomorphism

$$\inf : H^2_{\mathrm{ab}}(\mathbf{R}^2, N) = H^2_{\mathrm{ab}}(G_{\mathrm{ab}}, N) \to H^2_{\mathrm{pt}}(G,N),$$

from which it follows that $H^2_{\mathrm{pt}}(G,N) = \{0\}$ for all N. Thus, the transgression map $\mathrm{tg} : \hat{N} \to H^2(G,\mathbf{T})$ is the only obstruction for a central extension $1 \to N \to L \to G \to 1$ to be non-trivial. With a little bit of extra work one can show that $H^2(G,\mathbf{T})$ is isomorphic to the nasty non-Hausdorff group $\mathbf{R}/(\mathbf{Z} + \theta\mathbf{Z})$. In particular, G is not smooth, and the general structure theorem for the group algebras of central extensions of G as given in Theorem 3.5 does not apply. However, some weaker results can be deduced from [3].

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