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LOCALLY INNER ACTIONS ON $C_0(X)$-ALGEBRAS

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Abstract. We make a detailed study of locally inner actions on C^*-algebras whose primitive ideal spaces have locally compact Hausdorff complete regularizations. We suppose that G has a representation group and compactly generated abelianization G_{ab}. Then, if A is stable and if the complete regularization of $\text{Prim}(A)$ is X, we show that the collection of exterior equivalence classes of locally inner actions of G on A is parametrized by the group $\mathcal{E}_G(X)$ of exterior equivalence classes of $C_0(X)$-actions of G on $C_0(X,\mathcal{K})$. Furthermore, we exhibit a group isomorphism of $\mathcal{E}_G(X)$ with the direct sum $H^1(X,\widehat{\mathcal{G}}_{\text{ab}})\oplus C\big(X,H^2(G,\mathbb{T})\big)$. As a consequence, we can compute the equivariant Brauer group $\text{Br}_G(X)$ for G acting trivially on X.

Keywords: *Crossed products, locally inner action, $C_0(X)$-algebras, exterior equivalence, equivariant Brauer group.*

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1. INTRODUCTION

One of the original motivations for the study of C^*-algebras arose from the desire to understand the representation theory of locally compact groups. As is eloquently described in Rosenberg's survey article ([46], Section 3), the modern Mackey-Green machine shows that to make further progress in this direction, it will be necessary to have detailed knowledge of crossed product C^*-algebras which arise from actions with a "single orbit type" acting on a continuous-trace C^*-algebra. There is a considerable volume of work in this direction — for example, [41], [43], [42], [33], [14], [12] and other references cited in [46], Section 3. Notice that the case of nonvanishing Mackey obstructions is treated only in [14], [12], and that the majority of results assume that the group acting is abelian.

In this article, we consider a general family of dynamical systems which include all spectrum fixing actions of a wide class of locally compact groups on continuous-trace C^*-algebras. Since the crossed product by any action on a stable continuous-trace C^*-algebra with single orbit type and constant stabilizer N can be decomposed (via the Packer-Raeburn Stabilization Trick, [35], Theorem 3.4) into an action of N (which is trivial on $\widehat{A}$) and an action of G/N on $A \rtimes_\alpha N$ (with G/N acting freely on $(A \rtimes_\alpha N)^\wedge$), a detailed description of such actions and their crossed products provides a major step towards a general solution to Rosenberg's "Research Problem 1" of [46]. Following ideas developed in [41], [44], [8], [34], [37] (and others) we are going to describe our actions in terms of topological invariants living in Moore's group cohomology and in certain sheaf cohomology groups. In [15], we will use these topological invariants for a precise bundle-theoretic description of the corresponding crossed products.

Let us explain our results in more detail. If G is a second countable locally compact group and X is a second countable locally compact Hausdorff space, then we denote by $\mathcal{E}_G(X)$ the set of exterior equivalence classes of $C_0(X)$-linear (i.e., spectrum fixing) actions of G on $C_0(X,\mathcal{K})$. By taking diagonal actions on the balanced tensor product $C_0(X,\mathcal{K}) \otimes_{C_0(X)} C_0(X,\mathcal{K}) \cong C_0(X,\mathcal{K})$, we obtain an abelian group operation on $\mathcal{E}_G(X)$, and $\mathcal{E}_G(X)$ can be identified with a subgroup of the equivariant Brauer group $\mathrm{Br}_G(X)$ of all Morita equivalence classes of spectrum fixing G-actions on continuous trace algebras with spectrum $\widehat{A} = X$ ([8]). In fact, $\mathcal{E}_G(X)$ is the kernel of the forgetful homomorphism $F : \mathrm{Br}_G(X) \to \mathrm{Br}(X) \cong H^3(X,\mathbb{Z})$, sending a system (A, G, α) to the Dixmier-Douady invariant $\delta(A) \in H^3(X,\mathbb{Z})$. Since F has a natural splitting, $\mathrm{Br}_G(X) \cong \mathcal{E}_G(X) \oplus H^3(X,\mathbb{Z})$. Thus, the Dixmier-Douady invariant and the class in $\mathcal{E}_G(X)$ completely determine the class of a spectrum fixing automorphism group on a continuous-trace C^*-algebra with spectrum X.

If $[\beta] \in \mathcal{E}_G(X)$, then each $x \in X$ determines an element $[\omega_x] \in H^2(G,\mathbb{T})$ (Moore's group cohomology) called the *Mackey obstruction* for extending the evaluation map at x to a covariant representation of $(C_0(X,\mathcal{K}), G, \beta)$. The map $\varphi^\beta : X \to H^2(G,\mathbb{T})$ given by $\varphi^\beta(x) = [\omega_x]$ is continuous and we obtain a natural homomorphism

$$\Phi : \mathcal{E}_G(X) \to C(X, H^2(G,\mathbb{T})); \quad [\beta] \mapsto \varphi^\beta.$$

Notice that $\varphi^\beta = 0$ if and only if each irreducible representation of $C_0(X,\mathcal{K})$ can be extended to a covariant representation of $\big(C_0(X,\mathcal{K}), G, \beta\big)$; that is, if and only if β is *pointwise unitary*. In [34] (see also [37]), Packer observed that if G is an elementary abelian group (i.e., G is of the form $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^k \times F$ with F finite abelian), then Φ is surjective and admits a splitting map. Moreover, a theorem of Rosenberg (see Theorem 3.8) implies that under these hypotheses, $\ker \Phi$ coincides with the (equivalence classes of) locally unitary actions. Therefore the Phillips-Raeburn obstruction map (see Corollary 3.9) gives an isomorphism of $\ker \Phi$ with the isomorphism classes of principal $\widehat{G}$-bundles over X, or equivalently, with the sheaf cohomology group $H^1(X, \widehat{\mathcal{G}})$. (If G is an abelian group, we will use the corresponding calligraphic letter $\mathcal{G}$ to denote the sheaf of germs of continuous G-valued functions.) Thus as abelian groups, $\mathcal{E}_G(X) \cong H^1(X, \widehat{\mathcal{G}}) \oplus C\big(X, H^2(G,\mathbb{T})\big)$, and $\mathrm{Br}_G(X) \cong H^1(X, \widehat{\mathcal{G}}) \oplus C\big(X, H^2(G,\mathbb{T})\big) \oplus H^3(X;\mathbb{Z})$. Notice that the critical steps in Packer's argument are to find a splitting map for Φ and to identify $\ker \Phi$ using Rosenberg's theorem.

Our first main result is to produce a splitting map for Φ in the case that G has a representation group in the sense of Moore (see Definition 4.1). Such groups are called *smooth* and comprise a large class of locally compact groups including all compact groups, all discrete groups, and all compactly generated abelian groups (see Remark 4.2 and Corollary 4.6). Smooth groups G have the property that $H^2(G,\mathbb{T})$ is locally compact and Hausdorff. Thus if G_{ab} is compactly generated, then G satisfies the hypotheses of Rosenberg's theorem and allows us to identify $\ker\Phi$ with locally unitary actions of G. A suitable modification of the Phillips-Raeburn theory (see Section 3) gives an isomorphism of $\ker\Phi$ with $H^1(X,\widehat{\mathcal{G}}_{\mathrm{ab}})$ (Corollary 3.9). Thus we obtain the following result.

Theorem. (Theorem 5.4 and Corollary 5.5) *Suppose that G is smooth and that G_{ab} is compactly generated. Then for any trivial G-space X,*

$$\mathcal{E}_G(X)\cong H^1(X,\widehat{\mathcal{G}}_{\mathrm{ab}})\oplus C\big(X,H^2(G,\mathbb{T})\big),$$

and

$$\mathrm{Br}_G(X)\cong H^1(X,\widehat{\mathcal{G}}_{\mathrm{ab}})\oplus C\big(X,H^2(G,\mathbb{T})\big)\oplus H^3(X;\mathbb{Z}).$$

This gives new information even if G is abelian, since by Corollary 4.6 our result applies not only to elementary abelian groups, but to all second countable compactly generated abelian groups.

One reason that actions on continuous-trace C^*-algebras are more manageable than actions on arbitrary C^*-algebras is that, for suitable G (e.g., if the abelianization $G_{\mathrm{ab}}=G/\overline{[G,G]}$ is compactly generated), any spectrum fixing action of G on a continuous-trace C^*-algebra A is *locally inner* (see the proof of [45], Corollary 2.2). Thus it is natural to try to classify locally inner actions on arbitrary C^*-algebras rather than restricting ourselves to actions on continuous-trace algebras. We are able to provide this classification for a large class of C^*-algebras, namely those whose primitive ideal space $\mathrm{Prim}(A)$ has a (second countable) locally compact complete regularization X (this class includes all unital C^*-algebras, all C^*-algebras with Hausdorff primitive ideal spaces, and all quasi-standard C^*-algebras in the sense of Archbold and Somerset, [3]). If X is given, then we will denote by $\mathcal{CR}(X)$ the class of C^*-algebras with complete regularization X for $\mathrm{Prim}(A)$. For this class of algebras we obtain the following

Theorem. (Theorem 6.3) *Suppose that $A\in\mathcal{CR}(X)$ and $\alpha:G\to\mathrm{Aut}(A)$ is a locally inner action of a smooth group on A. If G_{ab} is compactly generated, then there is a unique $[\beta^\alpha]\in\mathcal{E}_G(X)$ such that $\alpha\otimes_X\mathrm{id}$ is exterior equivalent to $\mathrm{id}\otimes_X\beta^\alpha$ on $A\otimes_X C_0(X,\mathcal{K})$. In fact, the map $[\alpha]\mapsto[\beta^\alpha]$ is a well-defined injective map from the collection of exterior equivalence classes of locally inner actions on A to $\mathcal{E}_G(X)$. This correspondence is bijective if A is stable.*

It follows that if $A\in\mathcal{CR}(X)$ and if (A,G,α) is locally inner, then the crossed product $A\rtimes_\alpha G$ is Morita equivalent to one of the special form $\big(A\otimes_X C_0(X,\mathcal{K})\big)\rtimes_{\mathrm{id}\otimes_X\beta}G$, where β is in $\mathcal{E}_G(X)$. Having this, it is possible to describe the crossed product in terms of the invariants associated to β and a representation group for G; this we do in [15].

Our work is organized as follows. Section 2 is devoted to some preliminary results on $C_0(X)$-algebras and on the complete regularization of $\mathrm{Prim}(A)$. In Section 3 we consider locally unitary actions on $\mathcal{CR}(X)$-algebras, and extend the Phillips-Raeburn classification scheme to this setting. Since smooth groups play such an important rôle in the sequel, we devote Section 4 to developing some basic results about representation groups. The chief result connecting representation groups to the splitting of Φ is the characterization of smooth groups given in Lemma 4.3. In Section 5, we prove the first of our main results (Theorem 5.4) which describes $\mathcal{E}_G(X)$. Our description of locally inner actions is given in Section 6.

Since Rosenberg's theorem plays a key rôle, we provide a discussion of possible extensions of his theorem in Section 7. We give examples which show that the hypotheses are sharp — that is, the major assumptions that $H^2(G, \mathbb{T})$ is Hausdorff and that G_{ab} is compactly generated, are both necessary in general. On the other hand, we also show that Rosenberg's theorem holds for a strictly larger class of groups if we restrict ourselves to actions on continuous-trace C^*-algebras with locally connected spectrum (Theorem 7.4). This class of groups contains all connected nilpotent Lie groups and all $[\mathrm{FD}]^-$groups (Corollary 7.5) — a class of groups which contains all known examples of groups G for which $C^*(G)$ is a continuous-trace C^*-algebra.

2. PRELIMINARIES

If A is a C^*-algebra, then we will write $\mathrm{Prim}(A)$ for the space of primitive ideals of A with the Jacobson topology. This topology is badly behaved in general and may satisfy only the T_0-axiom for separability. On the other hand, $\mathrm{Prim}(A)$ is always locally compact (we do not require that compact or locally compact spaces be Hausdorff), and $\mathrm{Prim}(A)$ is second countable whenever A is separable ([10], Section 3.3). The Jacobson topology on $\mathrm{Prim}(A)$ not only describes the ideal structure of A, but also allows us to completely describe the center $\mathcal{Z}M(A)$ of the multiplier algebra $M(A)$ of A. If $a \in A$, then we will write $a(P)$ for the image of a in the quotient A/P, then the Dauns-Hofmann Theorem allows us to identify $C^{\mathrm{b}}\big(\mathrm{Prim}(A)\big)$ with $\mathcal{Z}M(A)$ as follows: if $f \in C^{\mathrm{b}}\big(\mathrm{Prim}(A)\big)$ and if $a \in A$, then $f \cdot a$ is the unique element of A satisfying $(f \cdot a)(P) = f(P)a(P)$ for all $P \in \mathrm{Prim}(A)$, and every element of $\mathcal{Z}M(A)$ is of this form (cf. [39], Corollary 4.4.8 or [31]). Note that A is a nondegenerate central Banach $C^{\mathrm{b}}\big(\mathrm{Prim}(A)\big)$-module.

Since the topology on $\mathrm{Prim}(A)$ can be awkward to deal with, a natural alternative is to use the following definition.

DEFINITION 2.2. Suppose that X is a locally compact Hausdorff space. A $C_0(X)$*-algebra* is a C^*-algebra A together with a $*$-homomorphism $\Phi_A : C_0(X) \to \mathcal{Z}M(A)$ which is *nondegenerate* in the sense that

$$\Phi_A\big(C_0(X)\big) \cdot A := \mathrm{span}\,\{\Phi_A(f)a : f \in C_0(X) \text{ and } a \in A\}$$

is dense in A.

$C_0(X)$-algebras have enjoyed a considerable amount of attention recently and there are a number of good treatments available [5], [4], [32]. We recall some of the basic properties here.

If (A, Φ_A) is a $C_0(X)$-algebra, then there is a continuous map $\sigma_A : \mathrm{Prim}(A) \to X$ such that $\Phi_A(f) = f \circ \sigma_A$. (Here and in the sequel, we *identify* $\mathcal{Z}M(A)$ with $C^{\mathrm{b}}\big(\mathrm{Prim}(A)\big)$ via the Dauns-Hofmann Theorem.) As the converse is clear, A is a $C_0(X)$-algebra if and only if there is a continuous map from $\mathrm{Prim}(A)$ to X. We will usually suppress Φ_A and σ_A and write $f \cdot a$ in place of $\Phi_A(f)a$ or $(f \circ \sigma_A) \cdot a$. Notice that A is a nondegenerate central Banach $C_0(X)$-module satisfying

$$(f \cdot a)^* = a^* \cdot \overline{f}. \tag{2.1}$$

Furthermore, any nondegenerate central $C_0(X)$-module satisfying (2.1) is a $C_0(X)$-algebra.

Suppose that U is open in X and that J is the ideal of functions in $C_0(X)$ vanishing off U. Then the Cohen factorization theorem ([7], [5], Proposition 1.8) implies that

$$\overline{J \cdot A} := \overline{\operatorname{span}}\,\{f \cdot a : f \in J \text{ and } a \in A\} = \{f \cdot a : f \in J \text{ and } a \in A\}.$$

(The point being that it is unnecessary to take either the closure or the span in the final set.) Anyway, $J \cdot A$ is an ideal of A which will be denoted by A_U. For each $x \in X$, we write $A(x)$ for the quotient of A by $A_{X\setminus\{x\}}$. If $a \in A$, then we write $a(x)$ for the image of a in $A(x)$. We refer to $A(x)$ as *the fibre of A over x*. Notice that it is possible that $A(x) = \{0\}$. Even so, we often view elements of A as "fields" in $\prod_{x\in X} A(x)$. This point of view is justified by the following.

LEMMA 2.1. ([5], [32]) *Suppose that A is a $C_0(X)$-algebra. For each $a \in A$, the map $x \mapsto \|a(x)\|$ is upper semicontinuous; that is, $\{x \in X : \|a(x)\| \geqslant \varepsilon\}$ is closed for all $\varepsilon \geqslant 0$. Furthermore,*

$$\|a\| = \sup_{x\in X} \|a(x)\|.$$

REMARK 2.3. The map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$ if and only if σ_A is open ([24], [32]). In this case, A is the section algebra of a C^*-bundle over the image of σ_A ([16], Section 1).

REMARK 2.4. Notice that if A is a $C_0(X)$-algebra then each $m \in M(A)$ defines a multiplier $m(x) \in M\big(A(x)\big)$. If $m \in M(A)$ and $a \in A$, then $ma(x) = m(x)a(x)$.

Given a C^*-algebra A, it is natural to look for a nice space X which makes A a $C_0(X)$-algebra. Since X will be the image of $\operatorname{Prim}(A)$ by a continuous map, it is reasonable to look for a "Hausdorffication of $\operatorname{Prim}(A)$". Regrettably, there are a horrifying number of alternatives to choose from (cf., e.g., [9], Chapter III, Section 3). For our purposes, the appropriate notion is the *complete regularization.* If P and Q belong to $\operatorname{Prim}(A)$, then we define $P \sim Q$ if $f(P) = f(Q)$ for all $f \in C^{\mathrm{b}}\big(\operatorname{Prim}(A)\big)$. Then $\sim$ is an equivalence relation and the set $\operatorname{Prim}(A)/{\sim}$ is denoted by $\operatorname{Glimm}(A)$ ([3]). If we give $\operatorname{Glimm}(A)$ the weak topology τ_{cr} induced by the functions in $C^{\mathrm{b}}\big(\operatorname{Prim}(A)\big)$, then $\big(\operatorname{Glimm}(A), \tau_{\mathrm{cr}}\big)$ is a completely regular space ([17], Theorem 3.7). The quotient map $q : \operatorname{Prim}(A) \to \operatorname{Glimm}(A)$ is called the *complete regularization map.* It is not clear that τ_{cr} coincides with the quotient topology τ_{q} on $\operatorname{Glimm}(A)$, although one certainly has $\tau_{\mathrm{cr}} \subseteq \tau_{\mathrm{q}}$. (These topologies do differ in general ([17], 3J.3); however, we know of no examples where they differ for $\operatorname{Glimm}(A)$.) In particular, q is continuous; moreover the map $f \mapsto f \circ q$ is an isomorphism of $C^{\mathrm{b}}\big(\operatorname{Glimm}(A)\big)$ and $C^{\mathrm{b}}\big(\operatorname{Prim}(A)\big)$ ([17], Theorem 3.9). Furthermore, τ_{cr} is the only completely regular topology on $\operatorname{Glimm}(A)$ such that the functions induced by $C^{\mathrm{b}}\big(\operatorname{Prim}(A)\big)$ are continuous ([17], Theorem 3.6).

Here it will be necessary to have the complete regularization $\bigl(\mathrm{Glimm}(A), \tau_{\mathrm{cr}}\bigr)$ be locally compact. Regrettably, this can fail to be the case ([9], Example 9.2). Even if the complete regularization is locally compact, we have been unable to show that it must be second countable if A is separable. Consequently, we must include both these assumptions in our applications.

DEFINITION 2.5. We will call a separable C^*-algebra a *$\mathcal{CR}$-algebra* if the complete regularization $X := \bigl(\mathrm{Glimm}(A), \tau_{\mathrm{cr}}\bigr)$ of $\mathrm{Prim}(A)$ is a second countable locally compact Hausdorff space. If X is a second countable locally compact Hausdorff space, then we will write $\mathcal{CR}(X)$ for the collection of $\mathcal{CR}$-algebras with complete regularization homeomorphic to X.

Despite the pathologies mentioned above, the class of $\mathcal{CR}$-algebras is quite large. It clearly contains all separable C^*-algebras with Hausdorff primitive ideal space $\mathrm{Prim}(A)$. If A is unital, then $\mathrm{Prim}(A)$ is compact. Since the complete regularization map is continuous, $\mathrm{Glimm}(A)$ is compact. Since $C^{\mathrm{b}}\bigl(\mathrm{Prim}(A)\bigr) = C^{\mathrm{b}}\bigl(\mathrm{Glimm}(A)\bigr)$ is actually a closed subalgebra of A in this case, $C^{\mathrm{b}}\bigl(\mathrm{Glimm}(A)\bigr)$ is separable and $\mathrm{Glimm}(A)$ is second countable. (There is an embedding of $\mathrm{Glimm}(A)$ into $C^{\mathrm{b}}\bigl(\mathrm{Glimm}(A)\bigr)^{\wedge}$ which is the Stone-Čech compactification $\beta\bigl(\mathrm{Glimm}(A)\bigr)$. Since a subset of a locally compact Hausdorff space is locally compact if and only if it is open in its closure, $\mathrm{Glimm}(A)$ is locally compact exactly when it is open in its Stone-Čech compactification.) Thus *every unital C^*-algebra is a $\mathcal{CR}$-algebra*. Another large class of $\mathcal{CR}$-algebras is provided by the *quasi-standard C^*-algebras* studied in [3]. Recall that a C^*-algebra is called quasi-standard if (1) defining $P \approx Q$ when P and Q cannot be separated by open sets in $\mathrm{Prim}(A)$ is an equivalence relation on $\mathrm{Prim}(A)$, and (2) the corresponding quotient map is open. If A is quasi-standard, then $\sim$ and $\approx$ coincide and A is $\mathcal{CR}$ ([3], Proposition 3.2). In fact a number of interesting group C^*-algebras turn out to be quasi-standard ([2], [22]).

Let $M(A)$ be the multiplier algebra of A. Recall that a net $\{T_i\}$ converges to T in the strict topology on $M(A)$ if and only if $T_i a \to Ta$ and $T_i^* a \to T^* a$ for all $a \in A$. If the net is bounded, then it suffices to take a in the unit ball of A. In fact, if each T_i is unitary, then it suffices to check only that $T_i a \to Ta$ for $a \in A$ with $\|a\| \leqslant 1$. Consequently if A is separable, then the unitary group $\mathcal{U}M(A)$ is a second countable topological group in the strict topology which admits a complete metric (compatible with the topology). That is, $\mathcal{U}M(A)$ is a Polish group. Since $\mathcal{ZU}M(A)$ is closed in $\mathcal{U}M(A)$, it too is a Polish group.

For notational convenience, let $X = \bigl(\mathrm{Glimm}(A), \tau_{\mathrm{cr}}\bigr)$ be the complete regularization of $\mathrm{Prim}(A)$. Then we can identify $\mathcal{Z}M(A)$ with $C^{\mathrm{b}}\bigl(X\bigr)$, and $\mathcal{ZU}M(A)$ with $C(X, \mathbb{T})$. However it is not immediately obvious how to describe the strict topology on $C(X, \mathbb{T})$. Our next result says that when X is second countable and locally compact, then the strict topology on $C(X, \mathbb{T})$ coincides with the compact-open topology (the topology of uniform convergence on compacta).

LEMMA 2.6. *Suppose that X is a second countable locally compact Hausdorff space and that $A \in \mathcal{CR}(X)$. Then $\mathcal{ZUM}(A)$ with the strict topology is homeomorphic to $C(X,\mathbb{T})$ with the compact-open topology.*

REMARK 2.7. The lemma holds for $X = (\mathrm{Glimm}(A), \tau_{\mathrm{cr}})$ whenever $C(X,\mathbb{T})$ is a Polish group in the compact-open topology. In general, X is a σ-compact, completely regular space. If $\tau_{\mathrm{q}} = \tau_{\mathrm{cr}}$, then X is compactly generated (or a k-space) by [24], 43H(3), and at least in a compactly generated space, the limit of continuous functions in the compact-open topology is continuous. In order that $C(X,\mathbb{T})$ be metric, it seems to be necessary that X be "hemicompact" ([48], 43G(3)). In any case, if X is hemicompact, then the compact open topology is metric and complete. In this case, $C(X,\mathbb{T})$ is Polish, at least *provided* that X is second countable — so that $C(X,\mathbb{T})$ is separable. However we have been unable to show that X is second countable — even if X is locally compact.

Proof. Suppose that $f_n \to f$ uniformly on compacta in $C(X,\mathbb{T})$. If $a \in A$ is nonzero and if $\varepsilon > 0$, then the image of

$$C = \{P \in \mathrm{Prim}(A) : \|a(P)\| \geqslant \varepsilon/2\}$$

is compact in X ([10], Section 3.3). Thus, there is an N such that $n \geqslant N$ implies that $|f_n(q(P)) - f(q(P))| < \varepsilon/\|a\|$ for all $P \in C$. If $P \notin C$, then for all n,

$$\|f_n \cdot a(P) - f \cdot a(P)\| \leqslant \varepsilon.$$

This proves that $f_n \to f$ strictly.

Since X is second countable and locally compact, there is a sequence of compact sets $\{K_n\}$ in X such that $X = \bigcup_n K_n$ and such that every compact set K in X is contained in some K_n. Then if $\{V_n\}$ is a countable basis for the topology of $\mathbb{T}$, we get a sub-basis $\{U_{n,m}\}$ for the compact-open topology on $C(X,\mathbb{T})$ by setting

$$U_{n,m} := \{f \in C(X,\mathbb{T}) : f(K_n) \subseteq V_m\}.$$

It follows that $C(X,\mathbb{T})$ is second countable in the compact-open topology. Using the K_n's, it is easy to construct a complete metric on $C(X,\mathbb{T})$ compatible with the compact-open topology. Thus, $C(X,\mathbb{T})$ is a Polish group in the compact open topology. Since the first part of the proof shows that the identity map is continuous from the compact-open topology to the strict topology, the result follows from the Open Mapping Theorem ([29], Proposition 5 (b)). ∎

An automorphism α of a C^*-algebra A is called *inner* if there is a $u \in \mathcal{UM}(A)$ such that $\alpha = \mathrm{Ad}\,(u)$. (Recall that $\mathrm{Ad}\,(u)(a) := uau^*$.) An action $\alpha : G \to \mathrm{Aut}(A)$ is called *inner* if α_s is inner for each $s \in G$. An action is called *unitary* if there is a strictly continuous homomorphism $u : G \to \mathcal{UM}(A)$ such that $\alpha_s = \mathrm{Ad}\,(u_s)$ for all $s \in G$. Unitary actions are considered trivial; for example, if α is unitary, then the crossed product $A \rtimes_\alpha G$ is isomorphic to $A \otimes_{\max} C^*(G)$. Also two actions $\alpha : G \to \mathrm{Aut}(A)$ and $\beta : G \to \mathrm{Aut}(A)$ are called *exterior equivalent* if there is a strictly continuous map $w : G \to \mathcal{UM}(A)$ such that

(1) $\alpha_s(a) = w_s\beta_s(a)w_s^*$ for all $a \in A$ and $s \in G$, and

(2) for all $s, t \in G$, $w_{st} = w_s\beta_s(w_t)$.

In this event, we call w a 1-cocycle. Actions $\alpha : G \to \operatorname{Aut}(A)$ and $\beta : G \to \operatorname{Aut}(B)$ are called *outer conjugate* if there is a $*$-isomorphism $\Phi : A \to B$ such that β and $\Phi \circ \alpha \circ \Phi^{-1}$ are exterior equivalent.

Although unitary actions are trivial from the point of view of dynamical systems, inner actions can be quite interesting. Another class of interesting actions are those which are locally inner or even locally unitary.

Definition 2.8. Let X be a second countable locally compact Hausdorff space and G a second countable locally compact group. Suppose that $A \in \mathcal{CR}(X)$ and that $\alpha : G \to \operatorname{Aut}(A)$ is an action. Then α is called *locally unitary* (*locally inner*) if every point in X has a neighborhood U such that A_U is invariant under α and the restriction α^U of α to A_U is unitary (inner).

Remark 2.9. If A has Hausdorff spectrum X, then the above definition coincides with the usual notion of a locally unitary action (cf. [41], Section 1 and [45], Section 1).

Recall from Lemma 2.6 that if $A \in \mathcal{CR}(X)$ then the group $\mathcal{ZUM}(A)$ of A, equipped with the strict topology, is isomorphic to the polish group $C(X, \mathbb{T})$ equipped with the compact-open topology. Thus we obtain a short exact sequence of polish groups

$$1 \longrightarrow C(X, \mathbb{T}) \longrightarrow \mathcal{U}M(A) \longrightarrow \operatorname{Inn}(A) \longrightarrow 1.$$

If $\alpha : G \to \operatorname{Aut}(A)$ is inner, then α defines a continuous *homomorphism* of G into $\operatorname{Inn}(A)$ with its Polish topology ([42], Corollary 0.2). Thus, we can choose a Borel map $V : G \to \mathcal{U}M(A)$ such that $V_e = 1$ and such that $\alpha = \operatorname{Ad} V$. (For example, [29], Proposition 4 implies there is a Borel section $c : \operatorname{Inn}(A) \to \mathcal{U}M(A)$ such that $c(\mathrm{id}) = 1$. Then $V = c \circ \alpha$ will do the job.) Then V determines a Borel cocycle $\sigma \in Z^2\big(G, C(X, \mathbb{T})\big)$ via the equation $V_s V_t = \sigma(s,t) V_{st}$ for $s, t \in G$. The class $[\sigma] \in H^2\big(G, C(X, \mathbb{T})\big)$ only depends on α and is the only obstruction to α being unitary (see [42], Corollary 0.12). In what follows we will refer to V as a *σ-homomorphism* of G into $\mathcal{U}M(A)$ which implements α.

Remark 2.10. Suppose that $\alpha : G \to \operatorname{Aut}\big(C_0(X, \mathcal{K})\big)$ is an inner automorphism group which is implemented by a σ-homomorphism as above. Then the Mackey obstruction for the induced action α^x on the fibre over x is the class of $\sigma(x)$, where $\sigma(x)$ is the cocycle in $Z^2(G, \mathbb{T})$ obtained by evaluation at x: $\sigma(x)(s,t) := \sigma(s,t)(x)$.

3. LOCALLY UNITARY ACTIONS

In this section we want to see that the Phillips-Raeburn classification scheme for locally unitary actions of abelian groups can be extended to all second countable locally compact groups acting on $\mathcal{CR}$-algebras.

Suppose that $A \in \mathcal{CR}(X)$ and that $\alpha : G \to \operatorname{Aut}(A)$ is locally unitary. Then there is a cover $\mathbf{U} = \{U_i\}_{i\in I}$ such that $\alpha^{U_i} = \operatorname{Ad}(u^i)$ for strictly continuous homomorphisms $u^i : G \to \mathcal{U}M(A_{U_i})$. For convenience, we will write U_{ij} for $U_i \cap U_j$, A_{ij} in place of $A_{U_{ij}}$, α^{ij} for $\alpha^{U_{ij}}$, and u^{ij} for $(u^i)^{U_{ij}}$. Even though $u^{ij} \neq u^{ji}$, both u^{ij} and u^{ji} implement α^{ij}. It follows that for each $s \in G$, $u^{ij}_s(u^{ji}_s)^*$ belongs to $\mathcal{ZUM}(A_{ij})$. In order to identify $\mathcal{ZUM}(A_{ij})$ with $C(U_{ij}, \mathbb{T})$ (with the compact-open topology), we need the following lemma.

LEMMA 3.1. *Suppose that $A \in \mathcal{CR}(X)$ and that U is open in X. Then $A_U \in \mathcal{CR}(U)$.*

Proof. Let $q : \operatorname{Prim}(A) \to X$ be the quotient map. Then we can identify $\operatorname{Prim}(A_U)$ with $q^{-1}(U)$, and $\operatorname{Glimm}(A_U)$ is the quotient of the latter with topology induced by $C^{\mathrm{b}}\big(q^{-1}(U)\big)$. We have to show that $\operatorname{Glimm}(A_U)$ can be identified with U with the relative topology. However since any $f \in C^{\mathrm{b}}\big(\operatorname{Prim}(A)\big)$ restricts to an element of $C^{\mathrm{b}}\big(q^{-1}(U)\big)$, it is clear that $P \sim Q$ in $q^{-1}(U)$ implies that $P \sim Q$ in $\operatorname{Prim}(A)$. On the other hand, suppose that $P \sim Q$ in $\operatorname{Prim}(A)$ and that $f \in C^{\mathrm{b}}\big(q^{-1}(U)\big)$. Since X is locally compact and U is an open neighborhood of $q(P) = q(Q)$, there is a $g \in C_c^+(X)$ with $g\big(q(P)\big) = 1$ and $\operatorname{supp}(g) \subseteq U$. Therefore, we may view $h = f(g \circ q)$ as an element of $C^{\mathrm{b}}\big(\operatorname{Prim}(A)\big)$. Since $h(P) = h(Q)$ by assumption, we must have $f(P) = f(Q)$. Thus, the two equivalence relations coincide on $q^{-1}(U)$ and we can identify U with $\operatorname{Glimm}(A_U)$ at least as a set.

Let τ_r be the relative topology on U. A similar argument to that in the previous paragraph shows that any element of $C^{\mathrm{b}}\big(q^{-1}(U)\big)$ agrees at least locally with an element of $C^{\mathrm{b}}\big(\operatorname{Prim}(A)\big)$. Thus $C^{\mathrm{b}}(U, \tau_r)$ and $C^{\mathrm{b}}(U, \tau_{\mathrm{cr}})$ coincide. Since both topologies on U are completely regular (Hausdorff) topologies, and therefore are determined by the zero sets of $C^{\mathrm{b}}(U)$ ([17], Theorem 3.7), the topologies coincide. ∎

Now the previous lemma allows us to conclude that $A_{ij} \in \mathcal{CR}(U_{ij})$ so that we may identify $\mathcal{ZUM}(A_{ij})$ with $C(U_{ij}, \mathbb{T})$ as claimed. Notice that $u^{ij}(x) = u^i(x)$. Since $C(U_{ij}, \mathbb{T})$ has the compact open topology and $s \mapsto u^{ij}_s(u^{ji}_s)^*$ is continuous, it follows that

$$(x, s) \mapsto u^i_s(x)u^j_s(x)^*$$

is jointly continuous from $U_{ij} \times G$ to $\mathbb{T}$. In particular, for each $x \in U_{ij}$, $s \mapsto u^i_s(x)u^j_s(x)^*$ is a continuous character $\gamma_{ij}(x)$ on G, and

$$u^i_s(x) = \gamma_{ij}(x)(s)u^j_s(x). \tag{3.1}$$

Since any character has to kill the closure of the commutator subgroup $[G, G]$, we will always view $\gamma_{ij}(x)$ as a character on the abelian group $G_{\mathrm{ab}} := G/\overline{[G, G]}$. The group G_{ab} is a locally compact abelian group usually called the *abelianization* of G. Notice that the joint continuity implies that the functions $\gamma_{ij} : U_{ij} \to \widehat{G}_{\mathrm{ab}}$ are continuous when $\widehat{G}_{\mathrm{ab}}$ is given the usual locally compact dual topology (of uniform convergence on compacta). A straightforward computation using the definition of the γ_{ij}'s shows that if $x \in U_{ijk}$, then

$$\gamma_{ij}(x)\gamma_{jk}(x) = \gamma_{ik}(x).$$

Thus the collection $\gamma = \{\gamma_{ij}\}$ defines a 1-cocycle in $Z^1(\mathbf{U}, \widehat{\mathcal{G}}_{\mathrm{ab}})$ and therefore a class ζ in $H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$. We claim this class depends only on (A, G, α). Suppose we had taken a different cover $\{V_j\}_{j \in J}$ and homomorphisms v^i. Passing to a common refinement allows us to assume that $I = J$ and that $U_i = V_i$. Then since u_i and v_i both implement α^i over U_i, an argument similar to that above implies they differ by a central multiplier $\lambda_i : U_i \to \widehat{G}_{\mathrm{ab}}$. Then it is easy to see that we get cohomologous cocycles. One usually writes $\zeta(\alpha)$ for the class ζ, and $\zeta(\alpha)$ is called the *Phillips-Raeburn obstruction*.

REMARK 3.2. When G is abelian and A is type I with spectrum X, then $\zeta(\alpha)$ is the classical Phillips-Raeburn obstruction of [41]. That is, $\zeta(\alpha)$ is the class of the principal $\widehat{G}$-bundle given by the restriction map $p : (A \rtimes_\alpha G)^\wedge \to X$ as in [41], Theorem 2.2.

PROPOSITION 3.3. *Let X be a second countable locally compact Hausdorff space. Suppose that $A \in \mathcal{CR}(X)$ and that $\alpha : G \to \mathrm{Aut}(A)$ is a second countable locally compact, locally unitary automorphism group. Then the transition functions* (3.1) *define a class $\zeta(\alpha)$ in $H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$ which depends only on (A, G, α). If (A, G, β) is another such system, then $\zeta(\alpha) = \zeta(\beta)$ if and only if α and β are exterior equivalent. In particular, α is unitary if and only if $\zeta(\alpha) = 1$ (We are writing the product in H^1 multiplicatively; therefore* 1 *denotes the trivial element.). Furthermore if A is stable, then every class in $H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$ is equal to $\zeta(\alpha)$ for some locally unitary action $\alpha : G \to \mathrm{Aut}(A)$.*

Proof of all but the last assertion. We have already seen that $\zeta(\alpha)$ depends only on (A, G, α). Now suppose that (A, G, β) is another locally unitary action with $\zeta(\beta) = \zeta(\alpha)$. Then we can find a cover $\{U_i\}_{i \in I}$ and $u^i, v^i : G \to \mathcal{U}M(A_{U_i})$ such that u^i implements α^i, v^i implements β^i, and such that

$$u^i_s(x)u^j_s(x)^* = \gamma_{ij}(x)(s) = v^i_s(x)v^j_s(x)^*. \tag{3.2}$$

Let $w^i_s(x) := u^i_s(x)v^i_s(x)^*$. Then $s \mapsto w^i_s(\cdot)$ is a strictly continuous map of G into $\mathcal{U}M(A_{U_i})$. Then it is easy to see that α^i is exterior equivalent to β^i via w^i. However, if $x \in U_{ij}$, then (3.2) implies that

$$w^i_s(x)w^j_s(x)^* = u^i_s(x)v^i_s(x)^*v^j_s(x)u^j_s(x)^* = 1.$$

Consequently, we can define $w_s(x) = w^i_s(x)$ if $x \in U_i$. Since each w^i defines a strictly continuous map into $\mathcal{U}M(A_{U_i})$ and since $x \mapsto \|a(x)\|$ vanishes at infinity for each $a \in A$, it is not hard to see that w is a strictly continuous map from G into $\mathcal{U}M(A)$. Therefore, α and β are exterior equivalent.

Conversely, if α and β are exterior equivalent via $w : G \to \mathcal{U}M(A)$, then with $\{U_i\}_{i \in I}$ and $u^i, v^i : G \to \mathcal{U}M(A_{U_i})$ as above, we must have unimodular scalars $\lambda_i(x)(s)$ for all $x \in U_i$ and $s \in G$ such that $\lambda_i(x)(s) = u^i_s(x)^*w_s(x)v^i_s(x)$.

As above, we may view these as continuous functions from U_i to $\widehat{G}_{\mathrm{ab}}$. Also, if $x \in U_{ij}$, then

$$\begin{aligned} u_s^i(x)^* u_s^j(x) &= u_s^i(x)^* w_s(x) v_s^i(x) v_s^i(x)^* v_s^j(x) \big(u_s^j(x)^* w_s(x) v_s^j(x)\big)^* \\ &= \lambda_i(x)(s)\overline{\lambda_j(x)(s)} v_s^i(x)^* v_s^j(x). \end{aligned}$$

It follows that $\zeta(\alpha) = \zeta(\beta)$. ∎

To prove that every class in $H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$ arises when A is stable, we want to recall some facts about balanced tensor products. Suppose that A and B are $C_0(X)$-algebras. Let I be the ideal in $A \otimes_{\max} B$ generated by

$$\{a \cdot f \otimes b - a \otimes f \cdot b : a \in A,\, b \in B, \text{ and } f \in C_0(X)\}.$$

The (*maximal*) $C_0(X)$-*balanced tensor product* of A and B is defined to be the quotient

$$A \otimes_X B := (A \otimes_{\max} B)/I.$$

REMARK 3.4. Balanced tensor products have been studied by several authors, and quite recently by Blanchard ([5], [4]). In particular if X is compact, then $A \otimes_X B$ coincides with Blanchard's $A \otimes_X^M B$. Moreover, $A \otimes_X B$ is a $C_0(X)$-algebra and, writing $a \otimes_X b$ for the image of $a \otimes b$ in $A \otimes_X B$, we have

$$f \cdot (a \otimes_X b) = f \cdot a \otimes_X b = a \otimes_X f \cdot b \quad \text{for all } f \in C_0(X). \tag{3.3}$$

We intend to discuss these and other properties of $\otimes_X$ elsewhere ([15], Section 2). Here we will be satisfied with the special cases outlined below.

In this work, we shall always assume that A and B are separable, and that B is nuclear — in fact, it will suffice to consider only the case where $B = C_0(X, \mathcal{K})$. Then [43], Lemma 1.1 applies and we can identify $\mathrm{Prim}(A \otimes_X B)$ with

$$\{(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B) : \sigma_A(P) = \sigma_B(Q)\}. \tag{3.4}$$

In this case, (3.3) is a straightforward consequence of (3.4) and the definition of I. Moreover, for all $x \in X$,

$$A \otimes_X B(x) \cong A(x) \otimes B(x), \tag{3.5}$$

and $(a \otimes_X b)(x) = a(x) \otimes b(x)$. (Note that we write simply $\otimes$ when one of the factors is nuclear.) If $B = C_0(X, \mathcal{K})$, then $\mathrm{Prim}\big(A \otimes_X C_0(X, \mathcal{K})\big)$ can be identified with $\mathrm{Prim}(A)$. Moreover since $C_0(X, \mathcal{K}) \cong C_0(X) \otimes \mathcal{K}$, the map $a \otimes_X (f \otimes T) \mapsto a \cdot f \otimes T$ extends to a $C_0(X)$-linear isomorphism of $A \otimes_X C_0(X, \mathcal{K})$ onto $A \otimes \mathcal{K}$. Notice that if U is open in X, then this isomorphism identifies $(A \otimes_X C_0(X, \mathcal{K}))_U$ with $A_U \otimes \mathcal{K}$. Furthermore, if A is stable, then we can choose an isomorphism of $A \otimes \mathcal{K}$ and A which induces the identity map on the primitive ideal spaces (assuming $\mathrm{Prim}(A \otimes \mathcal{K})$ has been identified with $\mathrm{Prim}(A)$) ([41], Lemma 4.3). Then $A \otimes_X C_0(X, \mathcal{K})$ is isomorphic to A and $(A \otimes_X C_0(X, \mathcal{K}))_U$ is identified with A_U.

LEMMA 3.5. (cf. [41], Proposition 3.10) *Suppose that X is a second countable locally compact Hausdorff space and that $A \in \mathcal{CR}(X)$. Then $A \otimes_X C_0(X,\mathcal{K}) \in \mathcal{CR}(X)$. If $\alpha : G \to \operatorname{Aut}(A)$ and $\beta : G \to \operatorname{Aut}\big(C_0(X,\mathcal{K})\big)$ are $C_0(X)$-automorphism groups, then the diagonal action $\alpha \otimes \beta$ on $A \otimes C_0(X,\mathcal{K})$ induces an action $\alpha \otimes_X \beta$ on $A \otimes_X C_0(X,\mathcal{K})$. If γ is exterior equivalent to α and δ is exterior equivalent to β, then $\alpha \otimes_X \beta$ is exterior equivalent to $\gamma \otimes_X \delta$. Finally, if α and β are locally unitary, then so is $\alpha \otimes_X \beta$; moreover,*

$$\zeta\big(\alpha \otimes_X \beta\big) = \zeta(\alpha)\zeta(\beta) \quad \textit{in } H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}}).$$

REMARK 3.6. If B is an arbitrary element of $\mathcal{CR}(X)$, then it seems to be difficult to decide whether $A \otimes_X B$ is in $\mathcal{CR}(X)$ — even if A and B are both nuclear. However if one of the algebras is nuclear and A or B has Hausdorff primitive ideal space, then one can replace $C_0(X,\mathcal{K})$ by B in the above and obtain the same results.

Proof. Since $\operatorname{Prim}(A \otimes_X C_0(X,\mathcal{K}))$ can be identified with $\operatorname{Prim}(A)$, $A \otimes_X C_0(X,\mathcal{K})$ is certainly in $\mathcal{CR}(X)$. If α_s and β_s are $C_0(X)$-linear, then $\alpha_s \otimes \beta_s$ maps the balancing ideal into itself and $\alpha \otimes_X \beta$ is a well-defined action on $A \otimes_X B$.

Now suppose that $u : G \to \mathcal{U}M(A)$ and $v : G \to \mathcal{U}M\big(C_0(X,\mathcal{K})\big)$ are strictly continuous 1-cocycles such that $\alpha_s(a) = u_s\gamma_s(a)u_s^*$ and $\beta_s(b) = v_s\delta_s(b)v_s^*$ for all $s \in G$, $a \in A$, and $b \in C_0(X,\mathcal{K})$. Since the image of $C_0(X)$ sits in the center of the respective multiplier algebras, it is clear that each u_s and v_s commutes with the $C_0(X)$-actions. Therefore $u_s \otimes v_s$ defines a well defined element $w_s := u_s \otimes_X v_s$ in $\mathcal{U}M\big(A \otimes_X C_0(X,\mathcal{K})\big)$. The continuity of $s \mapsto w_s t$ is clear for t in the algebraic tensor product $A \odot C_0(X,\mathcal{K})$. This suffices to show strict continuity as each w_s has norm one. Routine calculations show that w_s is a 1-cocycle implementing an exterior equivalence between $\alpha \otimes_X \beta$ and $\gamma \otimes_X \delta$.

Finally, suppose that α and β are locally unitary. Then we can find a cover $\mathbf{U} = \{U_i\}$ such that α^{U_i} is implemented by a homomorphism $u^i : G \to \mathcal{U}M(A_{U_i})$ and β^{U_i} by a homomorphism $v^i : G \to \mathcal{U}M\big(C_0(U_i,\mathcal{K})\big)$. Let $\gamma = \{\gamma_{ij}\}$ and $\eta = \{\eta_{ij}\}$ be the corresponding cocycles representing $\zeta(\alpha)$ and $\zeta(\beta)$. As above, we obtain a homomorphism $w^i = u^i \otimes_{U_i} v^i$ which implements $(\alpha \otimes_X \beta)^{U_i} = \alpha^{U_i} \otimes_{U_i} \beta^{U_i}$ on $A_{U_i} \otimes_{U_i} C_0(U_i,\mathcal{K}) \cong \big(A \otimes_X C_0(X,\mathcal{K})\big)_{U_i}$. Thus $\alpha \otimes_X \beta$ is locally unitary. Moreover since $w^i(x) = u^i(x) \otimes v^i(x)$,

$$w_s^i(x) = \gamma_{ij}(x)(s)u_s^j(x) \otimes \eta_{ij}(x)(s)v_s^j(x) = \big(\gamma_{ij}(x)(s)\eta_{ij}(x)(s)\big)w_s^j(x).$$

The result follows. ∎

Proof of the final assertion in Proposition 3.3. Let $\zeta_0 \in H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$. As we remarked above, when A is stable there is an isomorphism of A and $A \otimes_X C_0(X,\mathcal{K})$ carrying A_U onto $\big(A \otimes_X C_0(X,\mathcal{K})\big)_U$. Thus it will suffice to produce a locally unitary action α on $A \otimes_X C_0(X,\mathcal{K})$ with $\zeta(\alpha) = \zeta_0$. It follows from [41], Theorem 3.8 and Remark 3.2 that there is a locally unitary action $\widetilde{\beta} : G_{\mathrm{ab}} \to \operatorname{Aut}\big(C_0(X,\mathcal{K})\big)$ with $\zeta(\widetilde{\beta}) = \zeta_0$. Now we simply lift $\widetilde{\beta}$ to G. That is $\beta_s := \beta_{sH}$ where $H := \overline{[G,G]}$. It is straightforward to check that $\zeta(\beta) = \zeta(\widetilde{\beta}) = \zeta_0$. Now the result follows by applying Lemma 3.5 to $\alpha := 1 \otimes_X \beta$. ∎

REMARK 3.7. Since two actions with the same Phillips-Raeburn obstruction are exterior equivalent, the above argument makes it clear that *any* locally unitary action of G on a stable $\mathcal{CR}$-algebra A is lifted from an action of G_{ab}. In fact, there is a one-to-one correspondence between exterior equivalence classes of locally unitary actions of G on A and exterior equivalence classes of locally unitary actions of G_{ab} on A.

We end this section with a short discussion on locally unitary action on continuous-trace C^*-algebras. Assume that A is a separable continuous trace C^*-algebra with spectrum X. An action $\alpha : G \to \mathrm{Aut}(A)$ is called *pointwise unitary* if α is $C_0(X)$-linear and the action on each fibre $A(x)$ is unitary, or, equivalently, if each irreducible representation ρ can be extended to a covariant representation (ρ, V) of (A, G, α). In general, a pointwise unitary action need not be locally unitary (see Section 7). Despite this, pointwise unitary actions are locally unitary under mild additional hypotheses. The strongest result in this direction is due to Rosenberg.

THEOREM 3.8. ([45], Corollary 1.2) *Let A be a separable continuous-trace C^*-algebra with spectrum X and let G be a second countable locally compact group such that G_{ab} is compactly generated and $H^2(G, \mathbb{T})$ is Hausdorff. Then every pointwise unitary action of G on A is locally unitary.*

Thus as a direct corollary of this and Proposition 3.3, we obtain:

COROLLARY 3.9. *Let A and G be as above and assume in addition that A is stable. Then the Phillips-Raeburn obstruction map $\alpha \to \zeta(\alpha)$ induces a bijection between the exterior equivalence classes of pointwise unitary actions of G on A and $H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$.*

4. REPRESENTATION GROUPS

In this section we want to discuss the notion of representation groups of second countable locally compact groups as introduced by Moore in [28]. Recall that if $1 \to Z \to H \to G \to 1$ is a second countable locally compact central extension of G by the abelian group Z, then the transgression map $\mathrm{tg} : \widehat{Z} = H^1(Z, \mathbb{T}) \to H^2(G, \mathbb{T})$ is defined as follows: Let $c : G \to H$ be a Borel section for the quotient map $H \to G$ such that $c(eZ) = e$. Then $\sigma(s,t) := c(s)c(t)c(st)^{-1}$ is a cocycle in $Z^2(G, Z)$ (Moore-cohomology with values in the trivial G-module Z). If $\chi \in \widehat{Z}$, then $\sigma_\chi(s,t) := \chi(\sigma(s,t))$ defines a cocycle $\sigma_\chi \in Z^2(G, \mathbb{T})$ and then $\mathrm{tg}(\chi) := [\sigma_\chi]$ is the cohomology class of σ_χ in $H^2(G, \mathbb{T})$.

DEFINITION 4.1. (Moore) Let G be a second countable locally compact group and let H be a central extension of G by some abelian group Z such that the transgression map $\mathrm{tg} : \widehat{Z} \to H^2(G, \mathbb{T})$ is bijective. Then H (or rather the extension $1 \to Z \to H \to G \to 1$) is called a *representation group* for G. A group G is called *smooth* if it has a representation group H.

REMARK 4.2. The question of which groups have a representation group was studied extensively by Moore in [28]. If H is a representation group for G, then the transgression map $\mathrm{tg} : \widehat{Z} \to H^2(G,\mathbb{T})$ is actually a homeomorphism by [28], Theorem 2.2 and [30], Theorem 6, so that in this case $H^2(G,\mathbb{T})$ is always locally compact and Hausdorff. Conversely, if G is almost connected (i.e., G/G_0 is compact) and $H^2(G,\mathbb{T})$ is Hausdorff, then G is smooth by [28], Proposition 2.7. Since [27], Theorem A and following remark (2) imply that $H^2(G,\mathbb{T})$ is isomorphic to $\mathbb{R}^k$ for some $k \geqslant 0$ whenever G is a connected and simply connected Lie group, it follows that such groups are smooth. If G is a connected semisimple Lie group, then the universal covering group H of G is a representation group for G by [28], Proposition 3.4. Finally, every compact group is smooth (see the discussion preceding [28], Proposition 3.1), and every discrete group is smooth by [28], Theorem 3.1 (see also [36], Corollary 1.3).

We will see below that, in addition to the above, every second countable compactly generated abelian group is smooth (Corollary 4.6). To prove our results, we need the following characterization of smooth groups.

LEMMA 4.3. *Let G be a second countable locally compact group. Then G is smooth if and only if there exists a second countable locally compact Hausdorff topology on $H^2(G,\mathbb{T})$ and a Borel cocycle $\zeta \in Z^2\big(G, H^2(G,\mathbb{T})^\wedge\big)$ such that for each $[\omega] \in H^2(G,\mathbb{T})$ evaluation of ζ at $[\omega]$ gives an element in $Z^2(G,\mathbb{T})$ representing $[\omega]$.*

Proof. Assume $1 \to Z \to H \to G \to 1$ is a representation group for G. Since tg is an isomorphism, we can define an isomorphism $\mathrm{tg}_* : Z \to H^2(G,\mathbb{T})^\wedge$ by $\mathrm{tg}_*(z)([\omega]) = \mathrm{tg}^{-1}([\omega])(z)$. Let $c : G \to H$ be a Borel cross section with $c(e_G) = e_H$ (e_G denoting the unit in G). For $s,t \in G$, define $\zeta(s,t) := \mathrm{tg}_*(c(s)c(t)c(st)^{-1})$. Then $\zeta \in Z^2(G, H^2(G,\mathbb{T})^\wedge)$, and if $[\omega] \in H^2(G,\mathbb{T})$, then we obtain $\zeta(s,t)([\omega]) = \mathrm{tg}^{-1}([\omega])(c(s)c(t)c(st)^{-1})$. Thus, by the definition of tg (see the discussion above) $(s,t) \mapsto \zeta(s,t)([\omega])$ is a cocycle representing $[\omega]$. This is what we wanted.

For the converse, assume that there is a second countable locally compact Hausdorff topology on $H^2(G,\mathbb{T})$ and let $\zeta \in Z^2(G, H^2(G,\mathbb{T})^\wedge)$ be such that evaluation at each $[\omega] \in H^2(G,\mathbb{T})$ gives a representative for $[\omega]$. Let H denote the extension $G \times_\zeta H^2(G,\mathbb{T})^\wedge$ of G by $H^2(G,\mathbb{T})^\wedge$ given by ζ. This is the set $G \times H^2(G,\mathbb{T})^\wedge$ with multiplication defined by

$$(s,\chi)(t,\mu) = (st, \zeta(s,t)\chi\mu),$$

and equipped with the unique locally compact group topology inducing the product Borel structure on $G \times H^2(G,\mathbb{T})^\wedge$ (see [25], Theorem 7.1). Then

$$1 \to H^2(G,\mathbb{T})^\wedge \to H \to G \to 1$$

is a central extension of G by $H^2(G,\mathbb{T})^\wedge$. Let $c : G \to H$ denote the canonical section $c(s) = (s,1)$. Then the transgression map is a map from $H^2(G,\mathbb{T}) \cong (H^2(G,\mathbb{T})^\wedge)^\wedge$ to itself, and for each $[\omega] \in H^2(G,\mathbb{T})$ we obtain a representative for $\mathrm{tg}([\omega])$ by taking the cocycle

$$\nu(s,t) = (c(s)c(t)c(st)^{-1})([\omega]) = \zeta(s,t)([\omega]).$$

Therefore the transgression map $\mathrm{tg} : H^2(G,\mathbb{T}) \to H^2(G,\mathbb{T})$ is the identity, and H is a representation group for G as required. ∎

REMARK 4.4. If $1 \to Z \to H \to G \to 1$ is a representation group for G and if $\zeta \in Z^2(G, H^2(G,\mathbb{T})^\wedge)$ is constructed as above such that the evaluation map is the identity on $H^2(G,\mathbb{T})$, then $G \times_\zeta H^2(G,\mathbb{T})^\wedge$ is actually isomorphic to H. An isomorphism is given by $(s,[\sigma]) \mapsto c(s)(\mathrm{tg}_*)^{-1}([\sigma])$, where $c : G \to H$ denotes the Borel section defining ζ as above.

We now show that, under some weak additional assumptions, the direct product of smooth groups is again smooth.

PROPOSITION 4.5. *Suppose that G_1 and G_2 are smooth and let $B(G_1,G_2)$ denote the group of continuous bicharacters $\chi : G_1 \times G_2 \to \mathbb{T}$. If $B(G_1,G_2)$ is locally compact with respect to the compact open topology, then $G_1 \times G_2$ is smooth. In particular, if the abelianizations $(G_1)_{\mathrm{ab}}$, $(G_2)_{\mathrm{ab}}$ of G_1 and G_2 are compactly generated, then $G_1 \times G_2$ is smooth.*

Proof. In fact we are going to construct a representation group for $G_1 \times G_2$ as follows: Choose central extensions

$$1 \longrightarrow H^2(G_i,\mathbb{T})^\wedge \longrightarrow H_i \xrightarrow{q_i} G_i \longrightarrow 1$$

for $i = 1, 2$ such that the respective transgression maps are both equal to the identity (see Lemma 4.3 and Remark 4.4). By assumption $B(G_1,G_2)$ is locally compact, so the dual group $B(G_1,G_2)^\wedge$ is also a locally compact group. For each pair $(s_1,s_2) \in G_1 \times G_2$ define $\eta(s_1,s_2) \in B(G_1,G_2)^\wedge$ by $\eta(s_1,s_2)(\chi) = \chi(s_1,s_2)$, $\chi \in B(G_1,G_2)$. Let

$$H = H_1 \times H_2 \times B(G_1,G_2)^\wedge$$

with multiplication defined by

$$(h_1,h_2,\mu)(l_1,l_2,\nu) = \big(h_1l_1, h_2l_2, \mu\nu\eta(q_1(h_1),q_2(l_2))\big). \tag{4.1}$$

Then clearly

$$Z = H^2(G_1,\mathbb{T})^\wedge \times H^2(G_2,\mathbb{T})^\wedge \times B(G_1,G_2)^\wedge$$

is a central subgroup of H, and we obtain a short exact sequence

$$1 \longrightarrow Z \longrightarrow H \longrightarrow G_1 \times G_2 \longrightarrow 1.$$

We claim that H is a representation group for $G_1 \times G_2$.

For this recall that if $\omega_1 \in Z^2(G_1,\mathbb{T})$, $\omega_2 \in Z^2(G_2,\mathbb{T})$ and $\chi \in B(G_1,G_2)$, then $\omega_1 \otimes \omega_2 \otimes \chi$ defined by

$$\omega_1 \otimes \omega_2 \otimes \chi((s_1,s_2),(t_1,t_2)) = \omega_1(s_1,t_1)\omega_2(s_2,t_2)\chi(s_1,t_2)$$

is a cocycle in $Z^2(G_1 \times G_2,\mathbb{T})$. By [26], Theorem 9.6 and [23], Propositions 1.4 and 1.6 we know that the map

$$([\omega_1],[\omega_2],\chi) \longmapsto [\omega_1 \otimes \omega_2 \otimes \chi]$$

is an (algebraic) isomorphism of $H^2(G_1,\mathbb{T}) \times H^2(G_2,\mathbb{T}) \times B(G_1,G_2)$ onto $H^2(G_1 \times G_2,\mathbb{T})$, from which it follows that Z is isomorphic to $H^2(G_1 \times G_2,\mathbb{T})^\wedge$. Now choose Borel sections $c_i : G_i \to H_i$ and define $\zeta_i \in Z^2(G_i, H^2(G_i,\mathbb{T})^\wedge)$ by $\zeta_i(s,t) = c_i(s)c_i(t)c_i(st)^{-1}$, for $s,t \in G_i$. Since the transgression maps are both equal to the identity map, we see that evaluation of ζ_i at $[\omega_i] \in H^2(G_i,\mathbb{T})$ is a cocycle representing $[\omega_i]$. Defining $c : G_1 \times G_2 \to H_1 \times H_2 \times B(G_1, G_2)^\wedge$ by $c(s_1, s_2) = (c_1(s_1), c_2(s_2), 1)$, we easily compute

$$\begin{aligned}\zeta((s_1, s_2), (t_1, t_2)) &:= c(s_1, s_2)c(t_1, t_2)c(s_1t_1, s_2t_2)^{-1} \\ &= (\zeta_1(s_1, t_1), \zeta_2(s_2, t_2), \eta(s_1, t_2)).\end{aligned}$$

Thus, evaluating ζ at $[\omega_1 \otimes \omega_2 \otimes \chi] \in H^2(G_1 \times G_2, \mathbb{T})$ gives a cocycle representing this class. This proves the claim. The final assertion now follows from the fact that $B(G_1, G_2) = B((G_1)_{\mathrm{ab}}, (G_2)_{\mathrm{ab}})$ and [23], Theorem 2.1. ∎

COROLLARY 4.6. *Every second countable compactly generated abelian group is smooth.*

Proof. By the structure theorem for compactly generated abelian groups ([20], Theorem 9.8), we know that $G \cong \mathbb{R}^n \times K \times \mathbb{Z}^m$ for some $n, m \geqslant 0$ and some compact group K. By the results mentioned in Remark 4.2, it follows that $\mathbb{R}^n, K$ and $\mathbb{Z}^m$ are smooth. Now apply the proposition. ∎

The example given at the bottom of [28], p. 85 shows that there are nonsmooth abelian locally compact groups. The group constructed there is a direct product of $\mathbb{R}$ with an infinite direct sum of copies of $\mathbb{Z}$. Thus it also provides an example of two smooth groups whose direct product is not smooth. Thus the assumption on $B(G_1, G_2)$ in Proposition 4.5 is certainly not superfluous.

It is certainly interesting to see specific examples of representation groups. Some explicit constructions can be found in [28] and [36], Corollary 1.3 and Examples 1.4. For instance, if $G = \mathbb{R}^2$ (respectively $G = \mathbb{Z}^2$), then the three dimensional Heisenberg group (respectively discrete Heisenberg group) is a representation group for G. In the following example, we use Proposition 4.5 to construct representation groups for $\mathbb{R}^n$.

EXAMPLE 4.7. Let $G = \mathbb{R}^n$ and, as a set, let $H_n = \mathbb{R}^{n(n+1)/2}$. We write an element of H_n as $\mathfrak{s} = (s_i, s_{j,k})$, $1 \leqslant i \leqslant n$, $1 \leqslant j < k \leqslant n$. Define multiplication on H_n by $\mathfrak{s}\mathfrak{t} = ((st)_i, (st)_{j,k})$ with

$$(st)_i := s_i + t_i \quad \text{and} \quad (st)_{j,k} := s_{j,k} + t_{j,k} + s_j t_k.$$

Then H_n is clearly a central extension of $\mathbb{R}^n$ by $\mathbb{R}^{(n-1)n/2}$.

We claim that H_n is a representation group for $\mathbb{R}^n$. Since $H^2(\mathbb{R}, \mathbb{T})$ is trivial, this is certainly true for $n = 1$. For the step $n \to n + 1$ assume that H_n is a representation group for $\mathbb{R}^n$. For $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$ define $\chi_{\mathbf{s}} \in B(\mathbb{R}^n, \mathbb{R})$ by

$$\chi_{\mathbf{s}}((t_1, \ldots, t_n), r) := \mathrm{e}^{ir(s_1t_1 + \cdots + s_nt_n)}.$$

Since $\chi_{\mathbf{s}}(\mathbf{t}, r) = \chi_{r\mathbf{s}}(\mathbf{t}, 1)$, $s \mapsto \chi_{\mathbf{s}}$ is an isomorphism of $\mathbb{R}^n$ onto $B(\mathbb{R}^n, \mathbb{R})$, and we see that $B(\mathbb{R}^n, \mathbb{R})^\wedge$ is isomorphic to $\mathbb{R}^n$ via the map $\mathbf{t} \mapsto \eta_{\mathbf{t}}$ defined by $\eta_{\mathbf{t}}(\chi_{\mathbf{s}}) = \chi_{\mathbf{s}}(\mathbf{t}, 1)$, for $\mathbf{t} \in \mathbb{R}^n$. Moreover, if $(\mathbf{s}, r) \in \mathbb{R}^n \times \mathbb{R}$, and $\eta(\mathbf{s}, r) \in B(\mathbb{R}^n, \mathbb{R})^\wedge$ is defined by $\eta(\mathbf{s}, r)(\chi_{\mathbf{t}}) = \chi_{\mathbf{t}}(\mathbf{s}, r)$, then we get the identity $\eta(\mathbf{s}, r) = \eta_{r\mathbf{s}}$. It follows now from Proposition 4.5 and (4.1) that $H' = H_n \times \mathbb{R} \times \mathbb{R}^n$ with multiplication defined by

$$\big((s_i, s_{j,k}), r, \mathbf{t}\big)\big((s'_i, s'_{j,k}), r', \mathbf{t}'\big) = \big((s_i, s_{j,k})(s'_i, s'_{j,k}), r + r', \mathbf{t} + \mathbf{t}' + r'\mathbf{s}\big)$$

is a representation group for $\mathbb{R}^{n+1}$. Putting $s_{n+1} = r$ and $s_{j,n+1} = t_j$, $1 \leqslant j \leqslant n$, we see that this formula coincides with the multiplication formula for H_{n+1}.

Using similar arguments, it is not hard to show that a representation group for $\mathbb{Z}^n$ is given by the integer subgroup of H_n constructed above, i.e., assuming that all s_i and $s_{j,k}$ are integers. Notice that the group H_n constructed above is isomorphic to the connected and simply connected two-step nilpotent Lie group corresponding to the universal two-step nilpotent Lie algebra generated by $X_1, \ldots, X_n$ and the commutators $[X_j, X_k]$, $1 \leqslant j < k \leqslant n$. Note that any connected two-step nilpotent Lie group is a quotient of one of these groups (e.g., see [6], p. 409).

We conclude this section with a discussion of which conditions will imply that all representation groups of a given group G are isomorphic. Schur observed that even finite groups can have nonisomorphic representation groups ([47]), and Moore considers the case for G compact or discrete in [28], Section 3. Here we give a sufficient condition for the uniqueness of the representation group (up to isomorphism) valid for all smooth G.

PROPOSITION 4.8. *Let G be smooth and let $Z := H^2(G, \mathbb{T})^\wedge$. Then the representation groups of G are unique (up to isomorphism of groups) if every abelian extension $1 \to Z \to H \to G_{\mathrm{ab}} \to 1$ splits. In particular, if G_{ab} is isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m$ or if Z is isomorphic to $\mathbb{R}^n \times \mathbb{T}^m$, for some $n, m \geqslant 0$, then all representation groups of G are isomorphic.*

Proof. Let $1 \to Z \to H_1 \to G \to 1$ and $1 \to Z \to H_2 \to G \to 1$ be two representation groups of G. By Lemma (4.3) and Remark (4.4) we may assume that (up to isomorphism) both extensions are given by cocycles $\zeta_1, \zeta_2 \in Z^2(G, Z)$ such that the transgression maps $H^2(G, \mathbb{T}) = \widehat{Z} \to H^2(G, \mathbb{T})$ induced by ζ_1 and ζ_2 are the identity maps. Now let $\sigma = \zeta_1 \circ \zeta_2^{-1} \in Z^2(G, Z)$ and let $1 \to Z \to L \to G \to 1$ denote the extension defined by σ. We want to show that this extension splits (then $\sigma \in B^2(G, Z)$ and $[\zeta_1] = [\zeta_2] \in H^2(G, Z)$).

Since $\chi \circ \sigma = (\chi \circ \zeta_1) \cdot (\chi \circ \zeta_2^{-1})$ and $[\chi \circ \zeta_1] = [\chi \circ \zeta_2] \in H^2(G, \mathbb{T})$, it follows that the transgression map $\widehat{Z} \to H^2(G, \mathbb{T})$ induced by σ is trivial. But this implies that any character of Z can be extended to a character of L, which implies that $\widehat{L}_{\mathrm{ab}}$ is an extension $1 \to \widehat{G}_{\mathrm{ab}} \to \widehat{L}_{\mathrm{ab}} \to \widehat{Z} \to 1$. By assumption (using duality), this extension splits. Thus we find an injective homomorphism $\chi \mapsto \mu_\chi$ from $\widehat{Z} \to \widehat{L}_{\mathrm{ab}}$ such that each μ_χ is an extension of χ to L. Let $\widetilde{G} = \{s \in L : \mu_\chi(s) = 1 \text{ for all } \chi \in \widehat{Z}\}$. Then $\widetilde{G} \cap Z = \{e\}$ and $\widetilde{G} \cdot Z = L$. To see the latter, let $l \in L$ and let $z \in Z$ such that $\mu_\chi(l) = \chi(z)$ for all $\chi \in \widehat{Z}$. Then $lz^{-1} \in \widetilde{G}$. It follows that the quotient map $q : L \to G$ restricts to an isomorphism $\widetilde{G} \to G$. This proves all but the final statement.

By duality, it suffices to prove the final assertion only for G_{ab} isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m$. By induction, it suffices to consider only the cases $\mathbb{Z}$ and $\mathbb{R}$. Since the first is straightforward, we will show only that if G is abelian, then any continuous open surjection $q : G \to \mathbb{R}$ has a continuous section. By [20], Theorem 24.30, we may assume that $G = \mathbb{R}^m \times H$, where H has a compact, open subgroup. It follows that $q|\mathbb{R}^m$ is surjective. Thus there is an $x \in \mathbb{R}^m$ such that $q(x) = 1$. Then we can define $q^* : \mathbb{R} \to \mathbb{R}^m$ by $q(\lambda) = \lambda \cdot x$. ∎

5. THE BRAUER GROUP FOR TRIVIAL ACTIONS

In this section we want to give a precise description of the set $\mathcal{E}_G(X)$ of exterior equivalence classes of $C_0(X)$-linear actions $\beta : G \to \mathrm{Aut}\big(C_0(X,\mathcal{K})\big)$ in the case where G is smooth, G_{ab} is compactly generated, and X is a second countable locally compact Hausdorff space. This analysis also allows a description of the Brauer group $\mathrm{Br}_G(X)$ of [8] for a trivial G-space X, and, more generally, a description of the set of exterior equivalence classes of locally inner actions $\alpha : G \to \mathrm{Aut}(A)$ when $A \in \mathcal{CR}(X)$.

A C^*-dynamical system (A, G, α) is called a $C_0(X)$-*system* if A is a $C_0(X)$-algebra and each α_s is $C_0(X)$-linear. Two systems (A, G, α) and (B, G, β) are Morita equivalent if there is a pair (X, μ) consisting of an $A-B$-imprimitivity bimodule X and a strongly continuous action μ of G on X by linear transformations such that

$$\alpha_s\big({}_A\langle x, y\rangle\big) = {}_A\langle \mu_s(x), \mu_s(y)\rangle \quad \text{and} \quad \beta_s\big(\langle x, y\rangle_B\big) = \langle \mu_s(x), \mu_s(y)\rangle_B$$

for all $x, y \in \mathsf{X}$ and $s \in G$. The actions of A and B on X extend to the multiplier algebras $M(A)$ and $M(B)$. In particular, if A and B are $C_0(X)$-algebras, then X is both a left and a right $C_0(X)$-module. We say that two $C_0(X)$-systems (A, G, α) and (B, G, β) are $C_0(X)$-*Morita equivalent* if they are Morita equivalent via X and if $f \cdot x = x \cdot f$ for all $x \in \mathsf{X}$ and $f \in C_0(X)$. If we let G act trivially on X, then the equivariant Brauer group $\mathrm{Br}_G(X)$ of [8] is the collection of $C_0(X)$-Morita equivalence classes of $C_0(X)$-systems (A, G, α) where A is a separable continuous-trace C^*-algebra with spectrum X. Then $\mathrm{Br}_G(X)$ forms an abelian group ([8], Theorem 3.6). Recall that the group multiplication is defined using the balanced tensor product

$$[A, \alpha][B, \beta] = [A \otimes_X B, \alpha \otimes_X \beta]. \tag{5.1}$$

The identity is the class of $\big(C_0(X), \mathrm{id}\big)$ and the inverse of $[A, \alpha]$ is given by the class of the conjugate system $(\overline{A}, \overline{\alpha})$.

The collection of $[A, \alpha]$ in $\mathrm{Br}_G(X)$ such that the Dixmier-Douady class of A is zero is a subgroup. Note that each such element has a representative of the form $\big(C_0(X,\mathcal{K}), \alpha\big)$. That we can identify this subgroup with $\mathcal{E}_G(X)$ follows from the next proposition. In particular, $\mathcal{E}_G(X)$ is also an abelian group with multiplication given by (5.1) after identifying $C_0(X,\mathcal{K}) \otimes_X C_0(X,\mathcal{K})$ with $C_0(X)$ via a $C_0(X)$-isomorphism.

PROPOSITION 5.1. *Suppose that $\alpha, \gamma : G \to \mathrm{Aut}\big(C_0(X,\mathcal{K})\big)$ are $C_0(X)$-actions such that $[C_0(X,\mathcal{K}), \alpha] = [C_0(X,\mathcal{K}), \gamma]$ in $\mathrm{Br}_G(X)$. Then α and γ are exterior equivalent.*

Proof. Since $C_0(X,\mathcal{K})$ is stable, it follows from [8], Lemma 3.1 that γ is exterior equivalent to an action β of the form $\beta = \Phi \circ \alpha$ for some $C_0(X)$-linear automorphism Φ of $C_0(X,\mathcal{K})$. Thus it will suffice to see that β is exterior equivalent to α.

Since a $C_0(X)$-linear automorphism of $C_0(X,\mathcal{K})$ is locally inner ([40]), we may find an open cover $\{U_i\}_{i\in I}$ of X and continuous functions u^i from U_i to $U(\mathcal{H})$ for each $i \in I$ such that $\Phi(f)(x) = u_i(x)f(x)u_i(x)^*$ for all $f \in C_0(U_i,\mathcal{K})$. Moreover, on each overlap U_{ij}, there exist continuous functions $\chi_{ij} \in C(U_{ij},\mathbb{T})$

such that $u_j(x) = \chi_{ij}(x)u_i(x)$ for all $x \in U_{ij}$. Since β_s is a $C_0(X)$-automorphism, there are automorphisms β_s^x for each $x \in X$ such that $\beta_s(f)(x) = \beta_s^x\big(f(x)\big)$. Since $\alpha_s = \operatorname{Ad}\Phi \circ \beta_s$ it follows for all $x \in U_i$,

$$\alpha_s(f)(x) = u_i(x)^*\overline{\beta}_s^x\big(u_i(x)\big)\beta_s^x(f)(x)\overline{\beta}_s^x(u_i(x)^*)u_i(x),$$

where $\overline{\beta}_s^x$ is the canonical extension of β_s^x to $M\big(A(x)\big)$. If $x \in U_{ij}$, then

$$u_j(x)^*\overline{\beta}_s^x\big(u_j(x)\big) = \overline{\chi_{ij}(x)}u_i(x)^*\overline{\beta}_s^x\big(\chi_{ij}(x)u_i(x)\big) = u_i(x)^*\overline{\beta}_s^x\big(u_i(x)\big).$$

Consequently, we can define a map from G to $U(\mathcal{H})$ by $v_s(x) = u_i(x)^*\overline{\beta}_s^x\big(u_i(x)\big)$. Moreover, $s \mapsto v_s$ is strictly continuous, and we have $\alpha = \operatorname{Ad} v \circ \beta$. Thus we only need to verify that v is a 1-cocycle. For all $x \in X$ we get

$$\begin{aligned} v_{st}(x) &= u_i(x)^*\overline{\beta}_{st}^x\big(u_i(x)\big) = u_i(x)^*\overline{\beta}_s^x\big(\overline{\beta}_t^x\big(u_i(x)\big)\big) \\ &= u_i(x)^*\overline{\beta}_s^x\big(u_i(x)\big)\overline{\beta}_s^x\big(u_i(x)^*\overline{\beta}_t^x\big(u_i(x)\big)\big) = v_s(x)\overline{\beta}_s^x\big(v_t(x)\big), \end{aligned}$$

which implies $v_{st} = v_s\beta_s(v_t)$. ∎

REMARK 5.2. Suppose that $\gamma : G \to \operatorname{Aut}\big(C_0(X, \mathcal{K})\big)$ is a $C_0(X)$-automorphism group. In the sequel, we will write γ^o for a representative of the "inverse" automorphism group in $\mathcal{E}_G(X)$. That is, $[C_0(X, \mathcal{K}), \gamma]^{-1} := [C_0(X, \mathcal{K}), \gamma^o]$. Proposition 5.1 implies that γ^o is unique up to exterior equivalence and that $\gamma^o \otimes_X \gamma$ is exterior equivalent to $\operatorname{id} \otimes_X \operatorname{id}$.

The next lemma is a mild strengthening of [33], Lemma 3.3 to our setting.

LEMMA 5.3. *Suppose that $\beta : G \to \operatorname{Aut}\big(C_0(X, \mathcal{K})\big)$ is a $C_0(X)$-linear action and that $[\omega_x]$ is the Mackey obstruction for the induced automorphism group β^x on the fibre over x. Then the Mackey obstruction map $\phi^\beta : X \to H^2(G, \mathbb{T})$ given by $\phi^\beta(x) := [\omega_x]$ is continuous.*

Proof. Fix $x_0 \in X$ and suppose that $\{x_n\}$ is a sequence converging to x_0 in X. It will suffice to show that $[\omega_{x_n}]$ converges to $[\omega_{x_0}]$ in $H^2(G, \mathbb{T})$. Let M be the compact set $\{x_n\}_{n=1}^\infty \cup \{x_0\}$. Let β^M be the induced action on $C(M, \mathcal{K})$. Since β and β^M induce the same action on the fibres, we have $\phi^{\beta^M} = \phi^\beta|M$. Thus it will suffice to see that the former is continuous. But $H^2(M; \mathbb{Z})$ is trivial; any principal $\mathbb{T}$-bundle over M is locally trivial and therefore trivial. If follows from the Phillips-Raeburn exact sequence [40], Theorem 2.1 that β^M is inner. As in Remark 2.10, there is an obstruction to β^M being unitary given by a cocycle $\zeta \in Z^2\big(G, C(M, \mathbb{T})\big)$, and $\phi^{\beta^M}(x) = [\zeta(x)]$. Since for each $s, t \in G$, $\zeta(s, t)$ is continuous, it follows that $\zeta(x_n)$ converges to $\zeta(x)$ pointwise. Therefore $\zeta(x_n) \to \zeta(x)$ in $Z^2(G, \mathbb{T})$ ([29], Proposition 6). Since H^2 has the quotient topology, the result follows. ∎

Using the above lemma, the discussion in the introduction shows that there is a *homomorphism* Φ from $\mathcal{E}_G(X)$ to $C\big(X, H^2(G, \mathbb{T})\big)$ which assigns to each $[\alpha]$ in $\mathcal{E}_G(X)$ its "Mackey obstruction map" ϕ^α.

THEOREM 5.4. *Suppose that G is smooth. Then the homomorphism $\Phi : \mathcal{E}_G(X) \to C\bigl(X, H^2(G,\mathbb{T})\bigr)$ given by $[\beta] \mapsto \varphi^\beta$ is surjective and the short exact sequence*

$$1 \longrightarrow \ker\Phi \longrightarrow \mathcal{E}_G(X) \longrightarrow C\bigl(X, H^2(G,\mathbb{T})\bigr) \longrightarrow 1$$

splits. If, in addition, G_{ab} is compactly generated, then

$$\mathcal{E}_G(X) \cong H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}}) \oplus C\bigl(X, H^2(G,\mathbb{T})\bigr)$$

as abelian groups.

Proof. We have to construct a splitting homomorphism $\Phi^* : C\bigl(X, H^2(G,\mathbb{T})\bigr) \to \mathcal{E}_G(X)$ for Φ. Recall from [28], Theorem 5.1 and [21], Proposition 3.1 that there is a canonical homomorphism $\mu : H^2\bigl(G, C(X,\mathbb{T})\bigr) \to \mathcal{E}_G(X)$ defined as follows: Let $\sigma \in Z^2\bigl(G, C(X,\mathbb{T})\bigr)$, and let $L^{\sigma(x)}$ denote the left regular $\sigma(x)$-representation, where $\sigma(x)$ denotes evaluation of σ at $x \in X$. A representative for the class $\mu([\sigma]) \in \mathcal{E}_G(X)$ is then given by the action $\beta^\sigma : G \to \mathrm{Aut}\bigl(C_0(X, \mathcal{K}(L^2(G)))\bigr)$ defined by

$$\beta^\sigma_s(f)(x) = \mathrm{Ad}\, L_s^{\sigma(x)}\bigl(f(x)\bigr), \quad f \in C_0(X, \mathcal{K}(L^2(G))).$$

(If G is finite we have to stabilize this action in order to get an action on $C_0\bigl(X, \mathcal{K}\bigl(L^2(G)\bigr)\bigr) \otimes \mathcal{K} \cong C_0(X,\mathcal{K})$.)

By Lemma 4.3 we know that $H^2(G,\mathbb{T})$ is locally compact and that there exists an element $\zeta \in Z^2(G, H^2(G,\mathbb{T})^\wedge)$ such that evaluation of ζ at a point $[\omega] \in H^2(G,\mathbb{T})$ is a cocycle representing ω. If we give $C(H^2(G,\mathbb{T}),\mathbb{T})$ the compact-open topology, then we can view $H^2(G,\mathbb{T})^\wedge$ as a subset of $C(H^2(G,\mathbb{T}),\mathbb{T})$. Furthermore, if $\varphi \in C\bigl(X, H^2(G,\mathbb{T})\bigr)$, then $\zeta\circ\varphi(s,t)(x) := \zeta(s,t)\bigl(\varphi(x)\bigr)$ defines a Borel cocycle $\zeta\circ\varphi \in Z^2\bigl(G, C(X,\mathbb{T})\bigr)$.

We claim that $\Phi^*(\varphi) := \mu([\zeta\circ\varphi])$ defines a splitting homomorphism for Φ. To see that it is a homomorphism just notice that if $\varphi,\psi \in C\bigl(X, H^2(G,\mathbb{T})\bigr)$, then since $\zeta(s,t) \in H^2(G,\mathbb{T})^\wedge$, $\zeta(s,t)\bigl(\varphi(x)\bigr)\zeta(s,t)\bigl(\psi(x)\bigr) = \zeta(s,t)\bigl(\varphi(x)\psi(x)\bigr)$. Thus $\varphi \mapsto [\zeta\circ\varphi]$ is a homomorphism of $C\bigl(X, H^2(G,\mathbb{T})\bigr)$ into $H^2\bigl(G, C(X,\mathbb{T})\bigr)$. By the construction of μ we can choose a representative β for $\mu([\zeta\circ\varphi])$ such that β^x is implemented by a $\zeta\bigl(\varphi(x)\bigr)$-representation $V : G \to U(\mathcal{H})$. Since $\zeta\bigl(\varphi(x)\bigr)$ is a representative for $\varphi(x)$, it follows that $\Phi\circ\Phi^* = \mathrm{id}$.

We have shown that if G is smooth, then

$$1 \longrightarrow \ker\Phi \longrightarrow \mathcal{E}_G(X) \longrightarrow C\bigl(X, H^2(G,\mathbb{T})\bigr) \longrightarrow 1$$

is a split short exact sequence. If, in addition, G_{ab} is compactly generated, then we know from Corollary 3.9 that the Phillips-Raeburn obstruction $\beta \mapsto \zeta(\beta) \in H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$ of Proposition 3.3 defines a bijection of $\ker\Phi$ onto $H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$, which by Lemma 3.5 is multiplicative. This completes the proof. ∎

Let $F : \mathrm{Br}_G(X) \to H^3(X,\mathbb{Z})$ denote the forgetful homomorphism described in the introduction. It admits a natural splitting map, which assigns to an element $\delta \in H^3(X,\mathbb{Z})$ the (equivalence class of the) system $(A_\delta, G, \mathrm{id})$, where A_δ is the unique stable continuous-trace C^*-algebra with Dixmier-Douady invariant δ and id denotes the trivial action of G on A_δ. Since $\ker F$ is naturally isomorphic to $\mathcal{E}_G(X)$ by Proposition 5.1, we obtain the following as an immediate corollary.

COROLLARY 5.5. *Suppose that G is smooth and that G_{ab} is compactly generated. Then, for any trivial G-space X, we have a group isomorphism*

$$\mathrm{Br}_G(X) \cong H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}}) \oplus C\big(X, H^2(G, \mathbb{T})\big) \oplus H^3(X; \mathbb{Z}),$$

where $H^3(X;\mathbb{Z})$ denotes the third integral Čech cohomology.

We conclude this section with a discussion of some special cases.

EXAMPLE 5.6. If G is connected and $H^2(X,\mathbb{Z})$ is countable then it follows from [8] Section 6.3 that the homomorphism $\mu : H^2\big(G, C(X,\mathbb{T})\big) \to \mathcal{E}_G(X)$ described in the proof of Theorem 5.4 is actually an isomorphism (in particular, all $C_0(X)$-actions are inner). Under this isomorphism, the Mackey obstruction map $\Phi : \mathcal{E}_G(X) \to C\big(X, H^2(G,\mathbb{T})\big)$ corresponds to the evaluation map $H^2\big(G, C(X,\mathbb{T})\big) \to C\big(X, H^2(G,\mathbb{T})\big)$ and the kernel of Φ corresponds to the subgroup $H^2_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big)$ of pointwise trivial elements in $H^2\big(G, C(X,\mathbb{T})\big)$.

If G is not connected, there are usually lots of $C_0(X)$-linear actions of G on $C_0(X,\mathcal{K})$ which are not inner, for instance, if $G = \mathbb{Z}^n$, then $H^2\big(\mathbb{Z}^n, C(X,\mathbb{T})\big) \cong C\big(X, H^2(\mathbb{Z}^n,\mathbb{T})\big)$ by [34], Corollary 1.5, but $H^1(X, \widehat{\mathbb{Z}}^n)$ is often nontrivial (for instance for $G = \mathbb{Z}$ and $X = S^2$). In any case, if G is smooth and if ζ is as in Lemma 4.3, then the map $\varphi \mapsto \zeta \circ \varphi$ from $C\big(X, H^2(G,\mathbb{T})\big)$ to $H^2\big(G, C(X,\mathbb{T})\big)$ is a splitting homorphism for the exact sequence

$$1 \longrightarrow H^2_{\mathrm{pt}}\big(G, C(X,\mathbb{T})\big) \longrightarrow H^2\big(G, C(X,\mathbb{T})\big) \longrightarrow C\big(X, H^2(G,\mathbb{T})\big) \longrightarrow 1.$$

EXAMPLE 5.7. If G is smooth and G_{ab} is a vector group (i.e., G_{ab} is isomorphic to some $\mathbb{R}^l$ for $l \geqslant 0$) then $H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}}) = 0$ and $\mathcal{E}_G(X) \cong C\big(X, H^2(G,\mathbb{T})\big)$. This applies to all simply connected and connected Lie groups.

EXAMPLE 5.8. If G is any second countable locally compact group such that $H^2(G,\mathbb{T})$ is trivial (e.g., if $G = \mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, or any connected and simply connected semisimple Lie group), then G serves as a representation group for itself. Hence, if G_{ab} is also compactly generated, then $\mathcal{E}_G(X) \cong H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}})$. If, in addition, G_{ab} is a vector group, then it follows from Example 5.7 that $\mathcal{E}_G(X)$ is trivial.

EXAMPLE 5.9. It follows from the previous example, that if G is any connected and simply connected Lie group with $H^2(G,\mathbb{T})$ trivial, then $\mathcal{E}_G(X)$ is trivial. Since $H^2\big(G, C(X,\mathbb{T})\big)$ imbeds injectively into $\mathcal{E}_G(X)$ by [8], Section 6.3, it follows that for such groups $H^2\big(G, C(X,\mathbb{T})\big)$ is trivial for all X. For compact X this was shown in [19], Theorem 2.6.

6. LOCALLY INNER ACTIONS ON CR-ALGEBRAS

In this section we want to use our description of $\mathcal{E}_G(X)$ to describe locally inner actions on elements of $\mathcal{CR}(X)$. The next lemma provides an analogue for the Mackey-obstruction map $\beta \mapsto \varphi^\beta \in C\big(X, H^2(G,\mathbb{T})\big)$ of Theorem 5.4 in case of locally inner actions on general elements of $\mathcal{CR}(X)$.

Lemma 6.1. *Let G be a second countable locally compact group, $A \in \mathcal{CR}(X)$, and let $\alpha : G \to \mathrm{Aut}(A)$ be locally inner. For each $x \in X$ let U be an open neighborhood of x such that the restriction $\alpha^U : G \to \mathrm{Aut}(A_U)$ of α is inner, and let $[\sigma] \in H^2\big(G, C(U,\mathbb{T})\big)$ be the obstruction for α^U being unitary. Then $\varphi^\alpha(x) := [\sigma(x)]$ determines a well defined continuous map $\varphi^\alpha : X \to H^2(G,\mathbb{T})$. If β is exterior equivalent to α, then $\varphi^\alpha = \varphi^\beta$.*

Proof. We have to show that if U_1 and U_2 are two open neighborhoods of x such that, for $i = 1,2$, there exist cocycles $\sigma^i \in Z^2\big(G, C(U_i,\mathbb{T})\big)$ and σ^i-homomorphism $V^i : G \to \mathcal{U}M(A_{U_i})$ which implement $\alpha^i := \alpha^{U_i}$, then $[\sigma^1(x)] = [\sigma^2(x)]$ in $H^2(G,\mathbb{T})$. Let $U_{ij} := U_1 \cap U_2$ and let V^{ij}_s denote the image of V^i_s in $\mathcal{U}M(A_{U_{ij}})$, $i,j \in \{1,2\}$, and let σ^{ij} denote the restriction of σ^i to U_{ij}; that is, $\sigma^{ij}(s,t)(x) = \sigma^i(s,t)(x)$ for all $x \in U_1 \cap U_2$. Then V^{ij} is a σ^{ij}-homomorphism which implements the restriction $\alpha^{U_{ij}} : G \to \mathrm{Aut}(A_{U_{ij}})$. Thus it follows that $[\sigma^{12}] = [\sigma^{21}] \in H^2\big(G, C(U_{ij},\mathbb{T})\big)$, which in particular implies that $[\sigma^1(x)] = [\sigma^2(x)]$. Thus φ^α is well defined. The continuity of φ^α follows from the continuity of the evaluation map $x \mapsto [\sigma(x)]$ on U as shown in the proof of Lemma 5.3.

Finally, suppose that $\beta = \mathrm{Ad}\,(w) \circ \alpha$ for some 1-cocycle w. Since φ^α is defined locally, we can assume that $\alpha = \mathrm{Ad}\,(V)$ for some σ-homomorphism V. Then $\beta = \mathrm{Ad}\,(wV)$, and it is easily checked that wV is a σ-homomorphism. ∎

Notice that if $\beta : G \to \mathrm{Aut}\big(C_0(X,\mathcal{K})\big)$ is a $C_0(X)$-linear action, then the element φ^β constructed above is the same as the map φ^β which appeared in Theorem 5.4. Notice also that $\varphi^\alpha = 0$ if and only if all of the induced actions α^x of G on the fibres $A(x)$ of A are unitary.

Proposition 6.2. *Let G be a smooth group such that G_{ab} is compactly generated, and let $A \in \mathcal{CR}(X)$. Let $\Phi^* : C\big(X, H^2(G,\mathbb{T})\big) \to \mathcal{E}_G(X)$ denote the splitting homomorphism for the short exact sequence*

$$1 \longrightarrow H^1(X, \widehat{\mathcal{G}}_{\mathrm{ab}}) \longrightarrow \mathcal{E}_G(X) \longrightarrow C\big(X, H^2(G,\mathbb{T})\big) \longrightarrow 1$$

as constructed in the proof of Theorem 5.4. If $\alpha : G \to \mathrm{Aut}(A)$ is locally inner and $[\gamma] = \Phi^*(\varphi^\alpha)$, then $\alpha \otimes_X \gamma^o$ is locally unitary.

Proof. By the construction of Φ^* we know that there exists a Borel cocycle $\sigma \in H^2\big(G, C(X,\mathbb{T})\big)$ and a σ-homomorphism $W : G \to \mathcal{U}M\big(C_0(X,\mathcal{K})\big)$ such that $\gamma^o = \mathrm{Ad}\,W$ and such that $[\sigma(x)] = \varphi^\alpha(x)^{-1}$ in $H^2(G,\mathbb{T})$ for all $x \in X$. Since α is locally inner, we know further that for each $x \in X$ there exists a neighborhood U of x, a cocycle $\omega \in Z^2\big(G, C(U,\mathbb{T})\big)$, and a ω-homomorphism $V : G \to \mathcal{U}M(A_U)$ such that $\alpha^U = \mathrm{Ad}\,V$. Then $\varphi^\alpha(x) = [\omega(x)]$ for all $x \in U$ by Lemma 6.1. It follows that the restriction $(\alpha \otimes_X \gamma^o)^U$ of $\alpha \otimes_X \gamma^o$ to $\big(A \otimes_X C_0(X,\mathcal{K})\big)_U \cong A_U \otimes_{C_0(U)} C_0(U,\mathcal{K})$

is implemented by the $\omega\cdot\sigma$-homomorphism $s\mapsto V_s\otimes_U W_s$. Since $[\omega\cdot\sigma(x)]=0$ for all $x\in U$, it follows now from [45], Theorem 2.1 that there exists a neighborhood $U_1\subseteq U$ of x such that the restriction of $\omega\cdot\sigma$ to U_1 is trivial in $H^2\big(G,C(U_1,\mathbb{T})\big)$. But this implies that the restriction of $\alpha\otimes_X\gamma^o$ to $\big(A\otimes_X C_0(X,\mathcal{K})\big)_{U_1}$ is unitary. This completes the proof. ∎

Theorem 6.3. *Let G be a smooth group such that $G_{\rm ab}$ is compactly generated. Suppose that $A\in\mathcal{CR}(X)$, and that $\alpha:G\to\mathrm{Aut}(A)$ is locally inner. Then there exists a $C_0(X)$-linear action $\beta^\alpha:G\to\mathrm{Aut}\big(C_0(X,\mathcal{K})\big)$, unique up to exterior equivalence, such that the stabilized action $\alpha\otimes_X\mathrm{id}$ on $A\otimes_X C_0(X,\mathcal{K})$ ($\cong A\otimes\mathcal{K}$) is exterior equivalent to the diagonal action $\mathrm{id}\otimes_X\beta^\alpha$ of G on $A\otimes_X C_0(X,\mathcal{K})$.*

Moreover, if $\mathcal{LI}_G(A)$ denotes the set of exterior equivalence classes of locally inner G-actions on A, then $\alpha\mapsto\beta^\alpha$ factors through a well defined injective map $[\alpha]\mapsto[\beta^\alpha]$ of $\mathcal{LI}_G(A)$ into $\mathcal{E}_G(X)$, which is a bijection if A is stable.

Proof. Let $\alpha:G\to\mathrm{Aut}(A)$ be locally inner, let $\Phi^*:C\big(X,H^2(G,\mathbb{T})\big)\to\mathcal{E}_G(X)$ denote the splitting homomorphism of Theorem 5.4, and let $[\gamma]=\Phi^*(\varphi^\alpha)$. Then, by Proposition 6.2, $\alpha\otimes_X\gamma^o$ is a locally unitary action of G on $A\otimes\mathcal{K}$.

Let $\zeta(\alpha\otimes_X\gamma^o)\in H^1(X,\widehat{\mathcal{G}}_{\rm ab})$ denote the Phillips-Raeburn obstruction (see Proposition 3.3). If $\delta:G\to\mathrm{Aut}\big(C_0(X,\mathcal{K})\big)$ is locally unitary with $\zeta(\delta)=\zeta(\alpha\otimes_X\gamma^o)$, then it also follows from Proposition 3.3 that $\alpha\otimes_X\gamma^o$ is exterior equivalent to the diagonal action $\mathrm{id}\otimes_X\delta$ of G on $A\otimes_X C_0(X,\mathcal{K})$. Since taking diagonal actions on balanced tensor products preserves exterior equivalence by Lemma 3.5, and since $\gamma^o\otimes_X\gamma$ is exterior equivalent to the trivial action $\mathrm{id}\otimes_X\mathrm{id}$ (Remark 5.2), it follows that for $\beta'=\delta\otimes_X\gamma$ on $C_0(X,\mathcal{K})\otimes_X C_0(X,\mathcal{K})$,

$$\alpha\otimes_X\mathrm{id}\otimes_X\mathrm{id}\sim\alpha\otimes_X(\gamma^o\otimes_X\gamma)\sim(\alpha\otimes_X\gamma^o)\otimes_X\gamma\sim(\mathrm{id}\otimes_X\delta)\otimes_X\gamma\sim\mathrm{id}\otimes_X\beta',$$

where $\sim$ denotes exterior equivalence. Since $C_0(X,\mathcal{K})\otimes_X C_0(X,\mathcal{K})\cong C_0(X,\mathcal{K})$, it follows that $\alpha\otimes_X\mathrm{id}\sim\mathrm{id}\otimes_X\beta$ for some β on $C_0(X,\mathcal{K})$.

We have to show that β is unique up to exterior equivalence. For this observe that $\alpha\otimes_X\mathrm{id}\sim\mathrm{id}\otimes_X\beta$ implies that $\varphi^\alpha=\varphi^{\alpha\otimes_X\mathrm{id}}=\varphi^{\mathrm{id}\otimes_X\beta}=\varphi^\beta$ (Lemma 6.1). Thus, if β' is another $C_0(X)$-action of G on $C_0(X,\mathcal{K})$ such that $\alpha\otimes_X\mathrm{id}\sim\mathrm{id}\otimes_X\beta'$, then it follows that $\varphi^{\beta'}=\varphi^\alpha=\varphi^\beta$. Thus if $[\gamma]=\Phi^*(\varphi^\alpha)$ is as above, then

$$\mathrm{id}\otimes_X(\beta'\otimes_X\gamma^o)\sim\alpha\otimes_X\mathrm{id}\otimes_X\gamma^o\sim\mathrm{id}\otimes_X(\beta\otimes_X\gamma^o),$$

which implies that the Phillips-Raeburn obstructions $\zeta(\beta'\otimes_X\gamma^o)$, and $\zeta(\beta\otimes_X\gamma^o)$ coincide (Lemma 3.5). But then it follows from Proposition 3.3 that $\beta'\otimes_X\gamma^o\sim\beta\otimes_X\gamma^o$ which, via multiplication with γ, implies that $\beta\sim\beta'$.

It follows that there is a well defined map $[\alpha]\mapsto[\beta^\alpha]$ from $\mathcal{LI}_G(A)$ into $\mathcal{E}_G(A)$ which is determined by the property that $\alpha\otimes_X\mathrm{id}\sim\mathrm{id}\otimes_X\beta^\alpha$. Since $\beta\sim\beta'$ implies that $\mathrm{id}\otimes_X\beta\sim\mathrm{id}\otimes_X\beta'$, this map is injective. Finally, if A is stable, then we can define an inverse by choosing a fixed $C_0(X)$-isomorphism $\Theta:A\otimes\mathcal{K}\to A$ ([41], Lemma 4.3), and defining $[\beta]\to[\mathrm{Ad}\,\Theta\circ(\mathrm{id}\otimes_X\beta)]$ of $\mathcal{E}_G(X)$ onto $\mathcal{LI}_G(A)$. ∎

We immediately get the following corollary.

Corollary 6.4. *Let G be a smooth group such that $G_{\rm ab}$ is compactly generated. Let X be a second countable locally compact space and let $A\in\mathcal{CR}(X)$. Then $\alpha:G\to\mathrm{Aut}(A)$ is locally inner if and only if there exists $[\beta]\in\mathcal{E}_G(X)$ such that the stabilized action $\alpha\otimes_X\mathrm{id}$ is exterior equivalent to $\mathrm{id}\otimes_X\beta$.*

7. ROSENBERG'S THEOREM

One of the important ingredients for the proof of our results was Rosenberg's theorem (see Theorem 3.8) which implies that if

(a) G_{ab} is compactly generated and if
(b) $H^2(G, \mathbb{T})$ is Hausdorff,

then any pointwise unitary action on a separable continuous-trace C^*-algebra A is automatically locally unitary.

Our interest in smooth groups is partially explained by the fact that all smooth groups with G_{ab} compactly generated satisfy these assumptions. We give examples below which show that neither of conditions (a) and (b) can be weakened in general. On the other hand, if we assume that A has continuous trace with locally connected spectrum, then the class of groups with the property that pointwise unitary actions on A are automatically locally unitary is significantly larger than the class of groups which satisfy the conditions of Rosenberg's theorem (Theorem 7.4).

Example 7.1. Suppose that G is a second countable locally compact abelian group acting freely and properly on a separable locally compact space X such that X is *not* a locally trivial principal G-bundle. Although Palais's Slice Theorem ([38], Theorem 4.1) implies that G cannot be a Lie group, we can, for example, take $G = \prod_{n=1}^{\infty} \{1, -1\}$, $X = \prod_{n=1}^{\infty} \mathbb{T}$, and let G act by translation on X. Let α denote the corresponding action of G on $C_0(X)$ and let $A = C_0(X) \rtimes_\alpha G$. Then A has continuous trace by [18], Theorem 17, and the dual action $\widehat{\alpha}$ of $\widehat{G}$ is pointwise unitary ([33], Proof of Theorem 3.1). If $\widehat{\alpha}$ would be locally unitary, then $\big((C_0(X) \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}\big)^{\wedge} \cong X$ would be a locally trivial principal G-bundle with respect to the double dual action $\widehat{\widehat{\alpha}}$ ([41]). But by the Takesaki-Takai duality theorem, this implies that X is a locally trivial principal G bundle with respect to the original action; this contradicts our original assumption.

Since $\widehat{G}$ is discrete, $\widehat{G}$ is smooth and therefore $H^2(\widehat{G}, \mathbb{T})$ is Hausdorff. Of course, $\widehat{G}$ is not compactly generated as required in Rosenberg's theorem.

We are now going to construct in Example 7.3 a pointwise unitary action of a compactly generated group G on a continuous-trace algebra A which is not locally unitary. Of course, $H^2(G, \mathbb{T})$ will fail to be Hausdorff. Before we start, recall that if N is a closed normal subgroup of a second countable locally compact group G, then the inflation-restriction sequence

$$H^1(N, \mathbb{T})^G \xrightarrow{\mathrm{tg}} H^2(G/N, \mathbb{T}) \xrightarrow{\mathrm{inf}} H^2(G, \mathbb{T})$$

is exact at $H^2(G/N, \mathbb{T})$, where $H^1(N, \mathbb{T})^G$ denotes the group of G-invariant characters of N and inf denotes the inflation map ([27], p. 53).

EXAMPLE 7.2. (cf. [28], p. 85) Our group G will be a central extension of $\mathbb{T}^2$ by $\mathbb{R}^2$. For each $\lambda \in \mathbb{R}$ let ω_λ denote the two-cocycle

$$\omega_\lambda\big((s_1,t_1),(s_2,t_2)\big) = \mathrm{e}^{\mathrm{i}\lambda s_1 t_2}$$

on $\mathbb{R}^2$. Since the real Heisenberg group is a representation group for $\mathbb{R}^2$ (Example 4.7), $\lambda \mapsto [\omega_\lambda]$ is an isomorphism between $\mathbb{R}$ and $H^2(\mathbb{R}^2,\mathbb{T})$. Let θ be any irrational number and let ω_1 and ω_θ denote the cocycles in $Z^2(\mathbb{R}^2,\mathbb{T})$ corresponding to 1 and θ, respectively. Let $G_1 = \mathbb{R}^2 \times_{\omega_1} \mathbb{T}$ be the central extension of $\mathbb{R}^2$ by $\mathbb{T}$ corresponding to ω_1, and let $G = (\mathbb{R}^2 \times_{\omega_1} \mathbb{T}) \times_{\omega_\theta} \mathbb{T}$ denote the central extension of G_1 corresponding to the inflation of ω_θ to G_1. Then G is a central extension of $\mathbb{R}^2$ by $\mathbb{T}^2$, and is therefore a connected two-step nilpotent group of dimension four. Since the cocycles involved are continuous, G is homeomorphic to the direct product $\mathbb{R}^2 \times \mathbb{T}^2$ with multiplication given by

$$(s_1,t_1,z_1,w_1)(s_2,t_2,z_2,w_2) = (s_1+s_2, t_1+t_2, \mathrm{e}^{\mathrm{i}s_1t_2}z_1z_2, \mathrm{e}^{\mathrm{i}\theta s_1 t_2}w_1w_2).$$

There is a natural continuous section c from $\mathbb{R}^2 \cong G/\mathbb{T}^2$ onto G given by $c(s_1,s_2) := (s_1,s_2,1,1)$. Using the formula for the transgression map

$$\mathrm{tg} : \mathbb{Z}^2 \cong H^1(\mathbb{T}^2,\mathbb{T}) \to H^2(\mathbb{R}^2,\mathbb{T}),$$

a straightforward computation shows that $\mathrm{tg}(l,m) = [\omega_{l+\theta m}]$. Since $\mathbb{Z}+\theta\mathbb{Z}$ is dense in $\mathbb{R}$ and inf is continuous, the identity is not closed in $H^2(G,\mathbb{T})$; in other words, $H^2(G,\mathbb{T})$ is not Hausdorff.

EXAMPLE 7.3. We shall construct a pointwise unitary action of the group G from the previous example which is not locally unitary. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. We define $\alpha : G \to \mathrm{Aut}\big(C(X,\mathcal{K})\big)$ as follows. Since $\mathbb{Z}+\theta\mathbb{Z}$ is dense in $\mathbb{R}$, we find a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}+\theta\mathbb{Z}$ such that $\lambda_n \to 0$ in $\mathbb{R}$ while $\lambda_n \neq \lambda_m \neq 0$ for all $n,m \in \mathbb{N}$, with $n \neq m$. Putting $\lambda_0 = 0$ and $\lambda_{1/n} := \lambda_n$ we obtain a continuous map $x \mapsto \lambda_x$ of X to $\mathbb{Z}+\theta\mathbb{Z} \subseteq \mathbb{R}$. For each $x \in X$ let $\dot{V}_x : \mathbb{R}^2 \to U(L^2(\mathbb{R}^2))$ denote the regular ω_{λ_x}-representation of $\mathbb{R}^2$, which is given by the formula

$$(\dot{V}_x(s,t)\xi)(s',t') = \mathrm{e}^{\mathrm{i}\lambda_x s(t'-t)}\xi(s'-s,t'-t),$$

and let $V_x : G \to U\big(L^2(\mathbb{R}^2)\big)$ denote the inflation of $\dot{V}_x$ to G. Since $x \mapsto \lambda_x$ is continuous, it follows that we obtain a strongly continuous action $\alpha : G \to \mathrm{Aut}\big(C(X,\mathcal{K})\big)$ with $\mathcal{K} = \mathcal{K}\big(L^2(\mathbb{R}^2)\big)$ given by defining

$$\alpha_g(a)(x) = V_x(g)a(x)V_x(g)^*.$$

Since V_x is an $\mathrm{inf}(\omega_{\lambda_x})$-representation for each $x \in X$ and α is implemented pointwise by the representations V_x, and since each $[\omega_{\lambda_x}]$ lies in the range of the transgression map, it follows that α is pointwise unitary.

We claim that α is not locally unitary. Since X has only one accumulation point, α is locally unitary if and only if it is unitary. So assume that there were a strictly continuous homomorphism $U : G \to U\big(C(X,\mathcal{K})\big)$ which implements α. Thus, for each $x \in X$ and $g \in G$ we would obtain

$$U_x(g)a(x)U_x(g)^* = V_x(g)a(x)V_x(g)^*,$$

from which it follows that

$$V_x^*(g)U_x(g) = \gamma_x(g)1$$

for some $\gamma_x(g) \in \mathbb{T}$. Since, by construction, the maps $(x,g) \to V_x(g)$ and $(x,g) \to U_x(g)$ are strongly continuous, $(x,g) \to \gamma_x(g)$ defines a continuous map $\gamma : X \times G \to \mathbb{T}$. Moreover, since $V_x|\mathbb{T}^2 \equiv 1$, it follows that $\chi_x = \gamma_x|\mathbb{T}^2$ is a character of $\mathbb{T}^2$ for all $x \in X$. By continuity we have $\chi_{\frac{1}{n}} \to \chi_0$ in $\widehat{\mathbb{T}}^2$. Moreover, since V_0 is a unitary representation, it follows that V_0 and U_0 are both unitary representations which implement α at the point 0. But this implies that γ_0 is a character of G. Thus, multiplying each U_x with $\overline{\gamma}_0$, we may assume that $U_0 = V_0$. In particular, this implies that χ_0 is the trivial character of $\mathbb{T}^2$.

We finally show that $\chi_{\frac{1}{n}}$ is not trivial for all $n \in \mathbb{N}$. Since $\widehat{\mathbb{T}}^2 \cong \mathbb{Z}^2$ is discrete, this will contradict the fact that $\chi_{\frac{1}{n}} \to \chi_0$. Assume that $\chi_{\frac{1}{n}}$ is trivial for some $n \in \mathbb{N}$. Then $U_{\frac{1}{n}}|\mathbb{T}^2 \equiv 1$, from which it follows that $U_{\frac{1}{n}}$ is actually inflated from some unitary representation $\dot{U}_{\frac{1}{n}} : \mathbb{R}^2 \to U\big(L^2(\mathbb{R}^2)\big)$. Since, by construction, $V_{\frac{1}{n}}$ is inflated from the regular ω_{λ_n}-representation, say $\dot{V}_{\frac{1}{n}}$ of $\mathbb{R}^2$, it follows that $\dot{U}_{\frac{1}{n}}$ and $\dot{V}_{\frac{1}{n}}$ implement the same action of $\mathbb{R}^2$ on $\mathcal{K}\big(L^2(\mathbb{R}^2)\big)$, which contradicts the fact that $[\omega_{\lambda_n}]$, the Mackey-obstruction for the action implemented by $V_{\frac{1}{n}}$, is non-trivial in $H^2(\mathbb{R}^2, \mathbb{T})$.

Note that the space X in the above example is totally disconnected; in particular, the point 0 has no connected neighborhoods in X. The following theorem shows that this lack of connectedness plays a crucial rôle in our counterexample.

THEOREM 7.4. *Suppose that N is a closed normal subgroup of a second countable locally compact group G such that:*
(i) *G/N is compactly generated,*
(ii) *$H^2(G/N, \mathbb{T})$ is Hausdorff, and*
(iii) *$H^2(N, \mathbb{T})$ is Hausdorff and $N_{\mathrm{ab}} := N/\overline{[N,N]}$ is compact.*

Suppose further that A is a separable continuous-trace C^*-algebra such that $\widehat{A}$ is locally connected. Then any pointwise unitary action of G on A is automatically locally unitary.

Proof. Let α be a pointwise unitary action of G on A. Since the properties of being unitary and locally unitary are preserved under Morita equivalence of systems ([11], Proposition 3), we can replace (A, G, α) with $(A \otimes \mathcal{K}, G, \alpha \otimes \mathrm{id})$ and assume that A is stable. Clearly, $(A, N, \alpha|N)$ is pointwise unitary, so by Rosenberg's theorem, it is locally unitary. Since G must act trivially on $\widehat{A}$, we can replace A by an ideal and assume that $A = C_0(X, \mathcal{K})$ and that $\alpha|N = \mathrm{Ad}\,(u)$ for a strictly continuous homomorphism $u : N \to \mathcal{U}M\big(C_0(X, \mathcal{K})\big)$.

Localizing further if necessary, we claim that u is a Green twisting map; that is, $\alpha_s(u_n) = u_{sns^{-1}}$ for all $s \in G$ and $n \in N$. Since $M\big(C_0(X, \mathcal{K})\big)$ can be identified with the bounded strictly continuous functions from X to $B(H)$ ([1], Corollary 3.4), and since the strict topology on $U(\mathcal{H})$ coincides with the strong topology, it is not hard to see that we may view u as a strongly continuous function from $X \times N$ to $U(\mathcal{H})$ such that for all $n \in N$ and $x \in X$, we have $\alpha_n(a)(x) = u(x,n)a(x)u(x,n)^*$.

In order to show that u defines a Green twisting map for α, we need to show that $\alpha_s\bigl(u(\,\cdot\,,n)\bigr) = u(\,\cdot\,, sns^{-1})$ for all $s \in G$ and $n \in N$. However by assumption, for each $x \in X$ there is a unitary representation $V_x : G \to B(\mathcal{H})$ such that $\alpha_s(a)(x) = V_x(s)a(x)V_x(s)^*$ for all $a \in A$ and $s \in G$. Since both V_x and $u(x,\cdot)$ implement the same automorphism of $\mathcal{K}$, there is a character γ_x of N_{ab} such that $u(x,n) = \gamma_x(n)V_x(n)$ for all $n \in N$. Now if we abuse notation slightly and write $\alpha_s(u)(x,n)$ for $\alpha_s\bigl(u(\,\cdot\,,n)\bigr)(x)$, then

$$\begin{aligned}\alpha_s(u)(x,n) &= V_x(s)u(x,n)V_x(s)^* = \gamma_x(n)V_x(s)V_x(n)V_x(s^{-1}) \\ &= \gamma_x(n)V_x(sns^{-1}) = \gamma_x(n)\overline{\gamma_x(sns^{-1})}u(x,sns^{-1}).\end{aligned}$$

For each $x \in X$, $sN \in G/N$, and $n \in N$, define $\lambda(x,sN)(n) = \gamma_x(n)\overline{\gamma_x(sns^{-1})} = \gamma_x(n)\overline{s\cdot\gamma_x(n)}$, where $s\cdot\gamma := \gamma(s\cdot s^{-1})$. Clearly, u will be a twisting map exactly when we can arrange for λ to be identically one. (This invariant was also studied in [44], Section 5. The definition was slightly different there — partly to make equations such as (7.1) more attractive than we require here.) Since $A = C_0(X,\mathcal{K})$, it is not hard to see that the map

$$(x,sN,n) \mapsto u(x,sns^{-1})\alpha_s(u)(x,n) = \lambda(x,sN)(n)1_A$$

is continuous. Consequently, we can view λ as a continuous function from $X\times G/N$ into $\widehat{N}_{\mathrm{ab}}$. Notice that

$$\lambda(x,stN) = t\cdot\lambda(x,sN)\lambda(x,tN). \tag{7.1}$$

Fix $x_0 \in X$. Since we may pass to still another ideal of A, it will suffice to produce a neighborhood U of x_0 in X such that $\lambda(x,sN)(n) = 1$ for all $sN \in G$, $n \in N$, and $x \in U$. Of course, replacing $u(x,n)$ by $\overline{\gamma_{x_0}(n)}u(x,n)$, we may assume that $\lambda(x_0,sN)(n) = 1$ for all $s \in G$ and $n \in N$. In fact, since $\widehat{N}_{\mathrm{ab}}$ is discrete, given any $t \in G$, there is a neighborhood $U_t \times V_t \subseteq X \times G$ of (x_0,t) such that $\lambda(x,sN) = 1$ provided $(x,sN) \in U_t \times V_t$. In view of (7.1), $\lambda(x,sN) = 1$ for all $x \in U_t$ and sN in the subgroup of G/N generated by V_t. By condition (i), there is a compact set K which generates G/N. We can choose $t_1,\dots,t_n$ and neighborhoods $(U_{t_1}\times V_{t_1}),\dots,(U_{t_n}\times V_{t_n})$ such that $K \subseteq \bigcup_i V_{t_i}$. Then we can let $U = \bigcap_{i=1}^{n} U_{t_i}$. Then $\lambda(x,sN)(n) = 1$ for all $x \in U$, $s \in G$, and $n \in N$. Then after passing to the ideal of A corresponding to U, we can indeed assume that u is a Green twisting map.

Now let ρ_{x_0} be the element of $\widehat{A}$ corresponding to the point x_0 as chosen above. Then by the above constructions, there exists a covariant representation (ρ_{x_0},V_0) of (A,G,α) such that $V_0|N = \rho_{x_0}\circ u$, which just means that (ρ_{x_0},V_0) preserves the twist u in the sense of Green. Since A is stable, it follows from [11], Corollary 1 that (A,G,α,u) is exterior equivalent to $(A,G,\beta,1)$, for some action β of G on A. Then β is inflated from an action $\dot\beta$ of G/N (see Remark 1 on page 176 of [11]). We are now going to show that $\dot\beta$ is also pointwise unitary. Since $\widehat{A}$ is locally connected we may localize further in order to assume that $\widehat{A}$ is connected, and we may also assume that $\widehat{A}$ is compact. Let (ρ_{x_0},U_0) denote the representation of (A,G,β) corresponding to (ρ_{x_0},V_0) via the exterior equivalence between

(A, G, α, u) and $(A, G, \beta, 1)$. Then (ρ_{x_0}, U_0) preserves 1, since (ρ_{x_0}, V_0) preserves u. Thus it follows that U_0 is inflated from a representation $\dot{U}_0$ of G/N. Now, by assumption, β is pointwise unitary, which implies that $\dot{\beta}$ induces the trivial action of G/N on $\widehat{A}$. For each $\rho \in \widehat{A}$ let $[\omega_\rho] \in H^2(G/N, \mathbb{T})$ denote the Mackey obstruction to extend ρ to a covariant representation of $(A, G/N, \dot{\beta})$. Then $[\omega_{\rho_{x_0}}] = 0$ and the map $\rho \mapsto [\omega_\rho]$ is continuous by [33], Lemma 3.3 (or Lemma 5.3 above). Since $\widehat{A}$ is connected, it follows that its image, say M, is a compact and connected subset of $H^2(G/N, \mathbb{T})$. But since β is pointwise unitary, it follows that M lies in the kernel of the inflation map $\mathrm{inf} : H^2(G/N, \mathbb{T}) \to H^2(G, \mathbb{T})$, and hence in the image of the transgression map $\mathrm{tg} : H^1(N, \mathbb{T})^G \to H^2(G/N, \mathbb{T})$. By assumption, N_{ab} is compact, so $H^1(N, \mathbb{T})^G$ is discrete and countable (by the separability assumptions). Thus M is a countable and connected compact Hausdorff space, which implies that M consists of a single point. (For example, Baire's Theorem implies that a countable compact Hausdorff space has a clopen point.) But since $[\omega_{\rho_{x_0}}]$ is trivial, it follows that $[\omega_\rho]$ is trivial for all $\rho \in \widehat{A}$; in other words, $\dot{\beta}$ is pointwise unitary.

Now we can apply Rosenberg's theorem to the system $(A, G/N, \dot{\beta})$, from which follows that $\dot{\beta}$ is locally unitary. But this implies that β, and hence also α is locally unitary. ∎

We now recall that G is a $[\mathrm{FD}]^-$ *group* if $\overline{[G, G]}$ is compact and $G/\overline{[G, G]}$ is abelian. These groups are of particular interest since every type I $[\mathrm{FD}]^-$group has a continuous-trace group C^*-algebra ([13], Lemma 6) and there are no known examples of groups with continuous-trace C^*-algebra which are not $[\mathrm{FD}]^-$groups.

COROLLARY 7.5. *Suppose that G is a separable compactly generated* [FD]-*group, or that G is a connected nilpotent Lie group. Then every pointwise unitary action of G on a separable continuous-trace algebra with locally connected spectrum is locally unitary.*

Proof. If G is a separable compactly generated $[\mathrm{FD}]^-$group, then the theorem applies with the normal subgroup $N = \overline{[G, G]}$. So assume that G is a connected nilpotent Lie group. Then there exists a maximal torus T in the center of G such that G/T is simply connected (Any connected Lie group is a quotient of a simply connected Lie group by some central discrete group, thus the center of a connected nilpotent group is of the form $\mathbb{R}^l \times \mathbb{T}^m$ and the quotient of G by $\mathbb{T}^m$ is a simply connected nilpotent group), and hence $H^2(G/T, \mathbb{T})$ is Hausdorff, since G/T is smooth. ∎

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