Crossed Products by $C_0(X)$-Actions

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Suppose that G has a representation group H, that $G_{ab} := G/\overline{[G, G]}$ is compactly generated, and that A is a C^*-algebra for which the complete regularization of $\mathrm{Prim}(A)$ is a locally compact Hausdorff space X. In a previous article, we showed that there is a bijection $\alpha \mapsto (Z_\alpha, f_\alpha)$ between the collection of exterior equivalence classes of locally inner actions $\alpha: G \to \mathrm{Aut}(A)$, and the collection of principal $\hat{G}_{ab}$-bundles Z_α together with continuous functions $f_\alpha: X \to H^2(G, \mathbb{T})$. In this paper, we compute the crossed products $A \rtimes_\alpha G$ in terms of the data Z_α, f_α, and $C^*(H)$.
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1. INTRODUCTION

This paper is a continuation of our study of locally inner actions begun in [7]. In that article we gave a classification up to exterior equivalence of actions of a smooth group G on a $\mathscr{CR}$-algebra A. In this paper, we want to consider the structure of the corresponding crossed products.

As in [7], we are motivated by a desire to make progress along the lines of a research program outlined by Rosenberg in his survey article [29] (see "Research Problem 1" in Section 3 of that article). As detailed there, it is important to obtain information about crossed products of actions with "single orbit type" acting on continuous-trace C^*-algebras. Using the Packer–Raeburn stabilization trick, an action of G on a continuous-trace C^*-algebra A with a single orbit type and constant stabilizer N can be decomposed into a spectrum fixing action of N and an action of G/N which acts freely

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on $(A \rtimes_\alpha N)^\wedge$. Thus an important first step in this program is to consider spectrum fixing automorphism groups. Provided that the quotient G_{ab} of G by the closure of its commutator subgroup $[G, G]$ is compactly generated, spectrum fixing automorphism groups of continuous-trace C^*-algebras A are necessarily locally inner in that each point in $\hat{A}$ has a neighborhood U such that the action restricts to an inner action of the ideal A_U. (This follows from the proof of [28, Corollary 2.2].) Thus it is natural to try to classify locally inner actions on arbitrary C^*-algebras rather than restricting ourselves to actions on continuous-trace algebras. In [7], for suitable G, we were able to do precisely this for a large class of C^*-algebras: namely those algebras whose primitive ideal space $\mathrm{Prim}(A)$ has a second countable locally compact complete regularization X. Following [7, Definition 2.5], such algebras are called $\mathscr{CR}$-algebras. The collection of $\mathscr{CR}$-algebras whose primitive ideal spaces have complete regularization (homeomorphic to) X is denoted by $\mathscr{CR}(X)$. All unital C^*-algebras are $\mathscr{CR}$-algebras, as are the quasi-standard algebras of [1] ([7, Section 2]).

In this paper, we give a precise bundle-theoretic description of the crossed products corresponding to the dynamical systems classified in [7]. Our methods require that (virtually) everything in sight be separable; thus we assume from the onset that all our automorphism groups are second countable, and that the C^*-algebras on which they act be separable. If G is smooth (which, in particular, is true for all connected and simply connected Lie groups, all compact groups, all discrete groups, and all compactly generated abelian groups), G_{ab} is compactly generated, and $A \in \mathscr{CR}(X)$, then the collection $\mathscr{LI}_G(A)$ of exterior equivalence classes of locally inner actions of G on A is parameterized by

$$H^1(X, \hat{\mathscr{G}}_{\mathrm{ab}}) \oplus C(X, H^2(G, \mathbb{T})),$$

where $\hat{\mathscr{G}}_{\mathrm{ab}}$ denotes the sheaf of germs of $\hat{G}_{\mathrm{ab}}$-valued functions on X and $H^2(G, \mathbb{T})$ denotes the (topologized) second Moore-cohomology group of G with values $\mathbb{T}$ ([7, Theorem 6.3]). If (A, G, α) is such a locally inner system, our main result gives a description of the crossed product $A \rtimes_\alpha G$ in terms of the associated invariants $\zeta_H(\alpha) \in H^1(X, \hat{\mathscr{G}}_{\mathrm{ab}})$, $f_\alpha \in C(X, H^2(G, \mathbb{T}))$, and a representation group H for G as described below (Theorem 6.6).

The function $f_\alpha: X \to H^2(G, \mathbb{T})$ arises naturally. A $\mathscr{CR}$-algebra is naturally a $C_0(X)$-algebra, and therefore admits a fibering over X (see Section 2.1); $f_\alpha(x)$ is defined to be the inverse of the Mackey obstruction for the induced action α^x on the fibre A_x (see Definition 6.1). The construction of $\zeta_H(\alpha)$ is more subtle, although it reduces to the usual Phillips–Raeburn obstruction ([21], [7, Section 2]) when α is locally unitary. As indicated by the notation, it may depend on the choice of a representation group for G. A summary of the basic facts about smooth groups and representation groups is given

in [7, Section 4]. We state some of the basic results here for convenience. Recall that if

$$e \to C \to H \to G \to e \qquad (1.1)$$

is a locally compact central extension of G by an abelian group C, then any Borel section $c: G \to H$ satisfying $c(eC) = e$ determines a cocycle $\sigma(s, t) = c(s)\, c(t)\, c(st)^{-1}$ in the Moore group $Z^2(G, C)$. If $\chi \in \hat{C}$, then $\sigma_\chi = \chi \circ \sigma$ is a cocycle in $Z^2(G, \mathbb{T})$. The resulting map $\mathrm{tg}: \hat{C} \to H^2(G, \mathbb{T})$ is a continuous homomorphism with respect to the Moore topology on $H^2(G, \mathbb{T})$, and depends only on the extension. The map tg is called the *transgression map*. Moore called G *smooth* if G has a central extension (1.1), called a *representation group*, for which the transgression map is an isomorphism of topological groups. In that case, we can view f_α as a continuous map from X to $\hat{C}$. Since C is central in H, $C^*(H)$ admits a natural $C_0(\hat{C})$-action; that is, $C^*(H)$ is a $C_0(\hat{C})$-algebra. The pull back $f_\alpha^*(C^*(H)) := C_0(X) \otimes_{C_0(\hat{C})} C^*(H)$ is then a $C_0(X)$-algebra. Since A is also a $C_0(X)$-algebra, we can form the balanced tensor product

$$A \otimes_f C^*(H) := A \otimes_{C_0(X)} f_\alpha^*(C^*(H)). \qquad (1.2)$$

In the special case that $\zeta_H(\alpha)$ is trivial, our main theorem implies that $A \rtimes_\alpha G$ is isomorphic to (1.2). When $\zeta_H(\alpha)$ is nontrivial, then it is necessary to "twist" (1.2) by a principal $\hat{G}_{\mathrm{ab}}$-bundle Z over X whose isomorphism class corresponds to $\zeta_H(\alpha)$ in $H^1(X, \hat{\mathscr{G}}_{\mathrm{ab}})$. The details of this construction are given in Section 3 (see Definition 3.3). The basic idea is to view a $C_0(X)$-algebra, such as $A \otimes_f C^*(H)$, as a $\hat{G}_{\mathrm{ab}}$-bundle over X and form what corresponds to the usual bundle product: $Z * (A \otimes_f C^*(H))$. The latter is naturally a $C_0(X)$-algebra which admits a $\hat{G}_{\mathrm{ab}}$-action which we denote $Z * \alpha$. Then our main result goes as follows.

THEOREM (Theorem 6.6). *Let G be a smooth group with representation group H. Suppose that G_{ab} is compactly generated, that $A \in \mathscr{CR}(X)$, and that $\alpha: G \to \mathrm{Aut}(A)$ is a locally inner action. If $f_\alpha: X \to H^2(G, \mathbb{T})$ and $\zeta_H(\alpha)$ are as above, and if $q: Z \to X$ is a principal $\hat{G}_{\mathrm{ab}}$-bundle corresponding to $\zeta_H(\alpha)$, then there exists a $C_0(X)$-linear and $\hat{G}_{\mathrm{ab}}$-equivariant isomorphism between $A \rtimes_\alpha G$ and $Z * (A \otimes_{f_\alpha} C^*(H))$.*

A special case of interest arises when α is locally unitary. Then f_α is trivial and $\zeta_H(\alpha)$ is the (generalized) Phillips–Raeburn obstruction. (In fact, we do not require G to be smooth in this event.) Then Theorem 5.5 implies that

$$A \rtimes_\alpha G \cong A \otimes_{C_0(X)} (Z * (C_0(X, C^*(G)))).$$

Our basic motivation, and our basic strategy, for proving our results involves viewing $C_0(X)$-algebras as the C^*-analogue of topological bundles over X. Thus we begin in Section 2 with a review of some of the basic facts about bundle operations and their C^*-counterparts: $C_0(X)$-algebras, restrictions of $C_0(X)$-algebras, balanced tensor products, and pull-backs. In Section 3, we give our basic product constructions alluded to above.

Section 4 is devoted primarily to crossed products by inner actions. However, our methods require extensive use of the theory of Busby–Smith twisted crossed products. Much of the basis of this theory has been worked out by Packer and Raeburn [17, 19, 18]. We review some of the basic facts here and then formulate our results for twisted systems.

In Section 5, we consider twisted systems which are "locally equivalent." Our main result here (Theorem 5.3) is crucial and allows us to tie our analysis of inner systems to locally inner systems. As a rather special case, we derive the result on locally unitary actions mentioned above (Theorem 5.5). Our main results on locally inner actions are spelled out in Section 6.

In Section 7, we consider the special case of $\mathbb{R}^n$-actions. Here the special structure of $\mathbb{R}^n$ allows us to give more detailed information about the crossed products. In a future article, we plan to turn our attention to twisted transformation groups—such as arise in our study here (see Corollary 4.7). This leads naturally to the study of the group C^*-algebras of central extensions in view of Lemma 6.3(a).

2. BUNDLE OPERATIONS ON $C_0(X)$-ALGEBRAS

If X is a locally compact Hausdorff space, then the C^*-algebra analogue of a fibre bundle over X is a $C_0(X)$-*algebra* A; that is, a C^*-algebra A together with a ∗-homomorphism ϕ from $C_0(X)$ to the center $\mathscr{Z}M(A)$ of the multiplier algebra $M(A)$ of A, which is nondegenerate in that

$$\phi(C_0(X)) \cdot A := \overline{\operatorname{span}}\{\phi(f)a : f \in C_0(X) \text{ and } a \in A\} = A.$$

We will usually suppress the map ϕ and write $f \cdot a$ in place of $\phi(f)a$. If A and B are $C_0(X)$-algebras, then a homomorphism $\Psi : A \to B$ is called $C_0(X)$-*linear* if $\Psi(f \cdot a) = f \cdot \Psi(a)$ for all $f \in C_0(X)$ and $a \in A$. Two $C_0(X)$-algebras A and B are *isomorphic*, if there exists a $C_0(X)$-linear isomorphism $\Psi : A \to B$.

$C_0(X)$-algebras have enjoyed considerable interest of late, and there are several nice treatments available [14, 4]. We record some of their basic properties here for convenience. If A is a $C_0(X)$-algebra, if U is open in X, and if J is the ideal of functions in $C_0(X)$ vanishing off U, then

$$J \cdot A := \overline{\operatorname{span}\{f \cdot a : f \in J \text{ and } a \in A\}} \tag{2.1}$$

is an ideal in A. The fibre A_x of A over x is defined to be the quotient $A_x = A/I_x$, where $I_x := C_0(X\setminus\{x\}) \cdot A$. The spectrum $\hat{A}$ can then be written as a disjoint union $\coprod_{x \in X} \hat{A}_x$, and the projection $p: \hat{A} \to X$ is a continuous map. Thus $\hat{A}$ is a topological bundle over X in the weakest possible sense. Conversely, if $p: \hat{A} \to X$ is any continuous map, and if we identify $C^b(\hat{A})$ with $\mathscr{Z}M(A)$ via the Dauns–Hofmann Theorem, then p induces a non-degenerate $*$-homomorphism $\phi: C_0(X) \to C^b(\hat{A}) \cong \mathscr{Z}M(A)$ by defining $\phi(f) = f \circ p$, and then p coincides with the projection corresponding to the $C_0(X)$-structure on A induced by ϕ.

A $C_0(X)$-algebra can be viewed as the algebra of sections of an (upper-semicontinuous) C^*-bundle over X as follows. For each $x \in X$ and $a \in A$, let $a(x)$ denote the image of a in the fibre $A_x = A/I_x$. Then we have a faithful representation of A into the C^*-direct sum $\bigoplus_{x \in X} A_x$ given by $a \mapsto (a(x))_{x \in X}$. The set of sections $x \mapsto a(x) \in A_x$ for $a \in A$, satisfy

(C-1) $\|a\| = \sup_{x \in X} \|a(x)\|$;

(C-2) $x \mapsto \|a(x)\|$ is upper semicontinuous and vanishes at infinity—that is,

$$\{x \in X: \|a(x)\| \geq \varepsilon\} \text{ is compact for all } \varepsilon > 0;$$

(C-3) $(f \cdot a)(x) = f(x)\, a(x)$ for all $f \in C_0(X)$ and $a \in A$;

(C-4) $\{a(x): a \in A\} = A_x$ for all $x \in X$.

Conversely, if $\{A_x\}_{x \in X}$ is a family of C^*-algebras (zero or nonzero), then any C^*-subalgebra of $\bigoplus_{x \in X} A_x$ which is closed under pointwise multiplication with elements of $C_0(X)$, and which satisfies Conditions (C-2) and (C-4) above, becomes a $C_0(X)$-algebra by defining the $C_0(X)$-action on A by pointwise multiplication. A $C_0(X)$-algebra is called a continuous $C_0(X)$-bundle if the maps $x \mapsto \|a(x)\|$ are continuous for all $a \in A$. By Lee's theorem [12], this is equivalent to saying that the projection $p: \hat{A} \to X$ is open. If A is a $C_0(X)$-algebra and Y is a nonempty locally compact subset of X, then we define the *restriction* $A_Y := C_0(Y) \cdot A$ of A to Y by

$$C_0(Y) \cdot A := \left\{ b \in \bigoplus_{y \in Y} A_y : b(y) = f(y)\, a(y) \text{ for some } f \in C_0(Y) \text{ and } a \in A \right\}.$$

LEMMA 2.1. *Suppose that Y is a nonempty locally compact subset of X, and that A is a $C_0(X)$-algebra. Then the restriction A_Y is a $C_0(Y)$-algebra with $(A_Y)_y = A_y$ for all $y \in Y$. If U is open in X, then A_U can be identified*

with the ideal $C_0(U) \cdot A$ defined in (2.1). *If C is closed in X, then A_C is the image of A by the natural map of $\bigoplus_{x \in X} A_x$ onto $\bigoplus_{x \in C} A_x$. Moreover,*

$$0 \to A_{X \setminus C} \to A \to A_C \to 0 \qquad (2.2)$$

is an exact sequence of C*-algebras.

Proof. We identify $C_0(U)$ with an ideal in $C_0(X)$. Since $C_0(U) \cdot A$ is a nondegenerate Banach $C_0(U)$-module, the Cohen Theorem implies that every element of $C_0(U) \cdot A$ is of the form $f \cdot a$ for $f \in C_0(U)$ and $a \in A$. Thus we can identify A_U with $C_0(U) \cdot A$ as claimed. The assertion about closed sets follows from the Cohen Theorem applied to A together with the observation that restriction defines a surjection of $C_0(X)$ onto $C_0(C)$. To establish (2.2), it suffices to see that if $a \in A$ and if $a(x) = 0$ for all $x \in C$, then $a \in C_0(X \setminus C) \cdot A$. But this is clear from the compactness of

$$K = \{ x \in X : \|a(x)\| \geqslant \varepsilon \}$$

for all $\varepsilon > 0$ so that a can be approximated by elements of the form $g \cdot a$ with $g \in C_c(X \setminus C)$.

It is now now shown that the first assertion is true if Y is either open or closed. Since a subset of a locally compact space is locally compact only if it is locally closed, the result follows. ∎

The following lemma will be useful in the sequel.

LEMMA 2.2. *Let A be a $C_0(X)$-algebra and let $B \subset \bigoplus_{x \in X} A_x$ be such that each $b \in B$ satisfies condition* (C-2), *and such that for each $x \in X$ and $b \in B$ there exists an open neighborhood U of x such that $C_0(U) \cdot b = \{ f \cdot b : f \in C_0(U) \} \subset A$. Then $B \subseteq A$. If, in addition, B is a $C_0(X)$-submodule of $\bigoplus_{x \in X} A_x$ such that for each $x \in X$ there exists an open neighborhood U of x satisfying $C_0(U) \cdot B = A_U$, then B is a dense subspace of A.*

Proof. Let $b \in B$ and let $g \in C_c(X)$. Then, using condition (C-2) and a partition of unity for supp(g), it is not hard to show that $g \cdot b \in A$. Then, by taking an approximate unit in $C_0(X)$ which lies in $C_c(X)$, we see that condition (C-1) implies $b \in A$. Assume now that B satisfies the additional assumptions. Again using a partition of unity, it follows that each $a \in A$ with supp(a) compact is a linear combination of elements in B. Thus B contains the set of all elements $a \in A$ with compact support. Therefore B is dense in A. ∎

If A and B are C*-algebras, we let $i_A : M(A) \to M(A \otimes_{\max} B)$ and $i_B : M(B) \to M(A \otimes_{\max} B)$ denote the natural injections. If A is a $C_0(X)$-algebra and B

is a $C_0(Y)$-algebra, then $A \otimes_{\max} B$ becomes a $C_0(X \times Y)$-algebra via the composition of maps

$$C_0(X \times Y) \to \mathscr{Z}M(A) \otimes \mathscr{Z}M(B) \to \mathscr{Z}M(A \otimes_{\max} B).$$

Moreover, just as is shown in [4, Corollaire 3.16], the nice behavior of the maximal tensor product with respect to quotients implies that the fibres $(A \otimes_{\max} B)_{(x, y)}$ of $A \otimes_{\max} B$ are isomorphic to $A_x \otimes_{\max} A_y$. For any elementary tensor $a \otimes b$ we have $(a \otimes b)(x, y) = a(x) \otimes b(y) \in A_x \otimes_{\max} A_y$.

If A and B are both $C_0(X)$-algebras, then composition with i_A and i_B gives $A \otimes_{\max} B$ two $C_0(X)$-algebra structures. Since any quotient of a $C_0(X)$-algebra is a $C_0(X)$-algebra, the two $C_0(X)$-algebra structures will coincide on a given quotient exactly when elementary tensors of the form $f \cdot a \otimes b - a \otimes f \cdot b$ are mapped to zero.

DEFINITION 2.3 (cf., [4]). Let A and B be two $C_0(X)$-algebras and let I be the closed ideal of $A \otimes_{\max} B$ generated by

$$\{a \cdot f \otimes b - a \otimes f \cdot b : a \in A, b \in B, f \in C_0(X)\}.$$

Then $A \otimes_X B := (A \otimes_{\max} B)/I$ equipped with the $C_0(X)$-action given on the images $a \otimes_X b$ of elementary tensors $a \otimes b$ by $f \cdot (a \otimes_X b) = f \cdot a \otimes_X b = a \otimes_X f \cdot b$ is called the (maximal) $C_0(X)$-*balanced tensor product* of A and B.

It is possible to form other balanced tensor products. (A detailed account may be found in [3].) However, the maximal tensor product has good functorial properties which will be useful in the sequel. As in the above definition, we will denote the image of an elementary tensor $a \otimes b$ in $A \otimes_X B$ by $a \otimes_X b$. Notice that $A \otimes_{\max} B$ also has a $C_0(X)$-algebra structure arising from viewing $C_0(X)$ as a the quotient of $C_0(X \times X)$ by the ideal C_Δ of functions vanishing on the diagonal $\Delta := \{(x, x) : x \in X\}$. Our next result shows that this structure also induces the given structure on $A \otimes_X B$, and that $A \otimes_{\max} B$ coincides with Blanchard's $A \otimes_{C(X)}^M B$ when X is compact.

LEMMA 2.4 (cf., [3, Proposition 2.2]). *Let A and B be $C_0(X)$-algebras. Then $A \otimes_X B$ is $C_0(X)$-isomorphic to the restriction $(A \otimes_{\max} B)_\Delta$ of $A \otimes_{\max} B$ to $\Delta = \{(x, x) : x \in X\}$, where the $C_0(X)$-structure on $(A \otimes_{\max} B)_\Delta$ is defined via the canonical homeomorphism $x \mapsto (x, x)$ between X and Δ.*

In particular, each fibre $(A \otimes_X B)_x$ is isomorphic to $A_x \otimes_{\max} B_x$ and the elementary tensor $a \otimes_X b$ in $A \otimes_X B$ is given by the section $x \mapsto a(x) \otimes b(x) \in A_x \otimes_{\max} B_x$.

Proof. Since Δ is closed in $X \times X$, it follows from Lemma 2.2 that $(A \otimes_{\max} B)_\Delta$ is the quotient of $A \otimes_{\max} B$ by the ideal $J = C_\Delta \cdot (A \otimes_{\max} B)$. Since

$$(a \cdot f \otimes b - a \otimes f \cdot b)(x, x) = f(x)(a(x) \otimes b(x) - a(x) \otimes b(x)) = 0$$

for all $f \in C_0(X)$, $a \in A$, and $b \in B$, it follows that the balancing ideal I given in Definition 2.3 is contained in J. Conversely, since C_Δ is the closed ideal of $C_0(X \times X)$ generated by $\{hf \otimes g - f \otimes hg : h, f, g \in C_0(X)\}$ and since $(hf \otimes g - f \otimes hg) \cdot (a \otimes b) \in I$ for all elementary tensors $a \otimes b$, it follows that the quotient map $A \otimes_{\max} B \to A \otimes_X B$ maps $C_\Delta \cdot (A \odot B)$ to $\{0\}$. But this implies that $J \subseteq I$. $\blacksquare$

Remark 2.5. (a) Suppose that C is a C^*-algebra and that A and B are $C_0(X)$-algebras. Using the universal property of the maximal tensor product $A \otimes_{\max} B$ (see [31, Proposition 4.7]), it follows that if $\Phi_A : A \to M(C)$ and $\Phi_B : B \to M(C)$ are nondegenerate homomorphisms such that $\Phi_A(f \cdot a) \Phi(b) = \Phi_A(a) \Phi_B(f \cdot b)$ for all $a \in A$ and $b \in B$, then there is a unique nondegenerate homomorphism $\Phi_A \otimes_X \Phi_B : A \otimes_X B \to M(C)$ such that $\Phi_A = (\Phi_A \otimes_X \Phi_B) \circ i_A$ and $\Phi_B = (\Phi_A \otimes_X \Phi_B) \circ i_B$. Conversely, if $\Phi : A \otimes_X B \to M(C)$ is any nondegenerate homomorphism, then $\Phi = \Phi_A \otimes_X \Phi_B$ with $\Phi_A = \Phi \circ i_A$, $\Phi_B = \Phi \circ i_B$. Moreover, if C is a $C_0(X)$-algebra, then it is not to hard to check that $\Phi = \Phi_A \otimes_X \Phi_B$ is $C_0(X)$-linear if and only if Φ_A and Φ_B are $C_0(X)$-linear.

(b) If A and B are nuclear (and separable) $C_0(X)$-algebras, then the balanced tensor product $A \otimes_X B$ coincides with the construction given by Iain Raeburn and the second author in [24]. In particular, it follows that if $p : \mathrm{Prim}(A) \to X$ and $q : \mathrm{Prim}(B) \to X$ are the projections determined by the $C_0(X)$-structures of A and B, then $\mathrm{Prim}(A \otimes_X B)$ is homeomorphic to the *fibre* product

$$\mathrm{Prim}(A) \times_X \mathrm{Prim}(B) := \{(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B) : p(P) = q(Q)\}$$

[24, Lemma 1.1]. If A or B is type I, then we also have $(A \otimes_X B)^\wedge \cong \hat{A} \times_X \hat{B}$.

(c) An important special case occurs when $B = C_0(Y)$ for a locally compact space Y. If $p : Y \to X$ is a continuous map, (so that $C_0(Y)$ becomes a $C_0(X)$-algebra via the homomorphism $\phi : C_0(X) \to C^b(Y)$ defined by $\phi(g) = g \circ p$), then $A \otimes_X C_0(Y)$ is not only a $C_0(X)$-algebra, but there is also a canonical $C_0(Y)$-action on the balanced tensor product given by the canonical embedding of $C_0(Y)$ into $M(A \otimes_X C_0(Y))$. We will write $A \otimes_p C_0(Y)$ for the balanced tensor product $p^*A := A \otimes_X C_0(Y)$ viewed as a $C_0(Y)$-algebra; this is the *pull-back* of A along p as defined in [24]. If $y \in Y$, then the fibre $(A \otimes_p C_0(Y))_y$ is equal to $A_{p(y)}$, and the projection

$q: (A \otimes_p C_0(Y))^{\wedge} \cong Y \times_X \hat{B} \to Y$ is given by $q(y, \pi) = p(y)$. The justification for the pull-back terminology is [24, Proposition 1.3] where is shown that if A is the section algebra of a C^*-bundle, then p^*A is the section algebra of the pull-back bundle.

(d) More generally, suppose that B is a $C_0(X)$-algebra and that A is a $C_0(Y)$ algebra. Then if $p: Y \to X$ is continuous, we can view B as a $C_0(Y)$-algebra via composition with p. Since $B \otimes_Y A \cong (B \otimes_p C_0(Y)) \otimes_Y A$, we will write $B \otimes_p A$ in place of $B \otimes_Y A$.

3. THE BUNDLE PRODUCT CONSTRUCTIONS

A *topological bundle with group G* is a topological bundle $p: Y \to X$ such that Y is a G-space in such a way that each $s \in G$ acts as a bundle isomorphism of $p: Y \to X$. The C^*-algebraic analogue of a bundle with group G, is a C^*-dynamical system (A, G, α) in which A is a $C_0(X)$-algebra, and each α_s is a $C_0(X)$-automorphism. More simply, α is a strongly continuous homomorphism of G into $\operatorname{Aut}_{C_0(X)}(A)$.

As mentioned in the introduction, we want to build a dynamical system $(Z * A, G, Z * \alpha)$ from a "product" of a $C_0(X)$-system (A, G, α) and a principal G-space Z. By a *principal G-bundle* we understand a locally trivial G-bundle $p: Z \to X$ over X such that all fibres are isomorphic to G. Notice that X then has an open cover $\{U_i\}_{i \in I}$ such that there are G-homeomorphisms $h_i: U_i \times G \to p^{-1}(U_i)$ for each $i \in I$. On overlaps $U_{ij} := U_i \cap U_j$ we obtain continuous *transition functions* $\gamma_{ij}: U_{ij} \to G$ such that

$$h_j^{-1} \circ h_i(x, s) = (x, s\gamma_{ij}(x)) \qquad \text{for} \quad x \in U_{ij} \quad \text{and} \quad s \in G. \qquad (3.1)$$

Then, if x belongs to a triple overlap $U_{ijk} := U_i \cap U_j \cap U_k$,

$$\gamma_{ij}(x) \, \gamma_{jk}(x) = \gamma_{ik}(x).$$

Therefore $\gamma := \{\gamma_{ij}\}_{i, j \in I}$ defines a 1-cocycle in $Z^1(X, \mathscr{G})$, and the class of γ in the sheaf cohomology group $H^1(X, \mathscr{G})$ depends only on the isomorphism class of the principal bundle $p: Z \to X$. This gives the well known classification of principal G-bundles over X (cf., e.g., [32, Section 5.33] or [30]). (If G is a topological group, we use the caligraphic letter $\mathscr{G}$ to denote the corresponding sheaf of germs of G-valued functions.) If G is abelian, $H^1(X, \mathscr{G})$ is a group under pointwise product and the bundle product we want to investigate is analogous to the construction on the principal G-bundles corresponding to the product in $H^1(X, \mathscr{G})$.

The class of principal G-bundles is a subclass of the class of proper G-bundles. Suppose that Y is a locally compact G-space such that G acts

freely and properly on Y; that is, $s \cdot y = y$ implies $s = e$, and the map $(s, y) \mapsto (s \cdot y, y)$ is proper as a map from $G \times Y$ to $Y \times Y$. Then $X = G \backslash Y$ is a locally Hausdorff space and we say that Y is a *proper G-bundle over X*. A proper G-bundle is a principal G-bundle exactly when there are local (continuous) sections for $p: Y \to X$ [25, Proposition 4.3(3)]. If G is a Lie group, then it follows from Palais's slice theorem [20] that the proper G-bundles are precisely the principal G-bundles. Of course, there are groups G for which there exist non-principal proper G-bundles [26, Remark 2.5].

DEFINITION 3.1. Let G be an *abelian* locally compact group and let $q: Z \to X$ be a proper G-bundle over X.

(a) If Y is any G-space, then we define $Z \times_G Y$ to be the orbit space $G \backslash (Z \times Y)$, where the G-action is defined by $s \cdot (z, y) := (sz, s^{-1}y)$. We define a continuous map $i: Z \times_G Y \to X$ by $i([z, y]) := q(z)$. We define a left G-action on $Z \times_G Y$ by $s \cdot [x, y] := [s \cdot x, y]$, where $[z, y]$ denotes the orbit of $(z, y) \in Z \times Y$.

(b) If $p: Y \to X$ is any topological bundle over X with group G, we define $r: Z * Y \to X$ to be the topological bundle over X with group G such that $Z * Y := \{[z, y] \in Z \times_G Y: q(z) = p(y)\}$ and $r := i|_{Z * Y}$. The G-action is induced from the G-action on $Z \times_G Y$. We call $r: Z * Y \to X$ the *G-fibre product* of Z and Y.

Remark 3.2. (a) Notice that the above definitions of $Z \times_G Y$ and $Z * Y$ only make sense when G is abelian. For non-abelian G one could define similar spaces by taking the quotient of $Z \times Y$ by the diagonal action. However, there would be no analogue of the G-actions on $Z \times_G Y$ and $Z * Y$. It is straightforward to check that $Z \times_G Y$ is a fibre bundle over X with group G and fibre Y and that $Z * Y$ is a topological bundle with group G, which has the same fibres as Y. We view $Z * Y$ as the bundle Y *twisted* by Z. The G-isomorphism classes of $Z \times_G Y$ and $Z * Y$ depend only on those of Z and Y.

(b) If Z and Y are proper G-bundles, then it was shown in [26, Lemma 2.4] that $Z * Y$ is a proper G-bundle and that $[Z][Y] \mapsto [Z * Y]$ defines an abelian group structure on the set $\mathrm{HP}(X, \mathscr{G})$ of all isomorphism classes of proper G-bundles over X. If $q: Z \to X$ and $p: Y \to X$ are principal bundles corresponding to the classes $[\gamma_1]$ and $[\gamma_2]$ in $H^1(X, \mathscr{G})$, then $r: Z * Y \to X$ is a principal G-bundle corresponding to $[\gamma_1 \cdot \gamma_2]$. Thus $H^1(X, \mathscr{G})$ can be viewed as a subgroup of $\mathrm{HP}(X, \mathscr{G})$ [27, Remark 2.7].

Now we introduce the analogous C^*-algebraic constructions. Suppose that $q: Z \to X$ is a proper G-bundle and that (A, G, α) is a C^*-dynamical system with G abelian. Then $s \mapsto \alpha_s^{-1}$ is also a homomorphism and we can therefore employ the well-known construction of induced systems [23, 22, 24]

to (A, G, α^{-1}). In particular, we define $Z \times_G A$ to be $\mathrm{Ind}_G^Z(A, \alpha^{-1})$ (in the notation of [22]); that is, $Z \times_G A$ is the set of all bounded continuous functions $F: Z \to A$ satisfying

$$\alpha_s(F(z)) = F(s^{-1} \cdot z), \qquad \text{for all} \quad s \in G \quad \text{and} \quad z \in Z,$$

and such that $z \mapsto \|F(z)\|$ vanishes at infinity on $X = G \backslash Z$. Equipped with the pointwise operations and the supremum norm, $Z \times_G A$ becomes a C^*-algebra. We define a strongly continuous action $\mathrm{Ind}\,\alpha$ of G on $Z \times_G A$ by

$$(\mathrm{Ind}\,\alpha)_s (F)(z) = \alpha_s(F(z)) = F(s^{-1} \cdot z).$$

Strong continuity follows from straightforward compactness arguments using the fact that $z \mapsto \|f(z)\|$ vanishes at infinity on $G \backslash Z$. Note that $C_0(X)$ acts on $Z \times_G A$ via $(g \cdot F)(z) = g(q(z)) F(z)$, $g \in C_0(X)$, so that $(Z \times_G A, G, \mathrm{Ind}\,\alpha)$ is actually a $C_0(X)$-system with fibres $(Z \times_G A)_x \cong A$.

If (A, G, α) is itself a $C_0(X)$-system, then there is a $C_0(X \times X)$-action on $Z \times_G A$ given by

$$(h \cdot F)(z)(x) = h(q(z), x) F(z)(x), \qquad h \in C_0(X \times X).$$

We define $Z * A$ to be the restriction of $Z \times_G A$ to the diagonal $\varDelta$ of $X \times X$. Identifying X with $\varDelta$ gives $Z * A$ the structure of a $C_0(X)$-algebra with fibres $(Z*A)_x \cong A_x$. Lemma 2.1 implies that $Z * A$ may be written as the set of sections

$$\left\{ f: Z \to \bigoplus_{x \in X} A_x : f(z) = F(z)(q(z)) \in A_{q(z)} \text{ for some } F \in Z \times_G A \right\}.$$

Since $\mathrm{Ind}\,\alpha$ is $C_0(X \times X)$-linear, it follows that it restricts to a $C_0(X)$-linear action $Z * \alpha$ of G on $Z * A$. If $F \in Z \times_G A$, and $f(z) = F(z)(q(z))$, then

$$(Z * \alpha)_s (f)(z) = \alpha_s(F(z))(q(z)) = \alpha_s^{q(z)}(f(z)),$$

where $\alpha^{q(z)}$ is the induced action on the fibre $A_{q(z)}$.

DEFINITION 3.3. Suppose that G is an abelian group and $q: Z \to X$ is a proper G-bundle over X.

(a) If (A, G, α) is a C^*-dynamical system then $(Z \times_G A, G, \mathrm{Ind}\,\alpha)$ is called the $C_0(X)$-system *induced from* (A, G, α) *via* Z.

(b) If (A, G, α) is a $C_0(X)$-system, then $(Z * A, G, Z * \alpha)$ is called the *G-fibre product* of $q: Z \to X$ with (A, G, α).

Remark 3.4.   (a)   If γ is the diagonal action on $C_0(Z, A)$ given by $\gamma_s(F)(z) := \alpha_s^{-1}(F(s^{-1} \cdot z))$, then $Z \times_G A$ was denoted by $GC(Z, A)^\gamma$ in [24] and can be viewed as a generalized fixed point algebra for γ. The algebra $Z * A$ defined above first appeared in [24, Section 2] as $GC(Z, A)^\gamma/I$, where I is the kernel of the quotient map $Z \times_G A \to Z * A$. What is new with our construction are the actions Ind α and $Z * \alpha$.

(b)   Our definitions of $Z \times_G A$ and A^Z only make sense if G is abelian, since we need α^{-1} to be an action of G on A. Of course, we could have defined $Z \times_G A$ as the algebra $\mathrm{Ind}_G^Z(A, \alpha)$. This would have the advantage of working for nonabelian G, and would lead to a sensible definition of $Z * A$ in the general case. However, there would be no analogues for the actions Ind α and $Z * \alpha$ in the nonabelian case. In any event, our definition more closely parallels the classical bundle product, and leads to more elegant statements of our main results.

(c)   If (A, G, α) is a system with G abelian, and if $q: Z \to X$ is a proper G-bundle, then $(Z * C_0(X, A), G, Z * (\mathrm{id} \otimes \alpha))$ is canonically isomorphic to $(Z \times_G A, G, \mathrm{Ind}\, \alpha)$. The isomorphism is given by the map $\Psi: Z \times_G C_0(X, A) \to Z \times_G A; \Psi(F)(z) = F(z, q(z))$.

We now turn to the basic properties of the $C_0(X)$-systems $(Z \times_G A, G, \mathrm{Ind}\, \alpha)$ and $(Z * A, G, Z * \alpha)$. In so doing, we will see that these C^*-constructions from Definition 3.3 parallel the topological constructions of Definition 3.1. If (A, G, α) is a $C_0(X)$-system, then $\hat{A}$ is a topological bundle over X with group G with respect to the projection $p: \hat{A} \to X$ and the action of G defined by $s \cdot \pi = \pi \circ \alpha_s^{-1}$. If $(z, \pi) \in Z \times \hat{A}$, then it was shown in [24, Proposition 3.1] that (z, π) determines an irreducible representation $M(z, \pi) \in (Z \times_G A)^\wedge$ defined by $M(z, \pi)(F) = \pi(F(z))$. Moreover $M(z_1, \pi_1)$ is equivalent to $M(z_2, \pi_2)$ if and only if there exist an $s \in G$ such that $z_2 = s \cdot z_1$ and $\pi_2 = s^{-1} \cdot \pi_1$ (note that s^{-1} appears in the latter formula as we have replaced α by α^{-1} in the formulae from [24]). The representations of $Z * A$ are then given by those $M(z, \pi)$ which satisfy $q(z) = p(\pi)$. Thus we obtain

PROPOSITION 3.5 ([24, Proposition 3.1]).   *Let $q: Z \to X$ be a proper G-bundle with G abelian.*

(a)   *If (A, G, α) is a system, then $(Z \times_G A)^\wedge$ is naturally isomorphic to $Z \times_G \hat{A}$ as a fibre bundle over X with group G.*

(b)   *If (A, G, α) is a $C_0(X)$-system, then $(Z * A)^\wedge$ is naturally isomorphic to $Z * \hat{A}$ as topological bundles over X with group G.*

The next corollary follows immediately from the Gelfand theory.

COROLLARY 3.6. *Let Y be a locally compact G-space, and define $\tau: G \to$ $\text{Aut}(C_0(Y))$ by $\tau_s(f)(y) = f(s^{-1}y)$. Then $Z \times_G C_0(Y)$ is equivariantly isomorphic to $C_0(Z \times_G Y)$. Moreover, if $p: Y \to X$ is a locally compact topological bundle over X with group G, then $Z * C_0(Y)$ is G-equivariant to $C_0(Z * Y)$.*

PROPOSITION 3.7 (cf., [11, Proposition 2.15]). *Suppose that (A, G, α) is a $C_0(X)$-system such that A is actually a continuous $C_0(X)$-bundle; that is, A is the section algebra $C_0(X; \mathscr{A})$ of a C^*-bundle $p: \mathscr{A} \to X$. Then $\mathscr{A}$ is a (continuous) G-space with the action characterized by $s \cdot a(x) = \alpha_s(a)(x) = \alpha_s^x(a(x))$, $Z * \mathscr{A}$ is a C^*-bundle over X, and Z^*A is canonically isomorphic to $C_0(X, Z * \mathscr{A})$.*

Proof. We omit the proof that $\mathscr{A}$ is a continuous G-space with respect to the above given action and that $Z * \mathscr{A}$ is a C^*-bundle over X (for more details see [11, Proposition 2.15]). In order to see that $Z * A$ is isomorphic to $C_0(X, Z * \mathscr{A})$ let $F \in Z \times_G A$. Then F defines a section $f_F \in C_0(X; Z * \mathscr{A})$ by $f_F(q(z)) = [z, F(z)(q(z))]$. The collection $\Gamma = \{f_F : F \in Z \times_G A\}$ is dense in $C_0(X; Z * \mathscr{A})$ by [8, Corollary II.14.7], and it follows from the discussion preceeding Definition 3.3 that $Z * A \cong C_0(X; Z * \mathscr{A})$. ∎

We start to investigate the structure of $(Z * A, G, Z * \alpha)$ with some interesting special cases. First notice that if $\alpha: G \to \text{Aut}(A)$ is the trivial action, then $(Z * A, G, Z * \alpha)$ is isomorphic to (A, G, α) for all proper G-spaces $q: Z \to X$ (for a proof see [24, Proposition 3.2]). A similar result holds when Z is a trivial bundle:

LEMMA 3.8 (cf., [24, Proposition 3.2]). *Let (A, G, α) be a $C_0(X)$-system and let $q: Z \to X$ be a trivial G-bundle. Let $\varphi: X \to Z$ be a continuous section for $q: Z \to X$, and let $s(z)$ be the unique element in G which satisfies $z = s(z)\varphi(q(z))$ for each $z \in Z$. Then $\Phi(f)(x) = f(\varphi(x))$ defines an equivariant $C_0(X)$-isomorphism of the systems $(Z * A, G, Z * \alpha)$ and (A, G, α), with inverse given by $\Phi^{-1}(a)(z) = \alpha_{s(z)^{-1}}^x(a(q(z)))$.*

Proof. Define $\Psi: Z \times_G A \to C_0(X, A)$ by $\Psi(F)(x) = F(\varphi(x))$. Then it is easy to check that Ψ is a $C_0(X \times X)$-linear isomorphism with inverse given by $\Psi^{-1}(g)(z) = \alpha_{s(z)}^{-1}(g(q(z)))$. If $s \in G$ and $F \in Z \times_G A$, then

$$\Psi((\text{Ind } \alpha)_s(F))(x) = (\text{Ind } \alpha)_s(F)(\varphi(x)) = \alpha_s(F(\varphi(x))) = \alpha_s(\Phi(F)(x)).$$

Thus Ψ carries $\text{Ind } \alpha$ to $\text{id} \otimes \alpha$. Since Ψ is $C_0(X \times X)$-linear and the restriction $(C_0(X, A)_\Delta, G, (\text{id} \otimes \alpha)^\Delta)$ of $(C_0(X, A), G, \text{id} \otimes \alpha)$ to the diagonal Δ is isomorphic to (A, G, α), it follows that Ψ induces a G-equivariant and $C_0(X)$-linear isomorphism $\Phi: Z * A \to A$. Evaluation at the fibres reveals that Φ and Φ^{-1} are given by the formulas in the lemma. ∎

Remark 3.9.   The isomorphism of $Z * A$ and A given in Lemma 3.8 depends on the choice of section. If Φ_1 and Φ_2 are induced from two different continuous sections φ_1 and $\varphi_2 \colon X \to Z$, then let $\gamma_{12} \colon X \to G$ denote the transition function defined by $\varphi_1(x) = \gamma_{12}(x)\, \varphi_2(x)$. Then, for all $x \in X$, we get

$$
\begin{aligned}
\Phi_2(f)(x) &= f(\varphi_2(x)) = F(\varphi_2(x))(x) = F(\gamma_{12}(x)^{-1}\, \varphi_2(x))(x) \\
&= \alpha^x_{\gamma_{12}(x)}(F(\varphi_1(x))(x) = \alpha^x_{\gamma_{12}(x)}(f(\varphi_1(x))) \\
&= \alpha^x_{\gamma_{12}(x)}(\Phi_1(f)(x)).
\end{aligned}
$$

Again, suppose that $q \colon Z \to X$ is a proper G-bundle over X. If W is any locally compact subset of X, then the restriction $Z_W := q^{-1}(W)$ is a proper G-bundle over W, and our next result shows that $Z * A$ behaves well with respect to restrictions.

LEMMA 3.10.   *Let* (A, G, α) *be a* $C_0(X)$-*system and let* $q \colon Z \to X$ *be a proper* G-*bundle over* X. *If* W *is a locally compact subset of* X, *then* $((Z * A)_W,\ G,\ (Z * \alpha)_W)$ *and* $(Z_W * (A_W),\ G,\ Z_W * (\alpha^W))$ *are isomorphic as* $C_0(W)$-*systems. In particular,* $((Z * A)_x,\ G,\ (Z * \alpha)^x)$ *is isomorphic to* (A_x, G, α^x) *for all* $x \in X$.

Proof.   The second assertion is a consequence of the first and Lemma 3.8. The first assertion is straightforward when W is open or closed. Since Y is always the intersection of an open and a closed set, the result follows by iteration. ∎

When Z is a principal G-bundle, it will be convenient to have a description of $Z * A$ in terms of a representative $\gamma \in Z^1(X, \mathscr{G})$ for the class in $H^1(X, \mathscr{G})$ corresponding to Z.

PROPOSITION 3.11.   *Let* (A, G, α) *be a* $C_0(X)$-*system with* G *abelian and* X *paracompact. Let* $q \colon Z \to X$ *be a principal bundle and let* $\{U_i\}_{i \in I}$ *be a locally finite cover of* X *such that* $\gamma = \{\gamma_{ij}\}_{i,\, j \in I}$ *represents the class in* $H^1(X, \mathscr{G})$ *corresponding to* Z. *Then a* $C_0(X)$-*system* (B, G, β) *is* $C_0(X)$-*isomorphic to* $(Z * A, G, Z * \alpha)$ *if and only if there exist isomorphisms* $\Phi_i \colon B_{U_i} \to A_{U_i}$ *satisfying*

    (a)   *for all* $i \in I$, Φ_i *is* $C_0(U_i)$-*linear and* G-*equivariant, and*

    (b)   *for all* $i, j \in I$, $b \in B$, *and* $x \in U_{ij}$, $\Phi_j(b)(x) = \alpha^x_{\gamma_{ij}(x)}(\Phi_i(g)(x))$.

Proof.   Since γ is a representative for $q \colon Z \to X$ in $Z^1(X, \mathscr{G})$, there exist local sections $\varphi_i \colon U_i \to q^{-1}(U_i)$ such that $\varphi_i(x) = \gamma_{ij}(x)\, \varphi_j(x)$ for all $x \in U_{ij}$. It follows then from Remark 3.9 that the isomorphisms $\Phi_i \colon (Z * A)_{U_i} \to A_{U_i}$

of Lemma 3.8 corresponding to the local sections φ_i satisfy conditions (a) and (b).

Suppose now that (B, G, β) is an arbitrary $C_0(X)$-system and let $\Phi_i \colon B_{U_i} \to A_{U_i}$ be isomorphisms satisfying (a) and (b). For each $z \in q^{-1}(U_i)$ define $s_i(z) \in G$ by the equation $z = s_i(z)\,\varphi_i(q(z))$. It follows from Lemma 3.8 that

$$\Psi_i(b)(z) = \alpha_{s_i(z)^{-1}}^{q(z)}(\Phi_i(b)(q(z)))$$

defines a $C_0(U_i)$-linear and G-equivariant isomorphism $\Psi_i \colon B_{U_i} \to (Z * A)_{U_i}$ for all $i \in I$. Moreover, if $q(z) \in U_{ij}$, then $z = s_i(z)\,\varphi_i(q(z)) = s_j(z)\,\gamma_{ij}(q(z))\,\varphi_j(q(z))$ which implies that $s_j(z) = s_i(z)\,\gamma_{ij}(q(z))$ for all $z \in Z$. It follows that

$$\begin{aligned}
\Psi_i(b)(z) &= \alpha_{s_i(z)^{-1}}^{q(z)}(\Phi_i(b)(q(z))) = \alpha_{s_i(z)^{-1}}^{q(z)}(\alpha_{\gamma_{ij}(q(z))^{-1}}^{q(z)}(\Phi_j(b)(q(z)))) \\
&= \alpha_{(s_i(z)\,\gamma_{ij}(q(z)))^{-1}}^{q(z)}(\Phi_j(b)(q(z))) = \alpha_{s_j(z)^{-1}}^{q(z)}(\Phi_j(b)(q(z))) \\
&= \Psi_j(b)(z),
\end{aligned}$$

for all $z \in q^{-1}(U_{ij})$, $b \in B_{U_{ij}}$. Thus, if we define $\Psi \colon B \to Z * A$ by the formula $\Psi(b)(q(z)) = \Psi_i(b)(q(z))$, whenever $z \in q^{-1}(U_i)$, it follows from Lemma 2.2 that Ψ is an isomorphism between the $C_0(X)$-systems (B, G, β) and $(Z * A, G, Z * \alpha)$. ∎

We close this section with the following useful result which we will need later. It will be helpful to keep in mind that if A is a $C_0(X)$-algebra, and B any C^*-algebra, then $B \otimes_{\max} A$ is a $C_0(X)$-algebra with fibres $(B \otimes_{\max} A)_x \cong B \otimes_{\max} A_x$.

PROPOSITION 3.12. *Let $q \colon Z \to X$ be a proper G-bundle with G abelian. Suppose further that (A, G, α) is a $C_0(X)$-system and that B is a $C_0(X)$-algebra. Then*

$$(Z * (B \otimes_X A), G, Z * (\mathrm{id} \otimes_X \alpha)) \quad \text{and} \quad (B \otimes_X (Z * A), G, \mathrm{id} \otimes_X (Z * \alpha))$$

are isomorphic $C_0(X)$-systems.

Proof. We first show that for any system (A, G, α) and any C^*-algebra B the induced algebra $Z \times_G (B \otimes_{\max} A)$ is equivariantly isomorphic to $B \otimes_{\max} (Z \times_G A)$. For this let (i_B, i_A) denote the natural embeddings of B and A in $M(B \otimes_{\max} A)$, and define homomorphisms $\Phi_B \colon B \to M(Z \times_G (B \otimes_{\max} A))$ and $\Phi_{Z \times_G A} \colon Z \times_G A \to M(Z \times_G (B \otimes_{\max} A))$ by

$$(\Phi_B(b) \cdot H)(z) = i_B(b)(H(z)) \quad \text{and} \quad (\Phi_{Z \times_G A}(F) \cdot H)(z) = i_A(F(z))\,H(z),$$

where $b \in B$, $F \in Z \times_G A$, and $H \in Z \times_G (B \otimes_X A)$. It is then straightforward to check that Φ_B and $\Phi_{Z \times_G A}$ are commuting nondegenerate $*$-homomorphisms such that $\{ \Phi_B(b) \, \Phi_{Z \times_G A}(F) : b \in B$ and $F \in Z \times_G A \}$ generates a dense subalgebra of $Z \times_G (B \otimes_{\max} A)$. Thus, by the universal property of the maximal tensor product we obtain a surjective $*$-homomorphism $\Phi_B \otimes \Phi_{Z \times_G A}$: $B \otimes_{\max} (Z \times_G A) \to Z \times_G (B \otimes_{\max} A)$ which is clearly G-equivariant and $C_0(X)$-linear with respect to the $C_0(X)$-structures of the induced algebras. To see that $\Phi_B \otimes \Phi_{Z \times_G A}$ is an isomorphism, it suffices to see that the induced maps

$$(\Phi_B \otimes \Phi_{Z \times_G A})^x : (B \otimes_{\max} (Z \times_G A))_x \to (Z \times_G (B \otimes_{\max} A))_x$$

are isomorphisms for all x. To see this, note that both fibres $(B \otimes_{\max} (Z \times_G A))_x$ and $(Z \times_G (B \otimes_{\max} A))_x$ are isomorphic to $B \otimes_{\max} A$. If we do these identifications, then for $b \in B$ and $a \in A$ we can compute $(\Phi_B \otimes \Phi_{Z \times_G A})^x (a \otimes b)$ as follows: Choose $F \in Z \times_G A$ and $z \in Z$ with $q(z) = x$ and $F(z) = a$. Then

$$(\Phi_B \otimes \Phi_{Z \times_G A})^x (a \otimes b) = \Phi_B \otimes \Phi_{Z \times_G A}(b \otimes F)(z) = b \otimes F(z) = b \otimes a.$$

Thus $(\Phi_B \otimes \Phi_{Z \times_G A})^x$ is the identity on $A \otimes_{\max} B$ for each $x \in X$.

Now let (A, G, α) and B be as in the proposition. Then $B \otimes_{\max} (Z \times_G A)$ and $Z \times_G (B \otimes_{\max} A)$ are $C_0(X \times X \times X)$-algebras, and it follows directly from the definition that $\Phi_B \otimes \Phi_{Z \times_G A}$ is $C_0(X \times X \times X)$-linear. If $\Delta^3(X) = \{(x, x, x) : x \in X\}$ denotes the diagonal in $X \times X \times X$, then $B \otimes_X Z * A$ is the restriction of $B \otimes_{\max} (Z \times_G A)$ to $\Delta^3(X)$ and $Z * (B \otimes_X A)$ is the restriction of $Z \times_G (B \otimes_{\max} A)$ to $\Delta^3(X)$. Thus the result follows from the $C_0(X \times X \times X)$-linearity of $\Phi_B \otimes \Phi_{Z \times_G A}$.

4. CROSSED PRODUCTS BY INNER ACTIONS

In this section, we start with the investigation of crossed products by inner actions on $\mathscr{CR}(X)$-algebras. If $\alpha : G \to \mathrm{Inn}(A)$ is an inner action of a second countable group G on a $\mathscr{CR}(X)$-algebra A, then it follows from the discussion following [7, Remark 2.9] that there exists a cocycle $u \in Z^2(G, C(X, \mathbb{T}))$ and a Borel map $v : G \to \mathscr{U}M(A)$ satisfying

$$\alpha_s = \mathrm{Ad} \, v_s \quad \text{and} \quad v_s v_t = u(s, t) v_{st} \quad \text{for all } s, t \in G. \qquad (4.1)$$

Then [23, Corollary 0.12] implies that the cohomology class $[u] \in H^2(G, C(X, \mathbb{T}))$ is a complete invariant for the exterior equivalence class of α.

(Throughout, $C(X, \mathbb{T})$ is the trivial Polish G-module $C(X, \mathbb{T})$ equipped with the topology of uniform convergence on compact sets.)

To describe the crossed product $A \rtimes_\alpha G$ in terms of the cocycle u it is necessary to work with (Busby–Smith) twisted actions and twisted crossed products. A *twisted action* (α, u) of a second countable locally compact group G on a separable C^*-algebra A consists of a strongly measurable map $\alpha: G \to \operatorname{Aut}(A)$ and a strictly measurable map $u: G \times G \to \mathscr{U}M(A)$ satisfying

(a) $\alpha_e = \operatorname{id}$ and $u(e, s) = u(s, e) = 1$ for all $s \in G$;

(b) $\alpha_s(\alpha_t(a)) = u(s, t)\, \alpha_{st}(a)\, u(s, t)^*$ for all $s, t \in G$;

(c) $\alpha_r(u(s, t))\, u(r, st) = u(r, s)\, u(rs, t)$ for all $s, t, r \in G$.

The quadruple (A, G, α, u) is called a (Busby–Smith or Leptin) *twisted C^*-dynamical system*. If A is a $C_0(X)$-algebra, and α_s is $C_0(X)$-linear for all $s \in G$, then we will call (A, G, α, u) a *twisted $C_0(X)$-system*.

A *covariant homomorphism* of (A, G, α, u) into the multiplier algebra of a separable C^*-algebra B is a pair (Φ, v), where $\Phi: A \to M(B)$ is a non-degenerate homomorphism and $v: G \to \mathscr{U}M(B)$ is strictly measurable such that $v_e = 1$, and such that

$$\Phi(\alpha_s(a)) = v_s \Phi(a) v_s^*, \quad \text{and} \quad v_s v_t = \Phi(u(s, t)) v_{st} \quad \text{for all } s, t \in G, \quad a \in A.$$

The integrated form $\Phi \rtimes v: L^1(G, A) \to M(B)$ is then defined by $\Phi \rtimes v(f) = \int_G \Phi(f(s))\, v_s\, ds$. The following is a slight reformulation of Packer and Raeburn's definition of a crossed product for twisted systems.

DEFINITION 4.1 (cf., [18, Theorem 1.2]).   Let (A, G, α, u) be a twisted system. A *crossed product* for (A, G, α, u) consists of a triple (B, i_A, i_G) satisfying

(a) (i_A, i_G) is a covariant homomorphism of (A, G, α, u) into $M(B)$;

(b) $i_A \rtimes i_G(L^1(G, A))$ is a dense subalgebra of B;

(c)   if (Φ, v) is any covariant homomorphism of (A, G, α, u) into $M(C)$, for some separable C^*-algebra C, then there exists a nondegenerate $*$-homomorphism $\Phi \rtimes v: B \to M(C)$ such that $(\Phi \rtimes v) \circ i_A = \Phi$ and $(\Phi \rtimes v) \circ i_G = v$.

If (B, i_A, i_G) and (C, j_A, j_G) are two different crossed products of (A, G, α, u), then $j_A \rtimes j_G: B \to C$ is an isomorphism with inverse $i_A \rtimes i_G: C \to B$. Thus, the crossed product is unique up to isomorphism, and we will usually suppress the maps i_A and i_G and denote it by $A \rtimes_{\alpha, u} G$. Notice that ordinary (separable) C^*-dynamical systems and their crossed products are recovered as the special case where $u \equiv 1$, in which case we simply write (A, G, α) for the system and $A \rtimes_\alpha G$ for the crossed product.

Remark 4.2. (a) If (A, G, α, u) is a twisted $C_0(X)$-system, then $A \rtimes_{\alpha, u} G$ is a $C_0(X)$-algebra, where the action of $C_0(X)$ on $A \rtimes_{\alpha, u} G$ is defined via the composition of maps

$$C_0(X) \xrightarrow{\phi} \mathscr{Z}M(A) \xrightarrow{i_A} M(A \rtimes_{\alpha, u} G).$$

If W is a nonempty open subset W of X one can deduce from the definition of a crossed product that $((A \rtimes_{\alpha, u} G)_W, i_A^W, i_G^W)$ is a crossed product for $A_W \rtimes_{\alpha^W, u^W} G$, where i_A^W denotes the restriction of i_A to the ideal A_W of A composed with the natural map $\Phi: M(A \rtimes_{\alpha, u} G) \to M((A \rtimes_{\alpha, u} G)_W)$, and $i_G^W = \Phi \circ i_G$. Similarly, if (α^x, u^x) is the twisted action induced from (α, u) on the G-invariant quotient A_x, a crossed product for (A_x, G, α^x, u^x) is given by $((A \rtimes_{\alpha, u} G)_x, i_A^x, i_G^x)$, where i_A^x and i_G^x are the compositions of i_A and i_G with the quotient map $M(A \rtimes_{\alpha, u} G) \to M((A \rtimes_{\alpha, u} G)_x)$. In particular, the fibres $(A \rtimes_{\alpha, u} G)_x$ are isomorphic to $A_x \rtimes_{\alpha^x, u^x} G$. Note that $(i_A^x \times i_G^x) \circ (i_A^W \times i_G^W) = i_A^x \times i_G^x$ for $x \in W$ (see [14] for more details).

(b) If D is a $C_0(X)$-algebra and (Φ, v) is a covariant homomorphism of the $C_0(X)$-system (A, G, α, u) into $M(D)$, then $\Phi \rtimes v$ is $C_0(X)$-linear if and only if Φ is $C_0(X)$-linear. This follows immediately from the equality $(\Phi \rtimes v)(i_A \circ \phi(f)) = \Phi(\phi(f))$ for $f \in C_0(X)$.

If (A, G, α, u) is a twisted system, then there exists a "dual" action of the Pontryagin dual $\hat{G}_{ab}$ of the *abelianization* $G_{ab} = G/\overline{[G, G]}$ of G on $A \rtimes_{\alpha, u} G$. This action is defined by

$$(\alpha, u)_{\chi}^{\wedge} = i_A \rtimes (\bar{\chi} \cdot i_G),$$

where $\bar{\chi} \cdot i_G(s) = \overline{\chi(s)} \, i_G(s)$ (here we view $\chi \in \hat{G}_{ab}$ as a function on G), and $i_A \rtimes (\bar{\chi} \cdot i_G)$ denotes the integrated form of the covariant homomorphism $(i_A, \bar{\chi} \cdot i_G)$. If (A, G, α, u) is a $C_0(X)$-system, then $(A \rtimes_{\alpha, u} G, \hat{G}_{ab}, (\alpha, u)^{\wedge})$ is also a $C_0(X)$-system since i_A is $C_0(X)$-linear.

The following proposition is crucial as it will allow us to untangle certain diagonal twisted actions. It is the $C_0(X)$-analogue for twisted actions of the well known isomorphism of $(A \otimes_{\max} B) \rtimes_{\alpha \otimes \mathrm{id}_B} G$ with $(A \rtimes_\alpha G) \otimes_{\max} B$.

PROPOSITION 4.3. *Let (A, G, α, u) be a twisted $C_0(X)$-system, and let B be a separable $C_0(X)$-algebra. Let $(\mathrm{id}, 1)$ denote the trivial G-action on B, and let $(\alpha \otimes_X \mathrm{id}, u \otimes_X 1)$ denote the diagonal twisted action of G on $A \otimes_X B$. If i_A and i_G denote the canonical maps from A and G into $M(A \rtimes_{\alpha, u} G)$, then $(i_A \otimes_X \mathrm{id}, i_G \otimes_X 1)$ is a covariant homomorphism of $(A \otimes_X B, G, \alpha \otimes_X \mathrm{id}, u \otimes_X 1)$ into $M((A \rtimes_{\alpha, u} G) \otimes_X B)$. In particular, the integrated form $(i_A \otimes_X \mathrm{id}) \rtimes (i_G \otimes_X 1)$ is a $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant isomorphism of $(A \otimes_X B) \rtimes_{\alpha \otimes_X \mathrm{id}, u \otimes_X 1} G$ onto $(A \rtimes_{\alpha, u} G) \otimes_X B$.*

Proof. Put $C := (A \rtimes_{\alpha, u} G) \otimes_X B$. To show that $(C, i_A \otimes_X \mathrm{id}, i_G \otimes_X 1)$ is a crossed product for $(A \otimes_X B, G, \alpha \otimes_X \mathrm{id}, u \otimes_X 1)$, we have to verify conditions (a), (b), and (c) of Definition 4.1. Since (i_A, i_G) is a covariant homomorphism of (A, G, α, u), it follows that $(i_A \otimes_X \mathrm{id}, i_G \otimes_X 1)$ is a covariant homomorphism of $(A \otimes_X B, G, \alpha \otimes_X \mathrm{id}, u \otimes_X 1)$. This proves (a). For (b), let $\Phi: L^1(G, A) \odot B \to L^1(G, A \otimes_X B)$ be defined by $\Phi(f \otimes b)(s) = f(s) \otimes_X b$. Then

$$(i_A \otimes_X \mathrm{id}) \rtimes (i_G \otimes_X 1)(\Phi(f \otimes b)) = \int_G (i_A(f(s)) \otimes_X b)(i_G(s) \otimes_X 1) \, ds$$

$$= \left(\int_G i_A(f(s)) \, i_G(s) \, ds \right) \otimes_X b$$

$$= (i_A \rtimes i_G(f)) \otimes_X b,$$

and (b) follows from the fact that $i_A \rtimes i_G(L^1(G, A))$ is dense in $A \rtimes_{\alpha, u} G$.

For (c), suppose that (Ψ, v) is a covariant homomorphism of $(A \otimes_X B, G, \alpha \otimes_X \mathrm{id}, u \otimes_X 1)$ into $M(D)$ for some separable C^*-algebra D. By Remark 2.5, we have $\Psi = \Psi_A \otimes_X \Psi_B$ such that $\Psi_A(a \cdot f) \Psi_B(b) = \Psi_A(a) \Psi_B(f \cdot b)$ for $f \in C_0(X)$, $a \in A$, and $b \in B$. It is straightforward to check that (Ψ_A, v) is a covariant homomorphism of (A, G, α, u) into $M(D)$ which commutes with Ψ_B. Thus we obtain a homomorphism $(\Psi_A \rtimes v) \otimes \Psi_B$: $(A \rtimes_{\alpha, u} G) \otimes_{\max} B \to M(D)$. For $g \in L^1(G, A)$, $b \in B$, and $f \in C_0(X)$ we have

$$\Psi_A \rtimes v(f \cdot g) \, \Psi_B(b) = \int_G \Psi_A(g(s) \cdot f) \, v(s) \, ds \, \Psi_B(b)$$

$$= \int_G \Psi_A(g(s) \, \Psi(f) \, v(s) \, ds \, \Psi_B(b)$$

$$= \int_G \Psi_A(g(s)) \, v(s) \, ds \, \Psi_B(f \cdot b)$$

$$= \Psi_A \rtimes v(g) \, \Psi_B(f \cdot b).$$

Since $L^1(G, A)$ is dense in $A \rtimes_{\alpha, u} G$ this extends to all $g \in A \rtimes_{\alpha, u} G$. By Remark 2.5, there is a nondegenerate homomorphism $(\Psi_A \rtimes v) \otimes_X \Psi_B$: $C \to M(D)$ satisfying $(\Psi_A \rtimes v) \otimes_X \Psi_B(g \otimes_X b) = \Psi_A \rtimes v(g) \, \Psi_B(b)$ for all elementary tensors $g \otimes_X b$.

Finally, we compute

$$(\Psi_A \rtimes v) \otimes_X \Psi_B(i_A \otimes_X \mathrm{id}(a \otimes_X b)) = (\Psi_A \rtimes v)(i_A(a)) \, \Psi_B(b)$$

$$= \Psi_A(a) \, \Psi_B(b) = \Psi(a \otimes_X b),$$

which proves that $(\Psi \rtimes v) \circ (i_A \otimes_X \mathrm{id}) = \Psi$. Similarly,

$$(\Psi_A \rtimes v) \otimes_X \Psi_B(i_G(s) \otimes_X 1) = (\Psi_A \rtimes v)(i_G(s)) = v_s,$$

which proves that $(\Psi \rtimes v) \circ (i_G \otimes_X 1) = v$, as required.

At this point, we have proved that $(C, i_A \otimes_X \mathrm{id}, i_G \otimes_X 1)$ is a crossed product for $(A \otimes_X B, G, \alpha \otimes_X \mathrm{id}, u \otimes_X 1)$. In particular,

$$(i_A \otimes_X \mathrm{id}) \rtimes (i_G \otimes_X 1): (A \rtimes_{C_0(X)} B) \rtimes_{\alpha \otimes_X \mathrm{id},\, u \otimes_X 1} G \to (A \rtimes_{\alpha, u} G) \otimes_X B$$

is an isomorphism, which is $C_0(X)$-linear since $i_A \otimes_X \mathrm{id}$ is $C_0(X)$-linear. If $\chi \in \hat{G}_{\mathrm{ab}}$, then $\bar{\chi} \cdot (i_G \otimes_X 1) = (\bar{\chi} \cdot i_G) \otimes_X 1$, which implies that $(i_A \otimes_X \mathrm{id}) \rtimes (i_G \otimes_X 1)$ also preserves the dual action of $\hat{G}_{\mathrm{ab}}$. ∎

Remark 4.4. (a) Two twisted actions (α, u) and (β, v) of G on A are called *exterior equivalent* if there exists a strictly Borel map $w: G \to \mathscr{U}M(A)$ satisfying $\beta_s = \mathrm{Ad}\, w_s \circ \alpha_s$ and $v(s, t) = w_s \alpha_s(w_t)\, u(s, t)\, w_{st}^*$ for all $s, t \in G$ (see [17, Definition 3.1]). Note that if (A, G, α, u) is a twisted $C_0(X)$-system, and if (β, v) is exterior equivalent to (α, u), then $\beta_s(f \cdot a) = w_s \alpha_s(f \cdot a) w_s^* = f \cdot (w_s \alpha_s(a) w_s^*) = f \cdot \beta_s(a)$, so that each β_s is $C_0(X)$-linear. Further, if w_s^x denotes the image of w_s in $M(A_x)$, then w^x implements an exterior equivalence between (α^x, u^x) and (β^x, v^x).

(b) Suppose that $w: G \to \mathscr{U}M(A)$ implements an exterior equivalence between the twisted actions (α, u) and (β, v). Let $j_A: A \to M(A \rtimes_{\beta, v} G)$ and $j_G: G \to \mathscr{U}M(A \rtimes_{\beta, v} G)$ denote the canonical maps and let $\mu_G: G \to \mathscr{U}M(A \rtimes_{\beta, v} G)$ be defined by $\mu_G(s) = j_A(w_s^*)\, j_G(s)$. Then (j_A, μ_G) is a covariant homomorphism of (A, G, α, u), and $j_A \rtimes \mu_G$ is an isomorphism of $A \rtimes_{\alpha, u} G$ and $A \rtimes_{\beta, v} G$ (see [17, Lemma 3.3]). Moreover, if (A, G, α, u) is a twisted $C_0(X)$-system, then the isomorphism $j_A \rtimes \mu_G$ above is necessarily $C_0(X)$-linear (since j_A is $C_0(X)$-linear) and therefore implements an isomorphism between the $C_0(X)$-systems $(A \rtimes_{\alpha, u} G, \hat{G}_{\mathrm{ab}}, (\alpha, u)^\wedge)$ and $(A \rtimes_{\beta, v} G, \hat{G}_{\mathrm{ab}}, (\beta, v)^\wedge)$. Further, if $\chi \in \hat{G}_{\mathrm{ab}}$, then it follows from the definition of the dual actions that

$$\begin{aligned}(j_A \rtimes \mu_G) \circ (\alpha, u)_\chi^\wedge &= j_A \rtimes (\bar{\chi} \cdot \mu_G) = ((j_A \rtimes (\bar{\chi} \cdot ((j_A \circ w) \cdot j_G)) \\ &= j_A \rtimes ((j_A \circ w) \cdot (\bar{\chi} \cdot j_G)) = (\beta, v)_\chi^\wedge \circ (j_A \rtimes \mu_G), \quad (4.3)\end{aligned}$$

so that $j_A \rtimes \mu_G$ is $\hat{G}_{\mathrm{ab}}$-equivariant.

Twisted crossed products with $A = C_0(X)$ abelian and α trivial play a central rôle in our analysis. Such a crossed product is called a twisted transformation group C^*-algebra and is denoted by $C^*(G, X, u)$; a nice survey article is [15]. For our purposes, we need only remark that the condition on the twist u implies that u is a cocycle in the Moore cohomology group $Z^2(G, C(X, \mathbb{T}))$ for the trivial action of G on $C(X, \mathbb{T})$. Moreover, if

$u, v \in Z^2(G, C(X, \mathbb{T}))$, then (id, u) is exterior equivalent to (id, v) if and only if $[u] = [v]$ in $H^2(G, C(X, \mathbb{T}))$. In what follows we shall write $\hat{u}$ (instead of $(\mathrm{id}, u)^{\wedge}$ as in (4.2)) for the dual action of $\hat{G}_{\mathrm{ab}}$ on $C^*(G, X, u)$. It follows from Remark 4.4 that $(C^*(G, X, u), \hat{G}_{\mathrm{ab}}, u)$ only depends on the cohomology class of u.

DEFINITION 4.5. Suppose that A is a separable $C_0(X)$-algebra, and that $u \in Z^2(G, C(X, \mathbb{T}))$. A *u-homomorphism* is a strictly measurable map $v: G \to \mathscr{U}M(A)$ satisfying

$$v_e = 1 \qquad \text{and} \qquad v_s v_t = u(s, t) \cdot v_{st} \qquad \text{for all} \quad s, t \in G$$

(where we extended the action of $C_0(X)$ on A to the multiplier algebra $C^b(X) = M(C_0(X))$). If $\alpha: G \to \mathrm{Aut}(A)$ is an action, then we say that α is *implemented* by the *u*-homomorphism $v: G \to \mathscr{U}M(A)$ if $\alpha_s = \mathrm{Ad}\, v_s$ for all $s \in G$.

Notice that if v is a *u*-homomorphism and if $\phi: C_0(X) \to \mathscr{Z}M(A)$ is the homomorphism determined by the $C_0(X)$-action on A, then (ϕ, v) is a covariant homomorphism of $(C_0(X), G, \mathrm{id}, u)$ into $M(A)$. We will write $\bar{u}$ for the inverse of the cocycle $u \in Z^2(G, C(X, \mathbb{T}))$. If $\alpha: G \to \mathrm{Inn}(A)$ is implemented by the *u*-homomorphism $v: G \to \mathscr{U}M(A)$, then it follows from $\alpha_s = \mathrm{Ad}\, v_s$ and $1 = v_s v_t \bar{u}(s, t) v_{st}^*$ for $s, t \in G$, that v implements an exterior equivalence between the twisted actions $(\alpha, 1)$ and $(\mathrm{id}, \bar{u})$. We use this observation for the proof of

PROPOSITION 4.6. *Suppose that A is a $C_0(X)$-algebra, that $u \in Z^2(G, C(X, \mathbb{T}))$, and that $\alpha: G \to \mathrm{Aut}(A)$ is implemented by a u-homomorphism $v: G \to \mathscr{U}M(A)$. Then $A \rtimes_\alpha G$ is isomorphic to $C^*(G, X, \bar{u}) \otimes_X A$. In particular, if $i_G: G \to \mathscr{U}M(C^*(G, X, \bar{u}))$ is the canonical map, then $(1 \otimes_X \mathrm{id}_A, i_G \otimes_X v)$ is a covariant homomorphism of (A, G, α) into $M(C^*(G, X, \bar{u}) \otimes_X A)$ whose integrated form is a $C_0(X)$-linear covariant isomorphism of $(A \rtimes_\alpha G, \hat{G}_{\mathrm{ab}}, \hat{\alpha})$ onto $(C^*(G, X, \bar{u}) \otimes_X A, \hat{G}_{\mathrm{ab}}, \hat{\bar{u}} \otimes_X \mathrm{id})$.*

Proof. It follows from Proposition 4.3 that $(i_{C_0(X)} \otimes_X \mathrm{id}_A, i_G \otimes_X 1)$ is a covariant homomorphism of $(C_0(X) \otimes_X A, G, \mathrm{id} \otimes_X \mathrm{id}, \bar{u} \otimes_X 1)$ into $M(C^*(G, X, \bar{u}) \otimes_X A)$ whose integrated form is a $C_0(X)$-linear and $\hat{G}_{\mathrm{ab}}$-equivariant isomorphism of $(C_0(X) \otimes_X A) \rtimes_{\mathrm{id} \otimes_X \mathrm{id}, \bar{u} \otimes_X 1} G$ onto $C^*(G, X, \bar{u}) \otimes_X A$.

Let $\Phi: C_0(X) \otimes_X A \to A$ be the isomorphism defined on elementary tensors by $\Phi(f \otimes_X a) = f \cdot a$. Then Φ carries the trivial action $\mathrm{id} \otimes \mathrm{id}$ to the trivial action on A and we have $\Phi(\bar{u}(s, t) \otimes_X 1)a = \bar{u}(s, t) \cdot a$ for all $a \in A$. Thus, regarding $\bar{u}$ as a map $\bar{u}: G \times G \to \mathscr{U}M(A)$ via the $C_0(X)$-action on A, we see that Φ induces a $C_0(X)$-linear and $\hat{G}_{\mathrm{ab}}$-equivariant isomorphism between $(C_0(X) \otimes_X A) \rtimes_{\mathrm{id} \otimes_X \mathrm{id}, \bar{u} \otimes_X 1} G$ and $A \rtimes_{\mathrm{id}, \bar{u}} G$. Moreover, Φ carries

the covariant homomorphism $(i_{C_0(X)} \otimes_X \mathrm{id}_A, i_G \otimes_X 1)$ to the covariant homomorphism $(1 \otimes_X \mathrm{id}_A, i_G \otimes_X 1)$, which implies that $(1 \otimes_X \mathrm{id}_A) \rtimes (i_G \otimes_X 1)$ is a $C_0(X)$-linear and $\hat{G}_{\mathrm{ab}}$-equivariant homomorphism of $A \rtimes_{\mathrm{id}, \bar{u}} G$ onto $C^*(G, X, \bar{u}) \otimes_X A$. Since v is a u-homomorphism, it follows that v implements an exterior equivalence between the twisted actions $(\alpha, 1)$ and $(\mathrm{id}, \bar{u})$. Thus the result follows from Remark 4.4. ∎

If $\alpha: G \to \mathrm{Inn}(A)$ is any inner action of G on the $\mathscr{CR}(X)$-algebra A, then as we observed at the beginning of this section, there exists a unique class $[u] \in H^2(G, C(X, \mathbb{T}))$ and a u-homomorphism $v: G \to \mathscr{U}M(A)$ which implements α. Thus we get

COROLLARY 4.7. *Let A be a $\mathscr{CR}(X)$-algebra and let $\alpha: G \to \mathrm{Inn}(A)$ be an inner action of G on A. Let $u \in Z^2(G, C(X, \mathbb{T}))$ be associated to α as above. Then $(A \rtimes_\alpha G, \hat{G}_{\mathrm{ab}}, \hat{\alpha})$ is $C_0(X)$-isomorphic to $(C^*(G, X, \bar{u}) \otimes_X A, \hat{G}_{\mathrm{ab}}, \hat{\bar{u}} \otimes_X \mathrm{id})$.*

Remark 4.8. It is shown in [9, Proposition 3.1] that for every $u \in Z^2(G, C(X, \mathbb{T}))$, there exists a u-homomorphism v of G into $\mathscr{U}M(C_0(X, \mathscr{K}))$. From this it follows that for any stable $C_0(X)$-algebra A, there exists a u-homomorphism $w: G \to \mathscr{U}M(A)$: simply identify $A \cong A \otimes \mathscr{K}$ with $A \otimes_X C_0(X, \mathscr{K})$ and define $w = 1 \otimes_X v$. Thus, if $A \in \mathscr{CR}(X)$ is stable, then there exists a natural one-to-one correspondence between the exterior equivalence classes of inner G-actions on A and $H^2(G, C(X, \mathbb{T}))$, and by the above results the crossed products can be described in terms of the central twisted transformation group algebras $C^*(G, X, \bar{u})$.

5. $\hat{G}_{\mathrm{ab}}$-FIBRE PRODUCTS AND LOCALLY UNITARY ACTIONS ON $C_0(X, \mathscr{K})$

The exterior equivalence classes of locally unitary actions on A are classified by the isomorphism classes of principal $\hat{G}_{\mathrm{ab}}$-bundles, or equivalently, by classes in $H^1(X, \hat{\mathscr{G}}_{\mathrm{ab}})$ as described in Section 3 [7, Section 3]. If $A \in \mathscr{CR}(X)$, then an action $\alpha: G \to \mathrm{Aut}(A)$ is called *locally unitary* if each point in X has an open neighborhood W such that the restriction α^W of α to the ideal A_W of A is unitary. The class corresponding to a locally unitary action is determined as follows. Choose any locally finite open cover $(W_i)_{i \in I}$ of X such that each restriction $\alpha^i := \alpha^{W_i}$ is unitary. For each $i \in I$, set $A_i := A_{W_i}$ and let $w^i: G \to \mathscr{U}M(A_i)$ be a strictly continuous map such that $\alpha^i = \mathrm{Ad}\, w^i$. If $w^i(s, x)$ denotes the element of $M(A_x)$ induced by w^i, then there exist continuous functions $\gamma_{ij}: W_{ij} \to \hat{G}_{\mathrm{ab}}$ satisfying

(a) $w^i(s, x) = \gamma_{ij}(x)(s)\, w^j(s, x)$ for all $x \in W_{ij} := W_i \cap W_j$,

(b) $\gamma_{ij}(x)\, \gamma_{jk}(x) = \gamma_{ik}(x)$ for all $x \in W_{ijk} := W_i \cap W_j \cap W_k$.

The last property implies that $(\gamma_{ij})_{i,\,j\in I}$ is a cocycle in $Z^1(X,\hat{\mathscr{G}}_{ab})$, and by [7, Proposition 3.3] the class $\zeta(\alpha)$ of this cocycle in $H^1(X,\hat{\mathscr{G}}_{ab})$ is a complete invariant for the exterior equivalence class of α. If A is stable, then all classes in $H^1(X,\hat{\mathscr{G}}_{ab})$ appear this way. In order to state the main result of this section we need

DEFINITION 5.1. Let (α,u) and (β,v) be two twisted $C_0(X)$-linear actions of G on a $C_0(X)$-algebra A. Then we say that (α,u) is *locally exterior equivalent* to (β,v) if every point in X has an open neighborhood W such that (α^W,u^W) is exterior equivalent to (β^W,v^W).

Remark 5.2 (a) If $A\in\mathscr{CR}(X)$, then $\alpha\colon G\to\mathrm{Aut}(A)$ is locally unitary if and only if α is locally exterior equivalent to the trivial action id_A.

(b) Let (α,u) be a twisted action of G on A and let $\mathscr{L}\mathscr{U}M(A)=\mathscr{U}M(A)\cap\mathscr{Z}M(A)$. Then it follows from the definition of exterior equivalence that a strictly measurable map $\lambda\colon G\to\mathscr{U}M(A)$ implements an exterior equivalence of (α,u) with itself if and only if $\lambda_s\in\mathscr{L}\mathscr{U}M(A)$ and $\lambda_{st}=\lambda_s\alpha_s(\lambda_t)$ for all $s,t\in G$.

(c) Let A be a C^*-algebra. Then the isomorphism $\mathscr{Z}M(A)\to\mathscr{Z}M(A\otimes\mathscr{K})$ given by $z\mapsto z\otimes 1$ induces a homeomorphism between $\mathscr{Z}\mathscr{U}M(A)$ and $\mathscr{Z}\mathscr{U}M(A\otimes\mathscr{K})$ with respect to the strict topologies. To see this assume that $z_i\to z$ strictly in $\mathscr{Z}\mathscr{U}M(A)$. Then $z_ia\otimes c\to za\otimes c$ for every elementary tensor $a\otimes c\in A\otimes\mathscr{K}$. Since $\mathscr{Z}\mathscr{U}M(A\otimes\mathscr{K})$ is a bounded subset of $M(A\otimes\mathscr{K})$ this is enough to prove that $z_i\otimes 1\to z\otimes 1$ in $\mathscr{Z}\mathscr{U}M(A\otimes\mathscr{K})$. Conversely, if $z_i\otimes 1\to z\otimes 1$ strictly in $\mathscr{Z}\mathscr{U}M(A)$, then choose any $c\in\mathscr{K}$ with $\|c\|=1$ to deduce that $\|z_ia-za\|=\|z_ia\otimes c-za\otimes c\|\to 0$ for all $a\in A$.

THEOREM 5.3. *Let (α,u) and (β,v) be two locally exterior equivalent twisted $C_0(X)$-linear actions of G on a $C_0(X)$-algebra A. Suppose further that $\delta\colon G\to\mathrm{Aut}(C_0(X,\mathscr{K}))$ is locally unitary and that $q\colon Z\to X$ is a principal $\hat{G}_{ab}$-bundle corresponding to $\zeta(\delta)\in H^1(X,\hat{\mathscr{G}}_{ab})$. If $(\beta\otimes\mathrm{id},v\otimes 1)$ is exterior equivalent to $(\alpha\otimes_X\delta,u\otimes_X 1)$ as actions on $A\otimes\mathscr{K}\cong A\otimes_X C_0(X,\mathscr{K})$, then $(A\rtimes_{\beta,v}G,\hat{G}_{ab},(\beta,v)^\wedge)$ is $C_0(X)$-isomorphic to the G-fibre product $(Z*(A\rtimes_{\alpha,u}G),\hat{G}_{ab},Z*(\alpha,u)^\wedge)$.*

Proof. Since the isomorphism class of $(Z*(A\rtimes_{\alpha,u}G),\hat{G}_{ab},Z*(\alpha,u)^\wedge)$ only depends on the isomorphism class of $q\colon Z\to X$, it follows from the assumptions and the discussion at the beginning of this section that we can find a locally finite open cover $(W_i)_{i\in I}$ of X and a cocycle $(\gamma_{ij})_{i,\,j\in I}$ in $Z^1(X,\hat{\mathscr{G}}_{ab})$ (with respect to this cover) satisfying the conditions:

(a) There exist strictly continuous maps $w^i\colon G\to\mathscr{U}M(C_0(W_i,\mathscr{K}))$ such that $\delta^i=\mathrm{Ad}\,w_i$ for all $i\in I$, and $w^i(s,x)=\gamma_{ij}(x)(s)\,w^j(s,x)$ for all $x\in W_{ij}$.

(b)   There exist continuous local sections $\varphi_i: W_i \to q^{-1}(W_i)$ such that $\varphi_i(x) = \gamma_{ij}(x)\,\varphi_j(x)$ for all $x \in W_{ij}$.

(c)   There exist strictly measurable maps $\kappa^i: G \to \mathscr{U}M(A_i)$, $A_i := A_{W_i}$, implementing exterior equivalences between (β^i, v^i) and (α^i, u^i); that is, $\beta^i = \operatorname{Ad} \kappa^i \circ \alpha^i$ and $v^i(s, t) = \kappa^i_s \alpha^i_s(\kappa^i_t)\,u^i(s, t)(\kappa^i_{st})^*$ for all $s, t \in G$.

We want to use these data to define $C_0(W_i)$-linear and $\hat{G}_{ab}$-equivariant isomorphisms $\Phi^i: (A \rtimes_{\beta, v} G)_{W_i} \to (A \rtimes_{\alpha, u} G)_{W_i}$ which satisfy

$$\Phi^j(d)(x) = (\alpha^x, u^x)^{\wedge}_{\gamma_{ij}(x)}(\Phi^i(d))(x) \qquad \text{for all} \quad d \in A \rtimes_{\beta, v} G \quad \text{and} \quad x \in W_{ij}. \tag{5.1}$$

If this can be done, then since the bundle $q: Z \to X$ has transition functions $(\gamma_{ij})_{i,j \in I}$, the result follows from Proposition 3.11 and the observation that the action on $A_x \rtimes_{\alpha^x, u^x} G$ induced by $(\alpha, u)^{\wedge}$ is $(\alpha^x, u^x)^{\wedge}$.

We claim that we may assume that the exterior equivalences $\kappa_i: G \to \mathscr{U}M(A_i)$ satisfy the relation

$$\kappa_i(s, x) = \gamma_{ij}(x)(s)\,\kappa_j(s, x) \qquad \text{for all} \quad i, j \in I, \quad x \in W_{ij} \quad \text{and} \quad s \in G. \tag{5.2}$$

To make notation easier we think of $(\alpha \otimes_X \delta, u \otimes_X 1)$ as the family of actions $(\alpha^x \otimes \delta^x, u^x \otimes 1)_{x \in X}$, and we identify $A_i \otimes \mathscr{K}$ with $(A \otimes_X C_0(X, \mathscr{K}))_{W_i}$ $\cong A_i \otimes_{W_i} C_0(W_i, \mathscr{K})$. Then (by committing a criminal abuse of notation) we denote the restriction of $(\alpha \otimes_X \delta, u \otimes_X 1)$ to $A_i \otimes \mathscr{K}$ by $(\alpha^i \otimes \delta^i, u^i \otimes 1)$. By assumption there exists a strictly measurable map $\mu: G \to \mathscr{U}M(A \otimes \mathscr{K})$ which implements an exterior equivalence between $(\beta \otimes \operatorname{id}, v \otimes 1)$ and $(\alpha \otimes_X \delta, u \otimes_X 1)$, and we denote by μ^i the restriction of μ to $A_i \otimes \mathscr{K}$. Since on each W_i the map $w^i: G \to \mathscr{U}M(C_0(W_i, \mathscr{K}))$ implements an exterior equivalence between δ^i and the trivial action on $C_0(W_i, \mathscr{K})$, we can combine this with μ^i in order to obtain exterior equivalences σ^i between $(\beta^i \otimes \operatorname{id}, v^i \otimes 1)$ and $(\alpha^i \otimes \operatorname{id}, u^i \otimes 1)$ given by

$$\sigma^i(s, x) = \mu^i(s, x)(1 \otimes w^i(s, x)), \qquad (s, x) \in G \times W_i. \tag{5.3}$$

At this point we have two exterior equivalences between $(\beta^i \otimes \operatorname{id}, v^i \otimes 1)$ and $(\alpha^i \otimes \operatorname{id}, u^i \otimes 1)$, namely σ^i and $\kappa^i \otimes 1$. But then $s \mapsto \lambda^i(s) = (\kappa^i(s) \otimes 1)^*\,\sigma^i(s)$ is an exterior equivalence for $(\alpha^i \otimes 1, u^i \otimes 1)$ with itself. Thus, Remark 5.2(b) implies that λ^i takes values in $\mathscr{Z}\mathscr{U}M(A_i \otimes \mathscr{K})$ and satisfies $\lambda^i(st) = \lambda^i(s)(\alpha^i \otimes \operatorname{id})_s(\lambda^i(t))$. It follows from Remark 5.2(c) that there

exists a strongly measurable map $\tilde{\lambda}^i\colon G \to \mathscr{ZUM}(A_i)$ such that $\lambda^i(s) = \tilde{\lambda}^i(s) \otimes 1$ for all $s \in G$, and which satisfies $\tilde{\lambda}^i(st) = \tilde{\lambda}^i(s)\,\alpha^i_s(\tilde{\lambda}^i(t))$ for all $s, t \in G$. Therefore, $\tilde{\lambda}^i$ is an exterior equivalence between (α^i, u^i) and itself. Thus,

$$\tilde{\kappa}^i(s) = \kappa^i(s)\,\tilde{\lambda}^i(s) \qquad \text{for all} \quad s \in G$$

defines a new exterior equivalence between (β^i, v^i) and (α^i, u^i). Since $\mu^i(s, x) = \mu^j(s, x)$ if $x \in W_{ij}$, we get

$$\tilde{\kappa}^i(s, x)\,\tilde{\kappa}^j(s, x)^* \otimes 1 = \kappa^i(s, x)\,\tilde{\lambda}^i(s, x)\,\tilde{\lambda}^j(s, x)^*\,\kappa^j(s, x)^* \otimes 1$$
$$= (\kappa^i(s, x) \otimes 1)\,\lambda^i(s, x)\,\lambda^j(s, x)^*\,(\kappa^j(s, x)^* \otimes 1)$$

which, by (5.3), is

$$= \sigma^i(s, x)\,\sigma^j(s, x)^* = \mu^i(s, x)(1 \otimes w^i(s, x))(1 \otimes w^j(s, x)^*)\,\mu^j(s, x)^*$$
$$= \mu^i(s, x)(1 \otimes w^i(s, x)\,w^j(s, x)^*)\,\mu^j(s, x)^*$$
$$= \gamma_{ij}(x)(s)\,\mu^i(s, x)\,\mu^j(s, x)^* = \gamma_{ij}(x)(s) \otimes 1.$$

Thus we see that the $\tilde{\kappa}^i(x, s) = \gamma_{ij}(x)(s)\,\tilde{\kappa}^j(x, s)$ for all $i, j \in I$ and $x \in W_{ij}$. Thus we can replace κ^i by $\tilde{\kappa}^i$ so that (5.2) holds. This proves the claim.

We now identify $(A \rtimes_{\beta, v} G)_{W_i}$ and $(A \rtimes_{\alpha, u} G)_{W_i}$ with $A_i \rtimes_{\beta^i, v^i} G$ and $A_i \rtimes_{\alpha^i, u^i} G$ as in Remark 4.2, and we let $j^i_A = j^{W_i}_A$ and $j^i_G = j^{W_i}_G$ denote the natural embeddings of A_i and G in $M(A \rtimes_{\alpha^i, u^i})$. Then it follows from Remark 4.4 that

$$\Phi^i := j^i_A \rtimes (j^i_A \circ (\kappa^i)) \cdot j^i_G \colon A_i \rtimes_{\beta^i, v^i} G \to A_i \rtimes_{\alpha^i, u^i} G$$

is indeed a $C_0(W_i)$-linear and $\hat{G}_{\mathrm{ab}}$-equivariant isomorphism for all $i \in I$. Moreover, for all $x \in W_{ij}$ and $d \in A \rtimes_{\beta, v} G$ we get (again using Remark 4.2)

$$\Phi^j(d)(x) = (j^x_A \rtimes (j^x_A \circ \kappa^j(\cdot, x)) \cdot j^x_G)(d(x))$$
$$= (j^x_A \rtimes \gamma_{ij}(x)^{-1} \cdot (j^x_A \circ \kappa^i(\cdot, x)) \cdot j^x_G)(d(x))$$
$$= (\alpha^x, u^x)^{\wedge}_{\gamma_{ij}(x)}\,((j^x_A \rtimes (j^x_A \circ \kappa^i(\cdot, x)) \cdot j^x_G)(d(x)))$$
$$= (\alpha^x, u^x)^{\wedge}_{\gamma_{ij}(x)}\,(\Phi^i(d)(x)).$$

Thus the Φ^i satisfy equation (5.1) which completes the proof. ∎

CORROLARY 5.4. *Let (α, u) be a twisted $C_0(X)$-linear action of G on A and let $\delta\colon G \to \mathrm{Aut}(C_0(X, \mathscr{K}))$ be locally unitary. Let $q\colon Z \to X$ be a principal $\hat{G}_{\mathrm{ab}}$-bundle corresponding to $\zeta(\delta)$. Then there is a $C_0(X)$-linear and $\hat{G}_{\mathrm{ab}}$-equivariant isomorphism between $(A \otimes_X C_0(X, \mathscr{K})) \rtimes_{\alpha \otimes_X \delta,\, u \otimes_X 1} G$ and $Z * (A \rtimes_{\alpha, u} G) \otimes \mathscr{K}$.*

Proof. First recall that there exists a $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant isomorphism between $(A \rtimes_{\alpha, u} G) \otimes \mathcal{K}$ and $(A \otimes \mathcal{K}) \rtimes_{\alpha \otimes id, u \otimes 1} G$, and it follows from Proposition 3.12 that this isomorphism induces a $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant isomorphism between $Z * (A \rtimes_{\alpha, u} G) \otimes \mathcal{K}$ and $Z * ((A \otimes \mathcal{K}) \rtimes_{\alpha \otimes id, u \otimes 1} G)$ (with respect to the $\hat{G}_{ab}$-actions $Z * (\alpha, u)^{\wedge} \otimes id$ and $Z * ((\alpha \otimes id, u \otimes 1)^{\wedge})$). Since $(\alpha \otimes_X \delta, u \otimes_X 1)$ is locally exterior equivalent to $(\alpha \otimes id, u \otimes 1)$ we can apply Theorem 5.3 and the result follows. ∎

As a consequence of Theorem 5.3, we get a general description of crossed products by locally unitary actions.

THEOREM 5.5. *Let* $A \in \mathscr{CR}(X)$ *and let* $\alpha: G \to Aut(A)$ *be locally unitary. Let* $q: Z \to X$ *be a principal* $\hat{G}_{ab}$*-bundle corresponding to* $\zeta(\alpha)$, *and let* $\mu: \hat{G}_{ab} \to Aut(C^*(G))$ *denote the dual action of* $\hat{G}_{ab}$ *on* $C^*(G) = \mathbb{C} \rtimes_{id} G$. *Then* $(A \rtimes_\alpha G, \hat{G}_{ab}, \hat{\alpha})$ *is isomorphic to the* $C_0(X)$*-system* $(A \otimes_X (Z \times_{\hat{G}_{ab}} C^*(G)), \hat{G}_{ab}, id \otimes_X Ind \mu)$.

Proof. Since $C_0(X, \mathcal{K})$ is stable, we can choose a locally unitary action $\delta: G \to Aut(C_0(X, \mathcal{K}))$ such that $\zeta(\alpha) = \zeta(\delta)$. Then it follows from [7, Lemma 3.5] and Remark 5.2 that $\alpha \otimes id$ is exterior equivalent to $id \otimes_X \delta$ and that α is locally exterior equivalent to id. Thus, Theorem 5.3 implies that $(A \rtimes_\alpha G, \hat{G}_{ab}, \hat{\alpha})$ is isomorphic to $(Z * (A \rtimes_{id} G), \hat{G}_{ab}, Z * \widehat{id})$. But $(A \rtimes_{id} G, \hat{G}_{ab}, \widehat{id})$ is $C_0(X)$-isomorphic to $(A \otimes_{max} C^*(G), \hat{G}_{ab}, id \otimes \mu)$, and the latter may be written as $(A \otimes_X (C_0(X, C^*(G)), \hat{G}_{ab}, id \otimes_X \mu)$. But then we can apply Remark 3.4(c) and Proposition 3.12 to see that $(A \rtimes_\alpha G, \hat{G}_{ab}, \hat{\alpha})$ is $C_0(X)$-isomorphic to $(A \otimes_X (Z \times_{\hat{G}_{ab}} C^*(G)), \hat{G}_{ab}, id \otimes_X Ind \mu)$. ∎

Remark 5.6. (a) Suppose that $A \in \mathscr{CR}(X)$, $\alpha: G \to Aut(A)$ is locally unitary and $q: Z \to X$ is a $\hat{G}_{ab}$-principal bundle corresponding to $\zeta(\alpha)$. If either A or $C^*(G)$ is nuclear (in particular, if A is type I or G is amenable), then $Prim(A \rtimes_\alpha G)$ is isomorphic to $Prim(A) \times_X (Z \times_{\hat{G}_{ab}} Prim(C^*(G)))$ as a topological bundle over X with group $\hat{G}_{ab}$. This follows at once from Theorem 5.5 together with Remark 2.5 and Proposition 3.2. In particular, if $Prim(A)$ is Hausdorff (so that $Prim(A) = X$), then $Prim(A \rtimes_\alpha G)$ is isomorphic to $Z \times_{\hat{G}_{ab}} Prim(C^*(G))$ as a topological bundle over X with group $\hat{G}_{ab}$.

(b) If in addition, A is of type I, then $(A \rtimes_\alpha G)$ is isomorphic to $\hat{A} \times_X (Z \times_{\hat{G}_{ab}} \hat{G})$ as a topological bundle over X with group $\hat{G}_{ab}$. In particular, if $\hat{A}$ is Hausdorff (so that $\hat{A} = X$), then $(A \rtimes_\alpha G)^{\wedge}$ is isomorphic to $Z \times_{\hat{G}_{ab}} \hat{G}$ as a topological bundle over X with group $\hat{G}_{ab}$. Thus we get a complete bundle theoretic description of $(A \rtimes_\alpha G)$ in terms of the Phillips–Raeburn obstruction $\zeta(\alpha)$. Of course, if G is abelian, this coincides with the description given by Phillips and Raeburn in [21].

6. CROSSED PRODUCTS BY LOCALLY INNER ACTIONS OF SMOOTH GROUPS

In addition to our separability proviso, we shall want some additional assumptions and notations to be in effect throughout this section.

Standing Assumptions.   We assume that G is a smooth second countable locally compact group such that G_{ab} is compactly generated. We fix a representation group

$$e \to C \to H \to G \to e,$$

and identify C with $H^2(G, \mathbb{T})^\wedge$. Therefore, we may assume that the transgression map tg: $\hat{C} = H^2(G, \mathbb{T}) \to H^2(G, \mathbb{T})$ is the identity map [7, Remark 4.4]. We let $\sigma \in Z^2(G, H^2(G, \mathbb{T})^\wedge)$ be the corresponding cocycle in Moore cohomology. When convenient, we will view σ as an element of $Z^2(G, C(H^2(G, \mathbb{T}), \mathbb{T}))$. Finally, X will be a second countable locally compact space, and $A \in \mathscr{CR}(X)$. As in previous sections, we will identify $A \otimes \mathscr{K}$ and $A \otimes_X C_0(X, \mathscr{K})$.

Under these assumptions we were able to extend some results of Packer [16] and give a classification of the exterior equivalence classes $\mathscr{LI}_G(A)$ of locally inner actions of G on A in terms of the cohomology groups $H^1(X, \hat{\mathscr{G}}_{ab})$ and $H^2(G, \mathbb{T})$ [7, Theorem 6.3]. As a special case, it is useful to note that it follows from the argument in [28, Corollary 2.2] that if A has continuous trace, then the locally inner actions of G on A coincide with the $C_0(X)$-linear actions on A. In particular, $\mathscr{LI}_G(C_0(X, \mathscr{K}))$ coincides with the abelian group $\mathscr{E}_G(X)$ of exterior equivalence classes of $C_0(X)$-linear actions of G on $C_0(X, \mathscr{K})$. The group operation in $\mathscr{E}_G(X)$ is given by $[\gamma] \cdot [\delta] := [\gamma \otimes_X \delta]$ [7, Section 5].

In this section, we want to use our classification of locally inner actions to describe the $C_0(X)$-bundle structures of the crossed products $A \rtimes_\alpha G$ in terms of $C^*(H)$. To do this we have to recall the basic ingredients of our classification theory.

If $[\alpha] \in \mathscr{LI}_G(A)$, then $[\alpha]$ determines a continuous map $\varphi_\alpha \colon X \to H^2(G, \mathbb{T})$ such that the action $\alpha^x \colon G \to \operatorname{Aut}(A_x)$ is implemented by an $\varphi_\alpha(x)$-homomorphism $v^x \colon G \to \mathscr{U}M(A_x)$ (compare with Definition 4.5). Now let $\sigma \in Z^2(G, C(H^2(G, \mathbb{T}), \mathbb{T}))$ be the cocycle as in our standing assumptions. Then we can pull back σ to a cocycle $\varphi_\alpha^*(\sigma) \in Z^2(G, C(X, \mathbb{T}))$ given by

$$\varphi_\alpha^*(\sigma)(s, t)(x) = \sigma(s, t)(\varphi_\alpha(x)).$$

By [9, Proposition 3.1], there exists an inner action $\gamma \colon G \to \operatorname{Aut}(C_0(X, \mathscr{K}))$ which is implemented by a $\varphi_\alpha^*(\sigma)$-homomorphism $v \colon G \to \mathscr{U}M(C_0(X, \mathscr{K}))$. If $[\gamma^o]$ denotes the inverse of $[\gamma]$ in $\mathscr{E}_G(X)$, then $\alpha \otimes_X \gamma^o$ is locally unitary and, therefore, we obtain a class $\zeta_H(\alpha) := \zeta(\alpha \otimes_X \gamma^o) \in H^1(X, \hat{\mathscr{G}}_{ab})$ which

determines the exterior equivalence class of $\alpha \otimes_X \gamma^o$ (compare with the discussion in the previous section). It follows from [7, Theorem 6.3] that $[\alpha] \mapsto \zeta_H(\alpha) \oplus \varphi_\alpha$ defines an injection

$$\Phi^H: \mathscr{LI}_G(A) \to H^1(X, \hat{\mathscr{G}}_{ab}) \oplus C(X, H^2(G, \mathbb{T})).$$

Furthermore, Φ is a bijection whenever A is stable. (When $A = C_0(X, \mathscr{K})$, the map $\Phi^H: \mathscr{E}_G(X) \to H^1(X, \hat{\mathscr{G}}_{ab}) \oplus C(X, H^2(G, \mathbb{T}))$ is an isomorphism of abelian groups [7, Theorem 5.4].)

DEFINITION 6.1 ([7, Theorem 5.4 and Lemma 6.1]). Let

$$\varphi_\alpha \in C(X, H^2(G, \mathbb{T})) \qquad \text{and} \qquad \zeta_H(\alpha) \in H^1(X, \hat{\mathscr{G}}_{ab})$$

be as above. Then we say that $\varphi_\alpha: X \to H^2(G, \mathbb{T})$ is the *Mackey obstruction map* for α and we say that $\zeta_H(\alpha)$ is the *Phillips–Raeburn obstruction of* α *with respect to* H.

Remark 6.2. (a) Notice that the class of the cocycle σ in $H^2(G, C(H^2(G, \mathbb{T}), \mathbb{T}))$ in our standing assumptions depends on the choice of the representation group H. This implies that in general, the action γ constructed above, and therefore the class $\zeta_H(\alpha) \in H^1(X, \hat{\mathscr{G}}_{ab})$, depends on the choice of H. However, the class of σ is uniquely determined by the choice of H, which implies that the class of $\varphi_\alpha^*(\tilde{\sigma})$ in $H^2(G, C(X, \mathbb{T}))$ and hence the exterior equivalence class of γ is also uniquely determined by H (compare with Remark 4.8).

(b) If $\delta: G \to \mathrm{Aut}(C_0(X, \mathscr{K}))$ is a locally unitary action with $\zeta(\delta) = \zeta_H(\alpha) := \zeta(\alpha \otimes_X \gamma^o)$, then it follows from the above constructions and the classification of locally inner actions via $H^1(X, \hat{\mathscr{G}}_{ab})$, that $\alpha \otimes \gamma^o$ is exterior equivalent to $\mathrm{id}_A \otimes_X \delta$ as actions on $A \otimes \mathscr{K}$. Further, since $[\gamma^o]$ is the inverse of $[\gamma]$ in $\mathscr{E}_G(X)$, it follows that $\gamma^o \otimes_X \gamma$ is exterior equivalent to the trivial action on $C_0(X, \mathscr{K})$. Hence we see that $\alpha \otimes \mathrm{id}_{\mathscr{K}}$ is exterior equivalent to both $\mathrm{id}_A \otimes_X \delta \otimes_X \gamma$ and $\mathrm{id}_A \otimes_X \gamma \otimes_X \delta$.

The main idea in our description of crossed products by locally inner actions is to use the group C^*-algebra $C^*(H)$ as a universal bundle over the locally compact space $H^2(G, \mathbb{T})$ with fibres isomorphic to the twisted group algebras $C^*(G, \omega)$, $[\omega] \in H^2(G, \mathbb{T})$. The fact that a representation group H of G does provide a bundle over $H^2(G, \mathbb{T})$ with fibres $C^*(G, \omega)$ was first observed by Packer and Raeburn in [18, Section 1].

More generally, let $e \to N \to L \to G \to e$ be any locally compact central extension of G by an abelian group N. Let $i_L: L \to \mathscr{U}M(C^*(L))$ denote the canonical map and let $\phi: C^*(N) \to M(C^*(L))$ denote the integrated form of the restriction of i_L to N. Since, by assumption, N is central in L it

follows that ϕ takes image in $\mathscr{Z}M(C^*(L))$. Thus, if we identify $C^*(N)$ with $C_0(\hat{N})$ via the Gelfand transform, we see that $C^*(L)$ has a canonical structure as a $C_0(\hat{N})$-algebra. In order to see that the fibres are exactly what we want, it is convenient to write $C^*(L)$ as a central twisted crossed product as in the following lemma. Note that for any abelian locally compact group N, we may view N as a closed subgroup of $C(\hat{N}, \mathbb{T})$ by identifying N with the Pontryagin dual of $\hat{N}$.

LEMMA 6.3. *Let* $e \to N \to L \to G \to e$ *be a second countable central extension of* G *by the abelian group* N, *and let* $\sigma \in Z^2(G, N) \subseteq Z^2(G, C(\hat{N}, \mathbb{T}))$ *be given by*

$$\sigma(s, t)(\chi) = \chi(c(s)\, c(t)\, c(st)^{-1})$$

for some Borel section $c: G \to L$ *satisfying* $c(eN) = e$. *Let* $\phi: C_0(\hat{N}) \to \mathscr{Z}M(C^*(L))$ *denote the canonical map described above and let* $v: G \to \mathscr{U}M(C^*(L))$ *be given by* $v(s) = i_L(c(s))$. *Then the following assertions are true*:

(a) (ϕ, v) *is a covariant homomorphism of the twisted system* $(C_0(\hat{N}), G, \mathrm{id}, \sigma)$ *whose integrated form* $\phi \rtimes v$ *is a* $C_0(\hat{N})$-*isomorphism from* $C^*(G, \hat{N}, \sigma)$ *onto* $C^*(L)$.

(b) *For each* $\chi \in \hat{N}$, *the fibre* $C^*(L)_\chi$ *is isomorphic to the twisted group algebra* $C^*(G, \mathrm{tg}(\chi))$.

(c) *If* $\mu: G_{\mathrm{ab}} \to \mathrm{Aut}(C^*(L))$ *is given via restriction of the dual action of* $\hat{L}_{\mathrm{ab}}$ *to the closed subgroup* $\hat{G}_{\mathrm{ab}}$ *of* $\hat{L}_{\mathrm{ab}}$, *then* $\phi \rtimes v$ *intertwines* $\hat{\sigma}$ *and* μ.

(d) *If* G *is amenable, then* $C^*(L)$ *is a continuous* $C_0(\hat{N})$-*bundle*.

Proof. Part (a) of is a very special case of [17, Theorem 4.1], and can also be deduced from [17, Proposition 5.1] using the decomposition of group algebras by Green's twisted crossed products. Parts (b) and (d) follow from [19, Theorem 1.2], and (c) follows from the definitions of v and the dual actions.

Remark 6.4 (cf., [19, Section 1]). Let G, H and σ be as in our standing assumptions. Since the representation group is a central extension of G by C and since $\hat{C}$ has been identified with $H^2(G, \mathbb{T})$, it follows immediately from the lemma that $C^*(H)$ is a $C_0(H^2(G, \mathbb{T}))$-algebra which is $C_0(H^2(G, \mathbb{T}))$-linearly and $\hat{G}_{\mathrm{ab}}$-equivariantly isomorphic to $C^*(G, H^2(G, \mathbb{T}), \sigma)$. The fibres $C^*(H)_{[\omega]}$ are isomorphic to $C^*(G, \omega)$ for each $[\omega] \in H^2(G, \mathbb{T})$, and, if G is amenable, then $C^*(H)$ is a continuous $C_0(H^2(G, \mathbb{T}))$-bundle.

Recall that if A is a $C_0(X)$-algebra, B is a $C_0(Y)$-algebra and $f: X \to Y$ is a continuous map, then A becomes a $C_0(Y)$-algebra via composition

with f. Thus we can form the balanced tensor product $A \otimes_f B := A \otimes_Y B$ which becomes a $C_0(X)$-algebra via composition with the natural map $i_A: A \to M(A \otimes_f B)$. Therefore, for any $C_0(X)$-algebra A we have

$$A \otimes_X f^*B = A \otimes_X (C_0(X) \otimes_f B) \cong (A \otimes_X C_0(X)) \otimes_f B \cong A \otimes_f B, \quad (6.1)$$

where f^*B is the usual pull-back $C_0(X) \otimes_f B$ (Remark 2.5). Moreover, if $\beta: G \to \operatorname{Aut}(B)$ is a $C_0(Y)$-linear action of a group G on B, then we obtain a $C_0(X)$-linear action $\operatorname{id} \otimes_f \beta$ (resp., $f^*\beta$) of G on $A \otimes_f B$ (resp., f^*B), and the isomorphisms in Eq. (6.1) are $C_0(X)$-linear and G-equivariant.

LEMMA 6.5. *Let $f: X \to Y$ be a continuous map between the second countable locally compact spaces X and Y, and let $u \in Z^2(G, C(Y, \mathbb{T}))$. Let $f^*(u) \in Z^2(G, C(X, \mathbb{T}))$ be defined by $f^*(u)(s, t)(x) = u(s, t)(f(x))$. Then there exists a $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant isomorphism between $C^*(G, X, f^*(u))$ and $f^*(C^*(G, Y, u))$.*

Proof. Let $\Phi: C_0(X) \otimes_Y C_0(Y) \to C_0(X)$ denote the isomorphism given on elementary tensors by $\Phi(h \otimes_Y g)(x) = h(x) g(f(x))$. Then $\Phi(1 \otimes_Y u(s, t))(x) = u(s, t)(f(x)) = f^*(u)(s, t)$, which implies that Φ transforms the twisted action $(\operatorname{id} \otimes_Y \operatorname{id}, 1 \otimes u)$ to $(\operatorname{id}, f^*(u))$. Thus Proposition 4.3 implies that

$$C^*(G, X, f^*(u)) = C_0(X) \rtimes_{\operatorname{id}, f^*(u)} G \cong (C_0(X) \otimes_Y C_0(Y)) \rtimes_{\operatorname{id} \otimes_Y \operatorname{id}, 1 \otimes_Y u} G$$
$$\cong C_0(X) \otimes_Y (C_0(Y) \rtimes_{\operatorname{id}, u} G) = f^*(C^*(G, Y, u)).$$

Each of the above isomorphisms is $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant; the first because Φ is clearly $C_0(X)$-linear and the second due to Proposition 4.3. ∎

We are now prepared to state our main result.

THEOREM 6.6. *Let A, G and H be as in our standing assumptions and let $\alpha: G \to \operatorname{Aut}(A)$ be a locally inner action. Let $\varphi_\alpha \in C(X, H^2(G, \mathbb{T}))$ be the Mackey obstruction map for α and let $q: Z \to X$ be a principal $\hat{G}_{ab}$-bundle corresponding to the Phillips–Raeburn obstruction $\zeta_H(\alpha) \in H^1(X, \hat{\mathcal{G}}_{ab})$. Then there exists a $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant isomorphism between $A \rtimes_\alpha G$ and $Z * (A \otimes_f C^*(H))$, where $f: X \to H^2(G, \mathbb{T})$ is defined by $f(x) = \varphi_\alpha(x)^{-1}$ for all $x \in X$.*

Proof. Let $\sigma \in Z^2(G, C(H^2(G, \mathbb{T}), \mathbb{T}))$ be as in our standing assumptions, and let $\varphi_\alpha^*(\sigma) \in Z^2(G, C(X, \mathbb{T}))$ denote the pull-back of σ via φ_α. Choose a $\varphi_\alpha^*(\sigma)$-homomorphism $v: G \to \mathscr{U}M(C_0(X, \mathscr{K}))$ and let $\gamma: G \to \operatorname{Aut}(C_0(X, \mathscr{K}))$ denote the inner action implemented by v. Further let $\delta: G \to \operatorname{Aut}(C_0(X, \mathscr{K}))$ be a locally unitary action such that $\zeta(\delta) = \zeta(\alpha)$. Then it follows from Remark 6.2 that $\alpha \otimes \operatorname{id}_{\mathscr{K}}$ and $\operatorname{id}_A \otimes_X \gamma \otimes_X \delta$ are

exterior equivalent as actions on $A \otimes \mathcal{K}$. The discussion following Definition 4.5 implies that v implements an exterior equivalence between $\gamma = (\gamma, 1)$ and the twisted action $(\mathrm{id}_{C_0(X, \mathcal{K})}, \overline{\varphi_\alpha^*(\sigma)})$. If we identify $C_0(X, \mathcal{K})$ and $C_0(X) \otimes_X C_0(X, \mathcal{K})$ and view $f^*(\sigma)$ as taking values in $\mathcal{U}M(C_0(X))$, then γ is exterior equivalent to $(\mathrm{id}_{C_0(X)} \otimes_X \mathrm{id}_{C_0(X, \mathcal{K})}, f^*(\sigma) \otimes_X 1)$. If we let $\widetilde{f}^*(\sigma)$ be the cocyle taking values in $\mathcal{U}M(A)$ corresponding to $1 \otimes_X f^*(\sigma)$ via the identification of A with $A \otimes_X C_0(X)$, then we conclude that the action $\alpha \otimes \mathrm{id}_{\mathcal{K}}$ is exterior equivalent to the twisted action $(\mathrm{id}_A \otimes_X \delta, \widetilde{f}^*(\sigma))$

We claim that α is locally exterior equivalent to the twisted action $(\mathrm{id}_A, f^*(\sigma))$. To see this let $x \in X$ and choose an open neighborhood W of x such that $\alpha^W : G \to \mathrm{Aut}(A_W)$ is inner. Then there exists a cocycle $u \in Z^2(G, C(W, \mathbb{T}))$ and a u-homomorphism $w: G \to \mathcal{U}M(A_W)$ such that $\alpha^W = \mathrm{Ad}\, w$, and w implements an exterior equivalence between α^W and $(\mathrm{id}_A, \bar{u})$. We want to show that there exists a possibly smaller neighborhood W_1 of x such that the restrictions of $\bar{u}$ and $f^*(\sigma)$ to W_1 are cohomologous, or, equivalently, such that the product $u \cdot f^*(\sigma)$ restricted to W_1 is cohomologous to the trivial cocycle. This would imply the claim since the twisted actions $(\mathrm{id}_A, \bar{u})$ and $(\mathrm{id}_A, f^*(\sigma))$ would then be exterior equivalent when restricted to A_{W_1}. Since evaluation of $[u(y)] \in H^2(G, \mathbb{T})$ of u at a given point $y \in W$ must coincide with $\varphi_\alpha(y)$ (since both correspond to the same inner action on the fibre A_y), and since $[f^*(\sigma)(y)] = \varphi_\alpha(y)^{-1}$, it follows that $(u \cdot f^*(\sigma))(y)$ is cohomologous to the trivial cocycle for every $y \in W$, hence $u \cdot f^*(\sigma)$ is pointwise trivial. Thus, by Rosenberg's theorem [28, Theorem 2.1], it follows that $u \cdot f^*(\sigma)$ is locally trivial, which is precisely what we want.

Since $\alpha \otimes \mathrm{id}$ is exterior equivalent to $(\mathrm{id}_A \otimes_X \delta, \widetilde{f}^*(\sigma) \otimes 1)$ we can apply Theorem 5.3 to obtain a $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant isomorphism between $A \rtimes_\alpha G$ and $Z * (A \rtimes_{\mathrm{id}, \widetilde{f}^*(\sigma)} G)$. But $A \rtimes_{\mathrm{id}, \widetilde{f}^*(\sigma)} G$ is $C_0(X)$-linearly and $\hat{G}_{ab}$-equivariantly isomorphic to $A \otimes_X C^*(G, X, f^*(\sigma))$ by Proposition 4.3 (since $A \rtimes_{\mathrm{id}, \widetilde{f}^*(\sigma)} G \cong (C_0(X) \otimes_X A) \rtimes_{\mathrm{id} \otimes_X \mathrm{id}, f^*(\sigma) \otimes_X 1} G$). Finally, it follows from Remark 6.4 and Lemma 6.5 that $C^*(G, X, f^*(\sigma))$ is $C_0(X)$-linearly and $\hat{G}_{ab}$-equivariantly isomorphic to $f^*(C^*(H))$. Since $A \otimes_f C^*(H) = A \otimes_X f^*(C^*(H))$, this completes the proof. ∎

Remark 6.7. (a) Note that our theorem applies to all $C_0(X)$-linear actions of a smooth group G (with G_{ab} compactly generated) on a separable continuous trace algebra A with spectrum X.

(b) Since $A \otimes_f C^*(H)$ is isomorphic to $A \otimes_X f^*(C^*(H))$, since $Z * (A \otimes_X f^*(C^*(H)))$ is isomorphic to $A \otimes_X (Z * f^*(C^*(H)))$ by Proposition 3.12, and since all the isomorphisms are $C_0(X)$-linear and $\hat{G}_{ab}$-equivariant, we see that the crossed product $A \rtimes_\alpha G$ is obtained via the iteration of the following basic bundle operations: First take the pull-back $f^*(C^*(H))$ of the universal bundle $C^*(H)$ via the continuous map $f = \overline{\varphi_\alpha}$:

$G \to H^2(G, \mathbb{T})$. Then construct the G-fibre product $Z * f^*(C^*(H))$. Finally take the fibre product (i.e., the balanced tensor product) of $Z * f^*(C^*(H))$ with A.

(c)   If A is type I, then this bundle-theoretic description of $A \rtimes_\alpha G$ gives a bundle theoretic description of the spectrum $(A \rtimes_\alpha G)^\wedge$: it is isomorphic (as a topological bundle with group $\hat{G}_{ab}$ to $\hat{A} \times_X (Z * (f^*\hat{H}))$, and if $\hat{A} = X$ we get $(A \rtimes_\alpha G) = Z^*(f^*\hat{H})$. If A is nuclear (non-type I), we obtain a similar description of $\mathrm{Prim}(A \rtimes_\alpha G)$.

We close this section with some corollaries of Theorem 6.6 for certain special cases. For example, combining Theorem 6.6 with Lemma 3.8 immediately yields the following corollary. Note that $H^1(X, \hat{\mathcal{G}}_{ab})$ is always trivial if G_{ab} is a vector group, or if X is contractible (cf., e.g., [10, Corollary 4.10.3]).

COROLLARY 6.8.   *Suppose that $H^1(X, \hat{\mathcal{G}}_{ab})$ is trivial, and let A, G and H be as in our general assumptions. Then $A \rtimes_\alpha G$ is isomorphic to $A \otimes_X f^*(C^*(H))$ for any locally inner action $\alpha: G \to \mathrm{Aut}(A)$, where $f(x) = \varphi_\alpha(x)^{-1}$ for $x \in X$.*

So in the case where $H^1(X, \hat{\mathcal{G}}_{ab})$ is trivial we get a description of $A \rtimes_\alpha G$ by pulling back the universal bundle $C^*(H)$ via f and then taking the fibre product with A.

Another interesting situation occurs when φ_α is constant. If A has continuous trace, such systems were called *pointwise projective unitary* in [6]. Here the class $\zeta(\alpha)$ does not depend on the choice of the representation group H; if $[\omega] \in H^2(G, \mathbb{T})$ is the constant value of φ_α, then the pull-back of σ via φ_α would always give a cocycle cohomologous to the cocycle $\varphi_\alpha^*(\omega)$ defined by $\varphi_\alpha^*(\omega)(s, t)(x) = \omega(s, t)$.

COROLLARY 6.9.   *Let G be smooth, $A \in \mathscr{CR}(X)$, and let $\alpha: G \to \mathrm{Aut}(A)$ be a locally inner action of G on A so that there is a class $[\omega] \in H^2(G, \mathbb{T})$ with $\varphi_\alpha(x) = [\omega]$ for all $x \in X$. Let $q: Z \to X$ be a principal bundle corresponding to $\zeta(\alpha)$. Then $A \rtimes_\alpha G$ is $C_0(X)$-linearly and $\hat{G}_{ab}$-equivariantly isomorphic to $A \otimes_X (Z \times_{\hat{G}_{ab}} C^*(G, \bar{\omega}))$. Moreover, if A is type I, then $(A \rtimes_\alpha G)$ is isomorphic to $\hat{A} \times_X (Z \times_{\hat{G}_{ab}} \hat{G}_{\bar{\omega}})$, where $\hat{G}_{\bar{\omega}}$ denotes the set of equivalence classes of irreducible $\bar{\omega}$-representations of G.*

Proof.   If H is any representation group of G as in our standing assumptions, then since $f(x) = \varphi_\alpha(x)^{-1}$, it is easily seen that $f^*(C^*(H)) = C_0(X, C^*(G, \bar{\omega}))$. Thus the result follows from Theorem 6.6 and Proposition 3.12.   ∎

Note that the above result also holds true under the weaker assumption that α is *locally projective unitary* in the sense that there exists an action $\beta: G \to \mathrm{Aut}(\mathscr{K})$ with Mackey obstruction $[\bar{\omega}] \in H^2(G, \mathbb{T})$ such that $\alpha \otimes \beta$:

$G \to \operatorname{Aut}(A \otimes \mathscr{K})$ is locally unitary (compare with [6, p. 38]). The proof is similar to the proof of Theorem 5.5 and we omit the details.

7. SOME APPLICATIONS TO ACTIONS OF $\mathbb{R}^n$

As an application of our results, we want to consider locally inner actions of $\mathbb{R}^n$ on C^*-algebras A with Hausdorff spectrum X. (Since A is assumed to be separable, it is necessarily type I if its spectrum is Hausdorff.) When A has continuous-trace, this is equivalent to requiring that $(A, \mathbb{R}^n, \alpha)$ is a $C_0(X)$-system. A representation group H for $\mathbb{R}^n$ was explicitly constructed in [7, Example 4.7]. H is a simply connected and connected two-step nilpotent Lie group (in fact it is the universal two-step nilpotent group with n generators), and by [7, Proposition 4.8] it is unique up to isomorphism. Notice that in case $n = 2$, H is just the real Heisenberg group of dimension three. Since $H^1(X, \hat{\mathbb{R}}^n)$ is trivial for all second countable locally compact spaces X, the following is a consequence of Corollary 6.8.

THEOREM 7.1. *Let A be a separable type* I *C^*-algebra with Hausdorff spectrum X and let $\alpha: \mathbb{R}^n \to \operatorname{Aut}(A)$ be a locally inner action of $\mathbb{R}^n$ on A. Let $\varphi_\alpha: X \to H^2(\mathbb{R}^n, \mathbb{T})$ denote the Mackey obstruction map and let $f(x) = \varphi_\alpha(x)^{-1}$ for $x \in X$. Then $A \rtimes_\alpha \mathbb{R}^n$ is $C_0(X)$-linearly and $\hat{\mathbb{R}}^n$-equivariantly isomorphic to $A \otimes_X f^*(C^*(H))$ and $(A \rtimes_\alpha \mathbb{R}^n)\hat{\,}$ is isomorphic to $f^*(\hat{H})$ as a topological bundle over X with group $\hat{\mathbb{R}}^n$.*

We want to use our result to obtain a more detailed description of $A \rtimes_\alpha \mathbb{R}^n$ and its spectrum. Recall that if G is an abelian group and $\omega \in Z^2(G, \mathbb{T})$, then the *symmetry group* Σ_ω of ω is defined as $\Sigma_\omega := \{s \in G : \omega(s, t) = \omega(t, s) \text{ for all } t \in G\}$, and Σ_ω only depends on the cohomology class $[\omega] \in H^2(G, \mathbb{T})$. If $G = \mathbb{R}^n$ then any cocycle of G is cohomologous to a cocycle of the form $\omega(s, t) = e^{iJ(s, t)}$, where J is a skew symmetric form on $\mathbb{R}^n$. It follows that the symmetry group equals the radical of J, so that Σ_ω is actually a vector subgroup of $\mathbb{R}^n$. The symmetry groups play an important rôle in the representation theory of two-step nilpotent groups and crossed products by abelian groups (see for instance [2, 6, 5]). For example, if A is a separable (type I) C^*-algebra with Hausdorff spectrum $\hat{A} = X$ and if $\alpha: G \to \operatorname{Aut}(A)$ is a $C_0(X)$-linear action of the second countable abelian group G on A, then each fibre $\operatorname{Prim}(A_x \rtimes_{\alpha^x} G)$ of $\operatorname{Prim}(A \rtimes_\alpha G)$ over $x \in X$ is $\hat{G}$-equivariantly homeomorphic to $\hat{\Sigma}_x$, where Σ_x denotes the symmetry group of the Mackey-obstruction $\varphi_\alpha(x)$ (see [9, Theorem 1.1]).

Thus, if we view $\operatorname{Prim}(A \rtimes_\alpha G)$ as a topological bundle over X with group $\hat{G}$, then there is a nice description of the fibres. The problem is to get the global picture of the bundle. When $G = \mathbb{R}^n$, we will deduce a description from the

following fact, which was observed by Baggett and Packer in [2, Remark 2.5], and which can be deduced from Kirillov theory for nilpotent Lie groups.

PROPOSITION 7.2 (cf, [2]). *Let H be a connected and simply connected two-step nilpotent Lie group with center Z and quotient $G = H/Z$. Let* tg: $\hat{Z}$ $\to H^2(G, \mathbb{T})$ *denote the transgression map and let Σ_χ denote the symmetry group of* tg(χ) *for all $\chi \in \hat{Z}$. Then the topological bundle $\hat{H}$ over $\hat{Z}$ with group $\hat{G}$ is isomorphic to the quotient space $(\hat{G} \times \hat{Z})/\sim$, where $\sim$ denotes the equivalence relation*

$$(\mu, \chi) \sim (\mu', \chi') \Leftrightarrow \chi = \chi' \qquad and \qquad \bar{\mu}\mu' \in \Sigma_\chi^\perp,$$

and where $(\hat{G} \times \hat{Z})/\sim$ is equipped with the canonical structure as a topological bundle over X with group $\hat{G}$.

Combining this and Theorem 7.1 we get:

COROLLARY 7.3. *Let $\alpha: \mathbb{R}^n \to \operatorname{Aut}(A)$ be a locally inner action of $\mathbb{R}^n$ on a separable (type I) C^*-algebra with Hausdorff spectrum $\hat{A} = X$. For each $x \in X$ let Σ_x denote the symmetry group of $\varphi_\alpha(x) \in H^2(G, \mathbb{T})$. Then $(A \rtimes_\alpha \mathbb{R}^n)^\wedge$ is isomorphic to $(\hat{\mathbb{R}}^n \times X)/\sim$ as a topological bundle over X with group $\hat{\mathbb{R}}^n$, where $\sim$ is the equivalence relation*

$$(\mu, x) \sim (\mu', x') \Leftrightarrow x = x' \qquad and \qquad \bar{\mu}\mu' \in \Sigma_x^\perp.$$

Proof. Let H be the representation group of $\mathbb{R}^n$. Since H is a connected and simply connected two-step nilpotent Lie group with center $H^2(\mathbb{R}^n, \mathbb{T})$, it follows from Proposition 7.2 that $\hat{H}$ is isomorphic (as a bundle) to $(\hat{\mathbb{R}}^n \times H^2(\mathbb{R}^n, \mathbb{T}))/\sim$. Moreover, by Theorem 7.1 we know that $(A \rtimes_\alpha \mathbb{R}^n)$ is isomorphic to $f^*\hat{H} \cong f^*((\hat{\mathbb{R}}^n \times H^2(\mathbb{R}^n, \mathbb{T}))/\sim)$. Since $f(x) = \varphi_\alpha(x)^{-1}$ and the symmetry groups of $[\omega]$ and $[\omega]^{-1}$ coincide for all $[\omega] \in H^2(G, \mathbb{T})$, it is follows that $[\mu, x] \mapsto (x, [\mu, f(x)])$ is an isomorphism between $(\hat{\mathbb{R}}^n \times X)/\sim$ and $f^*((\hat{\mathbb{R}}^n \times H^2(\mathbb{R}^n, \mathbb{T}))/\sim)$. ∎

The previous result can fail for an arbitrary second countable compactly generated abelian group G; we know from the work of Phillips–Raeburn and Rosenberg that if $\alpha: G \to \operatorname{Aut}(A)$ is any $C_0(X)$-linear action of G on a continuous trace algebra A with spectrum X such that the Mackey obstruction map vanishes (i.e., α is pointwise unitary) that $(A \rtimes_\alpha G)$ can be any principal $\hat{G}$-bundle, while $(\hat{G} \times X)/\sim$ is just the trivial bundle $\hat{G} \times X$ in this case. On the other hand, it would be interesting to see whether the result remains to be true if we replace $\hat{G} \times X$ with an appropriate principal $\hat{G}$-bundle $q: Z \to X$. That is, it would be interesting to know under what circumstances the following question has a positive answer (see also [2, Remark 2.5]).

OPEN QUESTION.   Let $\alpha\colon G \to \operatorname{Aut}(A)$ be a $C_0(X)$-linear action of the second countable compactly generated abelian group G on the continuous trace algebra A with spectrum X. Does there always exist a principal $\hat{G}$-bundle $q\colon Z \to X$ such that $\operatorname{Prim}(A \rtimes_\alpha G)$ is isomorphic to $Z/\sim$ as a bundle over X? Here $\sim$ denotes the equivalence relation

$$z \sim z' \Leftrightarrow q(z) = q(z') \qquad \text{and} \qquad \bar{z}z' \in \Sigma_x^\perp,$$

where $\bar{z}z'$ denotes the unique element χ of $\hat{G}$ which satisfies $\chi \cdot z = z'$.

It is straightforward to check that $Z/\sim$ is just the twisted bundle $Z*(\hat{G} \times X)/\sim$. Clearly, the above problem is strongly related to the problem of describing the primitive ideal space of the group C^*-algebra of a two-step nilpotent group H with center Z and quotient $H/Z = G$ as a quotient space of a principal $\hat{G}$-bundle over $\hat{Z}$, as considered by Baggett and Packer in [2]. On the one hand, the problem for two-step nilpotent groups is a special case of the above, since by the Packer–Raeburn stabilization trick [17, Corollary 3.7], we can write $C^*(H) \otimes \mathscr{K}$ as a crossed product $C_0(\hat{Z}, \mathscr{K}) \rtimes_\beta G$, for some $C_0(\hat{Z})$-linear action β. On the other hand, if the result is true for the representation group H of G, which is always two-step nilpotent, then as in the proof of Corollary 7.3, one could get the same result for all $C_0(X)$-linear actions of G on separable continuous trace algebras with spectrum X (or, more generally, for locally inner actions on type I algebras with Hausdorff spectrum X).

We want to illustrate this for the special case $G = \mathbb{Z}^2$, where the (unique) representation group is the discrete Heisenberg group of rank three.

THEOREM 7.4.   *Let $\alpha\colon \mathbb{Z}^2 \to \operatorname{Aut}(A)$ be a $C_0(X)$-linear action of $\mathbb{Z}^2$ on the separable continuous trace algebra A with spectrum X. Let $\zeta(\alpha)$ be the Phillips–Raeburn obstruction of α as defined in Definition* 6.1, *and let $q\colon Z \to X$ denote the corresponding principal $\mathbb{T}^2 = \hat{\mathbb{Z}}^2$-bundle. Then* $\operatorname{Prim}(A \rtimes_\alpha \mathbb{Z}^2)$ *is isomorphic to $Z/\sim$ as a topological bundle over X with group* $\mathbb{T}^2$.

Proof.   Recall that the discrete Heisenberg group H is the set $\mathbb{Z}^3$ with multiplication given by $(n_1, m_1, l_1)(n_2, m_2, l_2) = (n_1 + n_2, m_1 + m_2, l_1 + l_2 + n_1 m_2)$. The center C of H is given by $\{(0, 0, l)\colon l \in \mathbb{Z}\}$. For each $t \in [0, 1)$ let $\chi_t(l) = e^{i2\pi tl}$ denote the character corresponding to t under the identification of $\hat{C}$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Using the section $c\colon \mathbb{Z}^2 \to H$; $c(n, m) = (n, m, 0)$, we easily compute that the cocycle $\omega_t \in Z^2(\mathbb{Z}^2, \mathbb{T})$ corresponding to $\chi_t \in \hat{C}$ is given by

$$\omega_t((n_1, m_1), (n_2, m_2)) = e^{i2\pi tn_1 m_2}.$$

If t is irrational, then the symmetry group $\Sigma_t = \Sigma_{\omega_t}$ is trivial, and if $t = p/q$, where p and q have no common factors, then it is not hard to show that

$\Sigma_t = q\mathbb{Z} \times q\mathbb{Z} \subseteq \mathbb{Z}^2$. Thus it follows that ω_t is identically 1 when restricted to the symmetry groups, and we can use [2, Theorem 2.3] to deduce that $\mathrm{Prim}(C^*(H))$ is isomorphic to $(\hat{\mathbb{Z}}^2 \times \hat{C})/\sim$ as a bundle over $\hat{C}$, where $\sim$ is the usual equivalence relation.

By Theorem 6.6 we have $A \rtimes_\alpha \mathbb{Z}^2 = Z * (f^* C^*(H))$, where $f(x)$ is the inverse of the Mackey obstruction $[\omega_x]$ for all $x \in X$. Hence

$$\mathrm{Prim}(A \rtimes_\alpha \mathbb{Z}^2) = Z * (f^* \mathrm{Prim}(C^*(H))) = Z * (f^*((\hat{\mathbb{Z}}^2 \times \hat{C})/\sim)))$$

$$= Z * ((\hat{\mathbb{Z}}^2 \times X)/\sim) = Z/\sim. \quad \blacksquare$$

Finally, we point out that our results are also helpful to the investigation of the structure of continuous trace subquotients of the crossed products $A \rtimes_\alpha \mathbb{R}^n$, where α is a $C_0(X)$-linear action on the continuous trace algebra A with spectrum X. For this we first recall the following result due to the first author:

PROPOSITION 7.5 ([5, Theorem 6.3.3]). *Let A be a separable continuous-trace algebra with spectrum $\hat{A} = X$ and let $\alpha: \mathbb{R}^n \to \mathrm{Aut}(A)$ be a $C_0(X)$-linear action. Further, let* $\dim : X \to \mathbb{Z}^+$ *be defined by letting* $\dim(x)$ *be the vector space dimension of Σ_x. Then $A \rtimes_\alpha \mathbb{R}^n$ has continuous trace if and only if* $\dim: X \to \mathbb{Z}^+$ *is continuous.*

More generally, if $\alpha: \mathbb{R}^n \to \mathrm{Aut}(A)$ is any $C_0(X)$-linear action of $\mathbb{R}^n$ on a continuous trace algebra A with spectrum X, then there exists a finite decomposition series of ideals

$$\{0\} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_l = A \rtimes_\alpha \mathbb{R}^n,$$

with $l \leqslant (n/2) + 1$ and all subquotients I_k/I_{k-1}, $1 \leqslant k \leqslant l$, given by crossed products with continuous trace as in the proposition above ([5, Theorem 6.3.3]). We are now going to use our results together with a recent result of Lipsman and Rosenberg to compute explicitly the Dixmier–Douady invariant of these subquotients.

THEOREM 7.6. *Let $\alpha: \mathbb{R}^n \to \mathrm{Aut}(A)$ be a $C_0(X)$-linear action of $\mathbb{R}^n$ on the separable continuous-trace C^*-algebra A with spectrum X, such that* $\dim: X \to \mathbb{Z}^+$ *is continuous. Let $Y := (A \rtimes_\alpha \mathbb{R}^n)^\wedge$ and let $p: Y \to X$ denote the canonical projection. Then $\delta(A \rtimes_\alpha \mathbb{R}^n) = p^* \delta(A)$, where $\delta(A \rtimes_\alpha \mathbb{R}^n) \in H^3(Y, \mathbb{Z})$ and $\delta(A) \in H^3(X, \mathbb{Z})$ denote the Dixmier–Douady invariants of A and $A \rtimes_\alpha \mathbb{R}^n$, respectively.*

Proof. First, since $\dim(x) \leqslant n$ for all x, X decomposes into a finite disjoint union of open subsets such that dim is constant on each subset. Therefore, we may assume that dim is constant with value k, say, on all of X. Let H be the representation group of $\mathbb{R}^n$ and let

$$D_k := \{[\omega] \in H^2(\mathbb{R}^n, \mathbb{T}) : \dim(\Sigma_{[\omega]}) = k\}.$$

Then it follows from [5, Theorem 6.3.3] that D_k is a locally closed (and hence locally compact) subset of $H^2(\mathbb{R}^n, \mathbb{T})$, and that the restriction $C^*(H)_k := C^*(H)_{D_k}$ of $C^*(H)$ to D_k is a continuous trace subquotient of $C^*(H)$. Thus, by Lipsman's and Rosenberg's result [13, Theorem 3.4], the Dixmier–Douady invariant of $C^*(H)_k$ is trivial and $C^*(H)_k$ is stably isomorphic to $C_0(\hat{H}_k, \mathcal{K})$, where $\hat{H}_k$ denotes the restriction of the topological bundle $\hat{H}$ over $H^2(\mathbb{R}^n, \mathbb{T})$ to D_k.

Since we already assumed that dim has constant value k on all of X, it follows that φ_α, and hence also the map $f : X \to H^2(\mathbb{R}^n, \mathbb{T})$ given by $f(x) = \varphi_\alpha(x)^{-1}$ takes values in D_k. Thus by Theorem 7.1 we get

$$(A \rtimes_\alpha \mathbb{R}^n) \otimes \mathcal{K} \cong (A \otimes_X f^*(C^*(H))) \otimes \mathcal{K} \cong A \otimes_X C_0(f^*\hat{H}_k, \mathcal{K}).$$

Now if $p : f^*\hat{H}_k \to X$ denotes the projection, we have

$$A \otimes_X C_0(f^*\hat{H}_k, \mathcal{K}) \cong p^*A \otimes_{f^*\hat{H}_k} C_0(f^*\hat{H}_k, \mathcal{K}) \cong p^*A \otimes \mathcal{K}.$$

Thus, [24, Proposition 1.4] implies that $\delta(A \rtimes_\alpha \mathbb{R}^n) = \delta(p^*A) = p^*(\delta(A))$.

Remark 7.7. Similar to the proof of [13, Lemma 3.3] one can show that if $\alpha : \mathbb{R}^n \to \operatorname{Aut}(A)$ is a $C_0(X)$-linear action of $\mathbb{R}^n$ on the separable continuous trace algebra A with spectrum X, then any continuous trace subquotient B of $A \rtimes_\alpha \mathbb{R}^n$ decomposes into a finite direct sum of ideals such that all these ideals are subquotients of some $A_{D_k} \rtimes_\alpha \mathbb{R}^n$, where D_k is a the locally closed subset of X such that the dimension function is constantly equal to k on D_k. From this and the above result it follows that if $\delta(A)$ is trivial, then any continuous trace subquotient of $A \rtimes_\alpha \mathbb{R}^n$ has also trivial Dixmier–Douady invariant. We omit the details.

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