Math. Ann. 302, 269–294 (1995)

Mathematische Annalen

Crossed products whose primitive ideal spaces are generalized trivial $\hat{G}$-bundles

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Received: 15 August 1993

Mathematics Subject Classification (1991): 46L05, 46L55

1 Introduction

Our object here is to investigate the structure of the crossed product $A \rtimes_\alpha G$ of a C*-algebra A by a locally compact abelian group G. One goal of any such program is to characterize the ideal structure via a description of the primitive ideal space together with its Jacobson topology. While it is theoretically possible to describe $\mathrm{Prim}(A \rtimes_\alpha G)$ as a set using the Mackey-machine as developed by Green, Gootman, and Rosenberg [15, 17], characterizing the topology is quite another matter – even when $A \cong C_0(\Omega, \mathcal{K})$ – due to the appearance of non-trivial Mackey obstructions while employing the Mackey-machine. In the event that all the Mackey obstructions vanish, considerable progress has been made in the case of constant stabilizers [23, 25, 21, 27, 28, 29, 30], and similar, but more modest, progress has been made when the stabilizers vary continuously [26]. When it is not assumed that the Mackey obstructions vanish, the situation is very murky. Although progress has been made in general [2], and especially in the case of constant stabilizer and constant (non-vanishing) Mackey obstruction [18, 6], the examples in [18, 6] illustrate the difficulties and subtleties inherent even in this restricted approach.

One of the reasons that the approach in the constant stabilizer (and constant Mackey obstruction) case has been so successful, is that one can employ the machinery of classical algebraic topology – especially the theory of principal bundles. In the general case, the "bundles" that will arise will not have isomorphic fibers, let alone be locally trivial or principal. However in [26], Iain Raeburn and the second author introduced the notion of locally σ-trivial G-spaces which we believe appropriately generalize the notion of principal

This research was supported by a grant from the University of Paderborn
The second author was partially supported by the National Science Foundation

bundles. To elaborate, recall that a locally compact G-space Ω is called σ-proper if the stabilizer map $x \mapsto S_x$ is continuous and if the map $(x,t) \mapsto (x, t{\cdot}x)$ defines a proper map of the quotient $\Omega \times G/\!\sim$ (where $(x,s) \sim (y,t)$ if and only if $x = y$ and $ts^{-1} \in S_x$) into $\Omega \times \Omega$ [26, Definition 4.1]. It is clear that σ-proper spaces are the natural extensions of free and proper G-spaces to non-freely acting transformation groups. This is evidenced by the rôle they and free and proper spaces play in the investigation of crossed products with continuous trace [16, 33, 21, 26, 4, 5, 1] Also recall that a G-space Ω is called σ-trivial if Ω/G is Hausdorff and if it is G-homeomorphic to $\Omega/G \times G/\!\sim$ (where $(G{\cdot}x, s) \sim (G{\cdot}y, t)$ if and only if $G{\cdot}x = G{\cdot}y$ and $ts^{-1} \in S_x$). Of course, Ω is called locally σ-trivial if Ω is the union of G-invariant open σ-trivial subsets [26, Definition 4.2]. The connection between σ-trivial spaces and σ-properness exactly parallels that between free and proper spaces and principal bundles: a σ-proper G-space is (locally) σ-trivial if and only if there are (local) cross sections for the orbit map [26, Proposition 4.3].

A significant class of examples of σ-trivial spaces are provided by the spectrums of transformation group C^*-algebras. It follows from [34] and [32, Theorem 5.3] that if $(C_0(\Omega) \rtimes_\tau G)^\wedge$ is Hausdorff, then it is a σ-trivial $\hat{G}$-space for the dual action. It was shown in [26], that when A is non-abelian, one can have $(A \rtimes_\alpha G)^\wedge$ locally, but not globally, σ-trivial. One of the original motivations for this paper was to classify those crossed products with σ-trivial primitive ideal space (with respect to the dual action). Unlike [26], we make *no* assumption on the action, the algebra A, or the Mackey obstructions.

The crucial concept is that of a (generalized) Green twisting map. Recall that if (A, G, α) is an abelian dynamical system, then a Green twisting map τ with domain N is a strictly continuous homomorphism $\tau : N \to \mathscr{U}(A)$ satisfying $\alpha_n(a) = \tau_n a \tau_n^*$ and $\alpha_s(\tau_n) = \tau_n$ for all $a \in A$, $n \in N$, and $s \in G$. If τ is a Green twisting map for (A, G, α), then Olesen and Pedersen [20, Theorem 2.4] have shown that $A \rtimes_\alpha G$ is covariantly isomorphic to an induced algebra $\operatorname{Ind}_{N^\perp}^{G}(A \rtimes_{\alpha,\tau} G)$, where $A \rtimes_{\alpha,\tau} G$ is Green's twisted crossed product (see [17, Sect. 1] for the relevant definitions and background). In fact, if $\hat{\alpha}$ denotes the dual $\hat{G}$-action on $A \rtimes_\alpha G$ and $N = \tilde{\Gamma}(\alpha)^\perp$, where $\tilde{\Gamma}(\alpha) = \{\gamma \in \hat{G} : \hat{\alpha}_\gamma(P) = P \text{ for all } P \in \operatorname{Prim}(A \rtimes_\alpha G)\}$, then $\tilde{\Gamma}(\alpha)$ is a subgroup of the usual Connes spectrum $\Gamma(\alpha)$ [22] and $A \rtimes_{\alpha,\tau} G$ is simple if and only if A is G-simple and $N = \{e\}$ is the trivial subgroup [20, Theorem 5.7]. In particular, $\operatorname{Prim}(A \rtimes_\alpha G)$ is $\hat{G}$-homeomorphic to $\hat{N}$ and is the simplest sort of σ-trivial $\hat{G}$-space.

To generalize these ideas, recall that if $R : \operatorname{Prim}(A) \to \Omega$ is a continuous, open G-invariant surjection onto a locally compact Hausdorff space Ω, then $A \rtimes_\alpha G$ is the section algebra of a C^*-bundle with fibers $A_x \rtimes_{\alpha^x} G$, where $A_x = A/\ker\big(R^{-1}(\{x\})\big)$ and α^x is the induced action [35]. Then, if $x \mapsto N_x$ is continuous from Ω into the normal subgroups of G, a generalized Green twisting map $\mathfrak{t}$ with domain $\Omega^N = \{(x,s) \in \Omega \times G : s \in N_x\}$ is a "compatible" choice of Green twisting maps $\mathfrak{t}^x$ for each quotient system (A_x, G, α^x) (see Definition 2.7 below). Naturally, there is an associated twisted crossed product $A \rtimes_{\alpha,\mathfrak{t}} G$ which behaves as if it were fibered over Ω with fibers $A_x \rtimes_{\alpha^x,\mathfrak{t}^x} G$ (see

Definition 2.9 below). Our main result (Theorem 4.1) is that $\mathrm{Prim}(A \rtimes_\alpha G)$ is a σ-trivial space if and only if (1) the quasi-orbit space $\mathcal{Q}^G(\mathrm{Prim}(A))$ is Hausdorff, (2) the map $x \mapsto C_x = \tilde{\mathrm{r}}(\alpha^x)^\perp$ is continuous, and (3) there is a generalized Green twisting map for (A, G, α) with domain Ω^C. In this event, $C_x^\perp = \Gamma(\alpha^x)$, the usual Connes spectrum. If A is type I and (A, G, α) is regular, then we can replace C with the symmetrizer map Σ (Corollary 4.3). Of course, as a special case of our results we obtain conditions (Corollary 4.7), which characterize when the primitive ideal space of $A \rtimes_\alpha G$ is a trivial $\hat{C}$-bundle for some closed subgroup C of G, which generalize those given in [27, Theorem 7.2].

The majority of the work required to prove Theorem 4.1 involves an extensive generalization of a theorem of Olesen and Pedersen [20, Theorem 2.4]. Namely, we show in Theorem 3.1 that there is a generalized Green twisting map t for (A, G, α) with domain Ω^N if and only if the dual system is covariantly isomorphic to a (generalized) induced system $\left(\mathrm{Ind}_{\Omega^{N^\perp}}^{\hat{G}} B, \hat{G}, \mathrm{Ind}\,\beta\right)$ as defined in [5, Sect. 3] (or at the end of Sect. 2 below). In fact, in complete analogy with [20, Theorem 2.4], we can take $B = A \rtimes_{\alpha,\mathrm{t}} G$.

Our work is organized as follows. In Sect. 2, we recall some of the basic definitions of subgroup actions and subgroup crossed products from [5]. We also give formal definitions of generalized Green twisting maps and the resulting twisted crossed products. In Sect. 3 we prove our first main result characterizing the crossed product of a twisted system as an induced algebra. Finally, in Sect. 4 we prove our main result characterizing crossed products with σ-trivial primitive ideal space. We close with a number of interesting special cases.

2 Subgroup actions and generalized Green twisting maps

For any locally compact group G we denote by $\mathfrak{K}(G)$ the space of all closed subgroups of G equipped with Fell's topology [7]. Then $\mathfrak{K}(G)$ is a compact Hausdorff space. If Ω is a locally compact space (here, and in the sequel, locally compact always means locally compact Hausdorff) and $H : \Omega \to \mathfrak{K}(G); x \mapsto H_x$ is any continuous map, then we define

$$\Omega^H = \{(x,s) \in \Omega \times G : s \in H_x\},$$

which is a closed subspace of $\Omega \times G$ and which may be thought as a trivial bundle over Ω with varying fibers H_x.

Suppose now that A is a C^*-algebra such that there exists a continuous, open and surjective map $P : \hat{A} \to \Omega$ for some locally compact space Ω. Then by Lee's theorem [19] (and [10, Proposition 1.6]) we may write A as the section algebra $\Gamma_0(E)$ of all continuous sections which vanish at infinity for some C^*-bundle $P : E \to \Omega$, such that the fibers A_x are isomorphic to $A/\ker P^{-1}(\{x\})$ for each $x \in \Omega$. Conversely, it was also shown by Lee that for any C^*-algebra A which is isomorphic to $\Gamma_0(E)$ for some C^*-bundle $p : E \to \Omega$ there exists a canonical continuous and open projection from $\hat{A}$ onto Ω, which is just P in the situation above. We now repeat the definition of a subgroup action as defined in [5].

Definition 2.1. *Suppose that A is a C*-algebra such that A is isomorphic to the section algebra $\Gamma_0(E)$ for some C*-bundle $p : E \to \Omega$. Assume further that $H : \Omega \to \mathfrak{K}(G); x \mapsto H_x$ is a continuous map such that for each $x \in \Omega$ there exists a strongly continuous action α^x of H_x on the fiber A_x such that the map*

$$\Omega^H \to E; (x,s) \mapsto \alpha^x_s\big(a(x)\big)$$

is continuous for each $a \in A$. Then $\alpha = (\alpha^x)_{x\in\Omega}$ is called a **subgroup action** *of Ω^H on A.*

If α is a subgroup action of Ω^H on A, then we may follow [5] and form the *subgroup crossed product* $A \rtimes_\alpha \Omega^H$ as follows: Let $q : \Omega^H \to \Omega$ denote the canonical projection, and let $\Gamma_c(q^*E)$ denote the set of continuous sections with compact support of the pull-back bundle q^*E. Note that the continuous sections of q^*E are exactly given by the continuous functions $f : \Omega^H \to E$ such that $f(x,s) \in A_x$ for all $x \in \Omega$. We define convolution, involution and norm on $\Gamma_c(q^*E)$ by the rules

$$f * g(x,s) = \int_{H_x} f(x,t)\alpha^x_t\big(g(x,t^{-1}s)\big)\, d_{H_x}t\,,$$

$$f^*(x,s) = \Delta_{H_x}(s^{-1})\alpha^x_s(f(x,s^{-1})^*)\,,$$

and

$$\|f\|_1 = \sup_{x\in\Omega} \int_{H_x} \|f(x,s)\|\, d_{H_x}s\,,$$

where Δ_{H_x} denotes the modular function on H_x and $(d_H)_{H\in\mathfrak{K}(G)}$ is a smooth choice of Haar measures on $\mathfrak{K}(G)$ [11, p. 908]. If $L^1(\Omega^H,A,\alpha)$ denotes the completion of $\Gamma_c(q^*E)$ with respect to $\|\cdot\|_1$, then the subgroup crossed product $A \rtimes_\alpha \Omega^H$ is defined as the enveloping C*-algebra of $L^1(\Omega^H,A,\alpha)$.

It has been shown in [5] that $(A \rtimes_\alpha \Omega^H)^\wedge$ can be identified with the set of pairs

$$\{(x,\rho\times V) : x \in \Omega, \rho\times V \in (A_x \rtimes_{\alpha^x} H_x)^\wedge\}$$

by defining

$$(x,\rho\times V)(f) = \rho\times V\big(f(x,\cdot)\big).$$

More generally, every representation of a fiber $A_x \rtimes_{\alpha^x} H_x$ defines a representation of $A \rtimes_\alpha \Omega^H$ in this way and the collection $\mathcal{S}(A \rtimes_\alpha \Omega^H)$ of all equivalence classes of those representation is called the space of *subgroup representations of* $A \rtimes_\alpha \Omega^H$. If we restrict to representations with dimension bounded by a fixed cardinal, say $\aleph$, then we may topologize $\mathcal{S}(A \rtimes_\alpha \Omega^H)$ by viewing $\mathcal{S}(A \rtimes_\alpha \Omega^H)$ as a subset of the space $\mathrm{Rep}(A \rtimes_\alpha \Omega^H)$ of all equivalence classes of representations of $A \rtimes_\alpha \Omega^H$ with dimension bounded by $\aleph$ equipped with Fell's inner hull kernel topology (see [8, 9]). Note that the canonical projection of $\mathcal{S}(A \rtimes_\alpha \Omega^H)$ onto Ω is always continuous. Moreover, it has been shown in [5, Corollary 4] that in case where H_x is amenable for all $x \in \Omega$, the restriction of this projection to $(A \rtimes_\alpha \Omega^H)^\wedge$ is also open, so that by Lee's theorem we may write $A \rtimes_\alpha \Omega^H$ as a section algebra $\Gamma_0(F)$ of a section bundle $r : F \longrightarrow \Omega$ with fibers $A_x \rtimes_{\alpha^x} H_x$.

The definition of a subgroup action on a C*-algebra as above was motivated by the desire to be able to restrict an action of G on a C*-algebra A to a set of continuously varying subgroups of G. Before we make this more precise we first want to introduce the notion of a regularization of a dynamical system.

Definition 2.2. *Let (A, G, α) be a dynamical system. A **regularization** (Ω, R) of (A, G, α) consists of a locally compact G-space Ω together with a G-equivariant continuous map $R : \mathrm{Prim}(A) \longrightarrow \Omega$. If, in addition, R is open and surjective, then (Ω, R) is called an **open regularization** of (A, G, α). If the action of G on Ω is trivial, then (Ω, R) is called a **G-invariant regularization**.*

Remark 2.3. Since Ω is Hausdorff, any continuous map from $\hat{A}$ into Ω must factor through $\mathrm{Prim}(A)$. Thus whenever convenient we can, and will, think of the regularizing maps R as being defined on $\hat{A}$.

Note that the definition of a regularization as given above is much weaker than the definition of regularizations as defined in [5]. Please see [5, Sect. 1] for situations where both definitions automatically coincide. We now come to the motivating example for subgroup actions.

Example 2.4. Suppose that (Ω, R) is an open regularization of (A, G, α). Assume that $H : \Omega \longrightarrow \mathfrak{K}(G)$ is a continuous map such that $H_x \subseteq S_x$ for all $x \in \Omega$, where S_x denotes the stabilizer of x. Then we may define for each $x \in \Omega$ an action α^x of H_x on the fiber $A_x = A/\ker R^{-1}(\{x\})$ of the corresponding C*-bundle $r : E \longrightarrow \Omega$ with $A \cong \Gamma_0(E)$, by first restricting α to H_x, and then taking the canonical action of H_x on the quotient of A by the H_x-invariant ideal $\ker R^{-1}(\{x\})$. Then $(\alpha^x)_{x \in \Omega}$ is a subgroup action of Ω^H on A. We will call this action the *restriction of α to Ω^H* and will denote it usually also by the letter α. This will cause no confusion as it will be clear from context whether α refers to an action of G on A, or the restriction of an action to Ω^H as defined above.

The construction of the subgroup action in the example above gives rise to many different constructions of subgroup algebras. One very important case is the following.

Example 2.5. Suppose that (A, G, α) is a dynamical system, Ω is a locally compact space and $H : \Omega \longrightarrow \mathfrak{K}(G)$ is a continuous map. Let $\mathrm{id} \otimes \alpha$ denote the diagonal action of G on $C_0(\Omega, A)$ with respect to the trivial action of G on Ω and let $R : C_0(\Omega, A)^\wedge \longrightarrow \Omega$ denote the canonical projection. Then (Ω, R) becomes an open regularization for $(C_0(\Omega, A), G, \mathrm{id} \otimes \alpha)$ and as in Example 2.4 we may form the subgroup action $\mathrm{id} \otimes \alpha$ of Ω^H on $C_0(\Omega, A)$ and the subgroup crossed product $C_0(\Omega, A) \rtimes_{\mathrm{id} \otimes \alpha} \Omega^H$, which we will denote in the following simply by $C^*(\Omega^H, A, \alpha)$.

Note that we could have constructed $C^*(\Omega^H, A, \alpha)$ easily by defining convolution and involution on $C_c(\Omega^H, A)$ by the usual operations on the fibers $C_c(H_x, A)$ with respect to the restriction of α to H_x. The space $\mathcal{S}(\Omega^H, A, \alpha)$ of subgroup representations of $C^*(\Omega^H, A, \alpha)$ consists of all pairs $(x, \rho \times V)$ with $x \in \Omega$ and $\rho \times V \in \mathrm{Rep}(A \rtimes_\alpha H_x)$. In the case where Ω is equal to $\mathfrak{K}(G)$ and H equals the identity $I : \mathfrak{K}(G) \to \mathfrak{K}(G)$, we obtain the subgroup algebra $C^*(\mathfrak{K}(G)^I, A, \alpha)$ whose subgroup representations give the collection of *all* subgroup representations for the system (A, G, α).

The following proposition shows that there is a very strong connection between the constructions in Examples 2.4 and 2.5.

Proposition 2.6. *Suppose that (Ω, R) is an open regularization of (A, G, α) and assume that $H : \Omega \to \mathfrak{K}(G)$ is a continuous map such that H_x is contained in the stabilizer S_x for all $x \in \Omega$. Let $A \rtimes_\alpha \Omega^H$ be the subgroup crossed product of A by Ω^H with respect to the restriction of α to Ω^H, and let $C^*(\Omega^H, A, \alpha)$ denote the subgroup algebra as constructed in Example 2.5. Let $\mathcal{S}_R^H$ denote the set of all irreducible representations of $C^*(\Omega^H, A, \alpha)$ which are given by pairs $(x, \rho \times V)$ such that $\rho \times V \in (A \rtimes_\alpha H_x)^\wedge$ satisfies $\ker \rho \supseteq \ker R^{-1}(\{x\})$ (i.e. $\rho \in \mathrm{Rep}(A_x)$). Then $C^*(\Omega^H, A, \alpha)/\ker \mathcal{S}_R^H$ is canonically isomorphic to $A \rtimes_\alpha \Omega^H$.*

Proof. Let $r : E \to \Omega$ be the C^*-bundle given by $R : \hat{A} \to \Omega$ and Lee's theorem, and let as usual q^*E denote the pull-back of E via the canonical projection $q : \Omega^H \to \Omega$. Then define

$$\Phi : C_c(\Omega^H, A) \to \Gamma_c(q^*E);\, \Phi(f)(x, s) = f(x, s)(x).$$

Then it is easily seen that Φ extends to a surjective $*$-homomorphism from $C^*(\Omega^H, A, \alpha)$ onto $A \rtimes_\alpha \Omega^H$ with kernel $\ker \mathcal{S}_R^H$ (compare with [5, Proposition 9]). □

Recall that a Green twisting map τ for a dynamical system (A, G, α) is a strictly continuous homomorphism $\tau : N_\tau \to \mathcal{U}(A)$ from a closed normal subgroup N_τ of G into the group of unitaries $\mathcal{U}(A)$ in the multiplier algebra $\mathcal{M}(A)$ of A such that the conditions

$$\alpha_n(a) = \tau_n a \tau_n^* \quad \text{and} \quad \alpha_s(\tau_n) = \tau_{sns^{-1}}$$

are satisfied for all $n \in N_\tau$, $a \in A$ and $s \in G$. The twisted crossed product $A \rtimes_{\alpha,\tau} G$ is defined as the quotient of $A \rtimes_\alpha G$ by the intersection of all kernels of representations $\pi \times U$ of $A \rtimes_\alpha G$ which preserve τ in the sense that

$$\pi(\tau_n) = U_n \quad \text{for all } n \in N_\tau .$$

Our first aim here is to generalize the notion of Green twisting maps to something which has as the domain a set of varying subgroups of G, rather than just one fixed normal subgroup N_τ of G. For this note first that if (Ω, R) is a G-invariant open regularization of a system (A, G, α), then $\ker R^{-1}(\{x\})$

is a G-invariant ideal in A for each $x \in \Omega$ and we have a canonical action α^x of G on each fiber $A_x = A/\ker R^{-1}(\{x\})$.

Definition 2.7. *Assume that (Ω, R) is a G-invariant open regularization of the dynamical system (A, G, α), $r : E \to \Omega$ the corresponding C*-bundle with fibers $A_x = A/\ker R^{-1}(\{x\})$, and let $N : \Omega \to \mathfrak{K}(G)$ be a continuous map such that N_x is normal in G for all $x \in \Omega$. A* ***generalized Green twisting map*** *with domain Ω^N for (A, G, α) is a map $\mathfrak{t} : \Omega^N \to \bigcup_{x\in\Omega} \mathcal{U}(A_x)$ such that*

(1) Each map $\mathfrak{t}^x : N_x \to \mathcal{U}(A_x)$; $\mathfrak{t}^x_n = \mathfrak{t}(x,n)$ is a Green twisting map for the system (A_x, G, α^x).

(2) The maps $\Omega^N \to E$ given by $(x,n) \mapsto \mathfrak{t}^x_n a(x)$ and $(x,n) \mapsto a(x)\mathfrak{t}^x_n$ are continuous for each $a \in A = \Gamma_0(E)$.

If $\mathfrak{t}$ is a generalized Green twisting map for (A, G, α), then we call $(A, G, \alpha, \mathfrak{t})$ a ***generalized twisted dynamical system.***

Remark 2.8. Note that if we do not require that $\alpha^x_s(\mathfrak{t}^x_n) = \mathfrak{t}^x_n$ for each $x \in \Omega$, $s \in G$ and $n \in N_x$ – that is, each $\mathfrak{t}^x$ is merely a strictly continuous homomorphism into $\mathcal{U}(A_x)$ implementing α^x – then we simply have a *unitary* subgroup action of Ω^N on A as defined in [26, Definition 5.1].

As in the case of ordinary twisted dynamical systems, there is a canonical procedure for defining twisted crossed products of generalized twisted dynamical systems. The easiest definition is the following.

Definition 2.9. *Suppose that $\mathfrak{t}$ is a generalized Green twisting map for (A, G, α) with domain Ω^N. We say that a representation $\pi \times U$ of (A, G, α)* ***preserves*** *$\mathfrak{t}$ if $\pi(\mathfrak{t}^x_s) = U_s$ for all $(x,s) \in \Omega^N$ such that $\ker\pi \supseteq \ker R^{-1}(\{x\})$. Let $I_\mathfrak{t}$ be the intersection of all kernels of representations of $A \rtimes_\alpha G$ which preserve $\mathfrak{t}$. Then we define the* ***generalized twisted crossed product*** *$A \rtimes_{\alpha,\mathfrak{t}} G$ as the quotient of $A \rtimes_\alpha G$ by $I_\mathfrak{t}$.*

The following observation will be useful in considering the structure of generalized twisted crossed products. The proof is straightforward.

Lemma 2.10. *Suppose that (Ω, R) is a G-invariant open regularization of (A, G, α). We identify G with the constant map $G : \Omega \to \mathfrak{K}(G); x \mapsto G$. Let $\alpha = (\alpha^x)_{x\in\Omega}$ be the subgroup action of Ω^G on A as in Example 2.4. Then*

$$\Psi : C_c(G,A) \to \Gamma_c(q^*E);\ \Psi(f)(x,s) = f(s)(x)$$

extends to an isomorphism between $A \rtimes_\alpha G$ and $A \rtimes_\alpha \Omega^G$.

Note that by the identification of $A \rtimes_\alpha G$ with $A \rtimes_\alpha \Omega^G$ via Lemma 2.10, we know that for every irreducible representation $\pi \times U$ of $A \rtimes_\alpha G$ there exists exactly one $x \in \Omega$ such that $\ker\pi \supseteq \ker R^{-1}(\{x\})$. Hence it follows that the collection of irreducible representations of $A \rtimes_\alpha G$ which preserve $\mathfrak{t}$ may be viewed as the union over all $x \in \Omega$ of the spaces $(A_x \rtimes_{\alpha^x,\mathfrak{t}^x} G)^\wedge$, where $A_x \rtimes_{\alpha^x,\mathfrak{t}^x} G$ denotes the usual twisted crossed product with respect to the ordinary twisted action $(\alpha^x, \mathfrak{t}^x)$ of G on A_x. It is straightforward to see that $I_\mathfrak{t}$ is equal to the intersection of all kernels of the irreducible representations which preserve t. We will later see that these are exactly the irreducible representations of $A \rtimes_{\alpha,\mathfrak{t}} G$.

As for ordinary twisted crossed products it is quite useful to have another realization of the generalized twisted crossed product. For this we have to define first what we will understand as a quotient of G by Ω^N.

Definition 2.11. *Assume that G is a locally compact group, Ω a locally compact space and $H : \Omega \to \mathfrak{K}(G)$ is a continuous map. Then we define an equivalence relation $\sim_H$ on $\Omega \times G$ by*

$$(x,s) \sim_H (y,t) \text{ if and only if } x = y \text{ and } s \in tH_x\,.$$

The quotient space $\Omega \times_H G = (\Omega \times G)/\sim_H$ under this equivalence relation is called the ***quotient of*** *G by Ω^H.*

Note that the elements of $\Omega \times_H G$ as in the definition above are just given by the pairs $(x, \dot{s})$, where $\dot{s}$ denotes the left coset sH_x of $s \in G$.

Assume now that $N : \Omega \to \mathfrak{K}(G)$ is a continuous map such that N_x is normal in G for all $x \in \Omega$. Then we may define (left) Haar measures on G/N_x by normalizing with respect to a fixed Haar measure on G and a smooth choice of Haar measures on $\mathfrak{K}(G)$ such that

$$\int_G g(s)\,ds = \int_{G/N_x}\int_{N_x} g(sn)\,d_{N_x}n\,d_{G/N_x}\dot{s}\,,$$

for any $g \in C_c(G)$. (While a similar family of measures is defined in [33, Lemma 2.20], the reader should be cautioned that in the case the subgroups S_x are normal the measures μ_x defined in [33] are *right* Haar measures (see [33, Lemma 2.21]).) We record some properties that these measures enjoy for later use.

Lemma 2.12. *Suppose we have chosen Haar measures on the quotient groups G/N_x as above.*

(1) *Let Λ be a locally compact space and suppose $F \in C_c(\Lambda, (\Omega \times_N G))$. Then*

$$f(\lambda, x) = \int_{G/N_x} F(\lambda, x, \dot{s})\,d_{G/N_x}\dot{s}$$

defines an element of $C_c(\Lambda \times \Omega)$.

(2) *For each $f \in C_c(\Omega \times_N G)$,*

$$\tilde{f}(x) = \int_{G/N_x} f(x, \dot{s})\,d_{G/N_x}\dot{s}$$

defines the element of $C_c(\Omega)$.

(3) *The map $(x, \dot{s}) \mapsto \Delta_{G/N_x}(\dot{s})$ is continuous on $\Omega \times_N G$.*

Proof. The first assertion can be proved along the same lines as [33, Lemmas 2.5(iv) and 2.22]. The second and third assertions follow from the first: (1)

follows immediately and (2) follows by considering $F(\dot r, x, \dot s) = f(x, \dot r\dot s)$ for appropriate choices of f in $C_c(\Omega \times_N G)$. $\square$

If $\mathfrak{t}$ is a generalized twist for (A, G, α) with domain Ω^N as in Definition 2.7, then we define $C_c(G, A, \mathfrak{t})$ as the set of all continuous A-valued functions f on G which satisfy

$$f(ns)(x) = f(s)(x)\mathfrak{t}^x_{n^{-1}}$$

for all $s \in G$ and $(x, n) \in \Omega^N$ such that the resulting function

$$(x, \dot s) \mapsto \|f(s)(x)\|$$

has compact support in $\Omega \times_N G$. Then, as usual, we define multiplication, involution and norm on $C_c(G, A, \mathfrak{t})$ by

$$f * g(s)(x) = \int_{G/N_x} f(t)(x)\alpha^x_t\big(g(t^{-1}s)(x)\big)\, d_{G/N_x}t\,, \tag{2.1}$$

$$f^*(s)(x) = \Delta_{G/N_x}(s^{-1})\alpha^x_s(f(s^{-1})(x)^*)\,, \tag{2.2}$$

and

$$\|f\|_1 = \sup_{x\in\Omega} \int_{G/N_x} \|f(t)(x)\|\, d_{G/N_x}t\,.$$

Note that if we define f^x by $f^x(s) = f(s)(x)$ for all $x \in \Omega$ and $f \in C_c(G, A, \mathfrak{t})$, then f^x is an element of the dense subspace $C_c(G, A_x, \mathfrak{t}^x)$ of the ordinary twisted crossed product $A_x \rtimes_{\alpha^x, \mathfrak{t}^x} G$ as constructed by Green in [17, Sect. 1]. Moreover, if we define $\|\cdot\|^x_1$ on $C_c(G, A_x, \mathfrak{t}^x)$ by

$$\|g\|^x_1 = \int_{G/N_x} \|g(s)\|\, d_{G/N_x}\dot s\,,$$

then it was shown also in [17, Sect. 1] that the twisted crossed product $A_x \rtimes_{\alpha^x, \mathfrak{t}^x} G$ is just the enveloping C*-algebra of the completion $L^1(G, A_x, \mathfrak{t}^x)$ of $C_c(G, A, \mathfrak{t})$ with respect to $\|\cdot\|^x_1$. Now it follows directly from Lemma 2.12 that for each $f \in C_c(G, A, \mathfrak{t})$ the map $\Omega \to \mathbb{R}; x \mapsto \|f^x\|^x_1$ is continuous with compact support. Thus we see that the completion $L^1(G, A, \mathfrak{t})$ of $C_c(G, A, \tau)$ with respect to $\|\cdot\|_1$ is isomorphic to $\Gamma_0(F)$, for some Banach bundle $p : F \to \Omega$ with fibers $B_x = L^1(G, A_x, \mathfrak{t}^x)$. Observe also that multiplication and involution on $C_c(G, A, \mathfrak{t})$ are just defined fiberwise by the multiplication and involution on the fibers $L^1(G, A_x, \mathfrak{t}^x)$. We use this observation to prove

Proposition 2.13. *Multiplication and involution on $C_c(G, A, \mathfrak{t})$, as defined in Equations* (2.1) *and* (2.2), *are well defined and extend to all of $L^1(G, A, \mathfrak{t})$. Thus $L^1(G, A, \mathfrak{t})$ is the section algebra $\Gamma_0(F)$ of a Banach $*$-algebra bundle $p : F \to \Omega$ with fibers $L^1(G, A_x, \mathfrak{t}^x)$. Moreover, the map*

$$\Phi : C_c(G, A) \to C_c(G, A, \tau);\ \Phi(f)(s)(x) = \int_{N_x} f(sn)(x)\mathfrak{t}^x_{sns^{-1}}\, d_{N_x}n$$

extends to a $*$-homomorphism from $A \rtimes_\alpha G$ onto the enveloping C-algebra C*$(G, A, \mathfrak{t})$ of $L^1(G, A, \mathfrak{t})$ with kernel $I_{\mathfrak{t}}$. Thus C*$(G, A, \mathfrak{t})$ is canonically isomorphic to $A \rtimes_{\alpha,\mathfrak{t}} G$. Moreover, the irreducible representations of $A \rtimes_{\alpha,\mathfrak{t}} G$ are exactly the irreducible representations of $A \rtimes_\alpha G$ which preserve* $\mathfrak{t}$.

Proof. That (2.1) and (2.2) define elements of $C_c(G,A,\mathfrak{t})$ is clear except for possibly the continuity of $f*g$. But if $f,g \in C_c(G,A,\mathfrak{t})$, then the function $(x,r) \mapsto f(r)(x)\alpha_r^x\big(g(r^{-1}s)(x)\big)$ defines an element $\phi_s \in C_c(\Omega \times_N G, A)$. In fact if $\rho : \Omega \times G \to \Omega \times_N G$ is the quotient map, and if $s_0 \in G$ is fixed, there are compact sets $C \subseteq \Omega$ and $K \subseteq G$ such that $\mathrm{supp}(\phi_s) \subseteq \rho(C \times K)$ for all s near s_0 [33, Lemma 2.3]. Now choose a generalized Bruhat approximate cross section ([33, Proposition 2.18]) $b \in C_c(\Omega \times G)$ such that

$$\int_{N_x} b(x,st)\, d_{N_x}t = 1$$

provided $x \in C$ and $s \in KN_x$. Then we can define a continuous function $F : G \to A$ by

$$F(s)(x) = \int_G \phi_s(x,\dot{r})b(x,r)\,dr\,. \tag{2.3}$$

Of course, near s_0, (2.3) equals

$$\int_{G/N_x} f(r)(x)\alpha_r^x\big(g(r^{-1}s)(x)\big)d_{G/N_x}\dot{r} = f*g(s)(x)\,.$$

Next observe that by the properties of a continuous choice of Haar measures, together with the continuity of $\mathfrak{t}$, it is straightforward to see that $\Phi(f) \in C_c(G,A,\mathfrak{t})$ for any $f \in C_c(G,A)$. In fact, Φ maps $C_c(G,A)$ onto $C_c(G,A,\mathfrak{t})$. To see this, consider $f \in C_c(G,A,\mathfrak{t})$ with support contained in $\rho(C \times K)$ as above. Also as above choose $b \in C_c(\Omega \times G)$ with

$$\int_{N_x} b(x,st)\, d_{N_x}t = 1$$

for all $x \in C$ and $s \in KN_x$. Then $F \in C_c(G,A)$ defined by $F(s)(x) = f(s)(x)b(x,s)$ satisfies $\Phi(F) = f$.

Now, for each $x \in \Omega$, let us look at the map

$$\Phi^x : C_c(G,A) \to L^1(G,A_x,\mathfrak{t}^x);\ \Phi^x(f) = \Phi(f)^x = \int_{N_x} f(sn)(x)\mathfrak{t}^x_{sns^{-1}}\, d_{N_x}n\,.$$

Observe that Φ^x is the composition of the natural maps $\Psi^x : C_c(G,A) \to C_c(G,A_x)$ given by $\Psi^x(f)(s) = f(s)(x)$ and $\varphi^x : C_c(G,A_x) \to L^1(G,A_x,\mathfrak{t}^x)$ which is given by integration against $\mathfrak{t}^x$. Since all these maps extend to $*$-homomorphisms from the appropriate L^1-completions onto their images, we conclude that each Φ^x extends to an $*$-homomorphism from $L^1(G,A)$ onto $L^1(G,A_x,\mathfrak{t}^x)$. Thus we see that Φ is a $*$-homomorphism with respect to multiplication and involution on $C_c(G,A,\mathfrak{t})$ as defined above.

It has now become clear that $L^1(G,A,\mathfrak{t})$ is equal to $\Gamma_0(F)$ for a Banach $*$-algebra bundle $p : F \to \Omega$ with fibers $L^1(G,A_x,\mathfrak{t}^x)$. But for these algebras it is clear that the irreducible representations of $L^1(G,A,\mathfrak{t})$ are canonically given by the representations of the fibers $L^1(G,A_x,\mathfrak{t}^x)$. This shows that the representations of $L^1(G,A,\mathfrak{t})$, and hence also of $C^*(G,A,\mathfrak{t})$ are given by the union over all $x \in \Omega$ of the spaces $(A_x \rtimes_{\alpha^x,\mathfrak{t}^x} G)^\wedge$.

Finally, if $f \in C_c(G,A)$ and if $\pi \times U \in (A_x \rtimes_{\alpha^x, t^x} G)^\wedge$ is viewed both as an element of $(A \rtimes_\alpha G)^\wedge$ on the one hand, and as an element of $C^*(G,A,t)^\wedge$ on the other, then it follows directly from the constructions that $\pi \times U(\Phi(f)) = \pi \times U(f)$ for all $f \in C_c(G,A)$. Thus Φ extends to a $*$-homomorphism from $A \rtimes_\alpha G$ onto $C^*(G,A,t)$ such that $\ker \Phi$ is equal to I_t. But this implies that Φ factors through an isomorphism between $A \rtimes_{\alpha,t} G$ and $C^*(G,A,t)$, which finishes the proof. $\square$

As in the case of ordinary Green twisting maps there is a canonical example for generalized Green twisting maps.

Example 2.14. Suppose that (Ω, R) is a G-invariant open regularization of (A, G, α), and let $N : \Omega \to \mathfrak{K}(G)$ be a continuous map such that N_x is a normal subgroup of G for all $x \in \Omega$. Then we may restrict α to Ω^N as in Example 2.4 and we may form the subgroup crossed product $A \rtimes_\alpha \Omega^N$. Let γ^N denote the action of G on $A \rtimes_\alpha \Omega^N$, which is defined by

$$\gamma_s^N(f)(x,n) = \delta(x,s)\alpha_s^x\big(f(x,n^{-1}s)\big)\,, \tag{2.4}$$

$f \in \Gamma_0(q^*E)$, where $\delta : \Omega \times G \to \mathbb{R}^+$ is defined by $\delta(x,s) = \Delta_{G/N_x}(\dot{s})\Delta_G(s^{-1})$. (Recall that δ is continuous by virtue of Lemma 2.12.)

Note that the quotient action γ^{N_x} of G on $A_x \rtimes_{\alpha^x} N_x$ is the action defined by Green in [17, Proposition 1]. Consequently, there is a Green twisting map $t^{N_x} : N_x \to \mathcal{U}(A_x \rtimes_{\alpha^x} N_x)$ for the quotient system $(A_x \rtimes_{\alpha^x} N_x,\, G,\, \gamma^{N_x})$.

Recall that $A \rtimes_\alpha \Omega^N$ is a section algebra $\Gamma_0(D)$ of a C^*-bundle $p : D \to \Omega$ with fibers $A_x \rtimes_{\alpha^x} N_x$ if and only if the canonical projection $P : (A \rtimes_\alpha \Omega^N)^\wedge \to \Omega$ is open (recall that this is always true if all N_x are amenable). If P is open, then we claim that the map

$$t^N : \Omega^N \to \bigcup_{x \in \Omega} \mathcal{U}(A_x \rtimes_{\alpha^x} N_x);\ t^N(x,n) = t_n^{N_x}$$

is a generalized Green twisting map for the system $(A \rtimes_\alpha \Omega^N,\, G,\, \gamma^N)$. In order to see this we only have to check the continuity of the maps from $\Omega^N \to D$ defined, for each $d \in A \rtimes_\alpha \Omega^N$, by

$$(x,n) \mapsto t_n^{N_x} d(x) \quad \text{and} \quad (x,n) \mapsto d(x) t_n^{N_x}\,.$$

But if $d = f \cdot a$ is defined as $f \cdot a(x,s) = f(x,s)a(x)$ for $f \in C_c(\Omega^N)$ and $a \in A$, we see that $\big(t_n^x f \cdot a(x)\big)(s) = f(x, n^{-1}s)\alpha_n^x\big(a(x)\big)$ and $(f \cdot a(x) t_n^x)(s) = \Delta_{N_x}(n^{-1}) f(xsn^{-1}) a(x)$, and it is clear that these expressions are continuous with respect to all variables x, s and n (see [33, Lemma 2.5(ii)]). Thus the two maps given above are easily seen to be continuous for such d. Since the set $\{f \cdot a : f \in C_c(\Omega^N), a \in A\}$ generates a dense subset of $A \rtimes_\alpha \Omega^N$, the continuity follows for all $d \in A \rtimes_\alpha \Omega^N$.

Next, let us consider the generalized twisted crossed product $(A \rtimes_\alpha \Omega^N) \rtimes_{\gamma^N, t^N} G$. In analogy to the situation of ordinary twisted crossed products [17, Proposition 1], we want to show that $(A \rtimes_\alpha \Omega^N) \rtimes_{\gamma^N, t^N} G$ is isomorphic to

$A \rtimes_\alpha G$. For this let $r : E \to \Omega$ denote the C*-bundle belonging to A and let $\Gamma_c(q^*(E))$ denote the dense subalgebra of $A \rtimes_\alpha \Omega^N$ of continuous sections with compact support of the pull-back bundle q^*E with respect to the projection $q : \Omega^N \to \Omega$. We define $\Psi : C_c(G,A) \to C_c(G, \Gamma_c(q^*E), t^N)$ by

$$\big(\Psi(f)(s)\big)(x,n) = \delta(x,s) f(ns)(x)\,.$$

Then it follows from the remarks preceding [2, Proposition 8] that Ψ is given fiberwise by the canonical isomorphism between the fibers $A_x \rtimes_{\alpha^x} G$ and $(A_x \rtimes_{\alpha^x} N_x) \rtimes_{\gamma^{N_x}, t^{N_x}} G$. Since it is also clear that $\Psi\big(C_c(G,A)\big)$ is invariant under pointwise multiplication with bounded continuous functions on Ω we conclude that $\Psi\big(C_c(G,A)\big)$ is dense in $(A \rtimes_\alpha \Omega^N) \rtimes_{\gamma^N, t^N} G$. Thus Ψ extends to the desired isomorphism between $A \rtimes_\alpha G$ and $(A \rtimes_\alpha \Omega^N) \rtimes_{\gamma^N, t^N} G$.

Before we close this section we want to recall from [5] some basic results about σ-trivial regularizations and induced dynamical systems. For this recall first that a locally compact G-space Ω is called a *σ-proper G-space* if the stabilizer map $S : \Omega \to \mathfrak{K}(G); x \mapsto S_x$ is continuous and the canonical map

$$\Omega \times_S G \to \Omega \times \Omega; (x, \dot{s}) \mapsto (x, sx)$$

is proper in the usual sense that inverse images of compact sets are compact. If Ω is a σ-proper G-space, then we say that Ω is *σ-trivial,* if there exists a continuous section $\mathfrak{s} : \Omega/G \to \Omega$ for the quotient map $\Omega \to \Omega/G$. Moreover, Ω is called *locally σ-trivial,* if Ω is a σ-proper G-space with local sections, which just means that each $x \in \Omega$ has a G-invariant neighborhood U such that U is a σ-trivial G-space.

Now let (A, G, α) be any dynamical system, and let (Ω, R) be a regularization of (A, G, α). Then we say that (Ω, R) is a *σ-trivial regularization* of (A, G, α) if Ω is a σ-trivial G-space.

It was shown in [5] that the dynamical systems which possess a σ-trivial *open* regularization are exactly those which are induced from a subgroup action as follows. Start with any action β of Ω^H on a C*-algebra B as in Definition 2.1, where $B = \Gamma_0(E)$ for some C*-bundle $p : E \to \Omega$, and $H : \Omega \to \mathfrak{K}(G)$ is a continuous map. Then the *induced C*-algebra* $\operatorname{Ind}_{\Omega^H}^G D$ is defined in [5, Definition 5] as

$$\begin{aligned}\operatorname{Ind}_{\Omega^H}^G B = \{\, f \in C_b(G,B) : f(sh^{-1})(x) = \alpha_h^x\big(f(s)(x)\big) \text{ for all } (x,h) \in \Omega^H\,, \\ \text{and such that } (x,\dot{s}) \to \|f(s)(x)\| \text{ is a function in } C_c(\Omega \times_H G)\}\,.\end{aligned}$$

The induced action $\operatorname{Ind}\beta$ of G on $\operatorname{Ind}_{\Omega^H}^G B$ is given by $\big(\operatorname{Ind}\beta_s(f)\big)(t) = f(s^{-1}t)$ for all $s, t \in G$.

For each $\pi \in \hat{B}$ and $s \in G$ let $M(\pi, s)$ denote the irreducible representation of $\operatorname{Ind}_{\Omega^H}^G B$ given by

$$M(\pi,s)(f) = \pi\big(f(s)\big), \quad f \in \operatorname{Ind}_{\Omega^H}^G B\,.$$

Then $M(\pi,s) = M(\rho,t)$ in $(\mathrm{Ind}_{\Omega H}^{G} B)^{\wedge}$ if and only if $(\pi,s) \sim (\rho,t)$, where $\sim$ denotes the equivalence relation on $\hat{B} \times G$ which is given by

$$(\pi,s) \sim (\rho,t) \text{ if and only if } P(\pi) = P(\rho) = x \text{ and there exists}$$
$$h \in H_x \text{ such that } (\pi,s) = (\rho \circ \beta_h^x, th)\,.$$

Here, $P : \hat{B} \to \Omega$ denotes the canonical projection corresponding to $p : E \to \Omega$. Moreover, each representation of $\mathrm{Ind}_{\Omega H}^{G} B$ is given in this way and $(\mathrm{Ind}_{\Omega H}^{G} B)^{\wedge}$ is homeomorphic to $(\hat{B} \times G)/\!\sim$ (see [5, Proposition 10]).

Hence we observe easily that there exists a canonical continuous open and surjective G-map $R : (\mathrm{Ind}_{\Omega H}^{G} B)^{\wedge} \to \Omega \times_H G$, which is given by mapping $M(\pi,s) \in (\mathrm{Ind}_{\Omega H}^{G} B)^{\wedge}$ to the pair $(P(\pi), \dot{s}) \in \Omega \times_H G$. Thus the pair $(\Omega \times_H G, R)$ becomes an open σ-trivial regularization of $(\mathrm{Ind}_{\Omega H}^{G} B,\ G,\ \mathrm{Ind}\,\beta)$.

For the converse, let (Ω, R) be any open σ-trivial regularization of the dynamical system (A, G, α). Then $\Omega = \Lambda \times_H G$, where Λ is the image of a section for Ω/G and H is the restriction of the stabilizer map to Λ. Let $I = \ker R^{-1}(\Lambda)$ and let $B = A/I$. Then it is easily seen that the canonical action of Ω^S on A given by Example 2.4 restricts to a subgroup action of Λ^H on B, and it was shown in [5, Theorem 3] that there is a canonical G-equivariant isomorphism between A and $\mathrm{Ind}_{\Lambda H}^{G} B$ which is given by

$$\big(\Phi(a)\big)(s) = \alpha_{s^{-1}}(a) + I\,.$$

One reason that induced systems are interesting in the investigation of crossed products is the following result.

Proposition 2.15 ([5, Corollary 3]). *Let β be an action of Ω^H on B. Then $\mathrm{Ind}_{\Omega H}^{G} B \rtimes_{\mathrm{Ind}\,\beta} G$ is Morita equivalent to $B \rtimes_{\beta} \Omega^H$.*

3 Twisted systems and induced crossed products

Our main result in this section is the following theorem which is a generalization of a result of Pedersen and Olesen [20] (which is essentially (1) $\Rightarrow$ (3) in the theorem below in case of ordinary Green twisting maps) to generalized twisted systems. In the following we say that a covariant system (A, G, α) is *abelian*, if G is an abelian group.

Theorem 3.1. *Suppose that (A, G, α) is an abelian dynamical system and that Ω is a locally compact Hausdorff space. Let $N : \Omega \to \mathfrak{K}(G)$ be a continuous map. Then the following conditions are equivalent.*

(1) *There exists an open G-invariant regularization (Ω, R) for (A, G, α) for which there is a generalized Green twisting map with domain Ω^N.*

(2) *There exists an open σ-trivial regularization $\hat{R} : (A \rtimes_\alpha G)^{\wedge} \to \Omega \times_{N^\perp} \hat{G}$ for the dual system $(A \rtimes_\alpha G,\ \hat{G},\ \hat{\alpha})$.*

(3) *The dual system $(A \rtimes_\alpha G,\ \hat{G},\ \hat{\alpha})$ is isomorphic to an induced system $(\mathrm{Ind}_{\Omega^{N^\perp}}^{\hat{G}} B,\ \hat{G},\ \mathrm{Ind}\,\beta)$, for some C^*-algebra B and some action β of $\Omega^{N^\perp}$ on B.*

Moreover, if the conditions (1) *to* (3) *are satisfied, then the algebra B in* (3) *may be chosen to be $A \rtimes_{\alpha,\mathrm{t}} G$, where the action of β^x of $N_x^{\perp}$ is given on the fiber $A_x \rtimes_{\alpha^x,\mathrm{t}^x} G$ by the dual action of $N_x^{\perp}$ on this fiber.*

Note that the equivalence of (2) and (3) is a consequence of [5, Theorem 3] (see also the remarks at the end of Sect. 1). Thus in order to prove the equivalences of our theorem it is enough to prove the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). Before we start with the proofs we have to recall some facts about the space $\mathrm{Rep}(D)$ of equivalence classes of $*$-representations (with bounded dimension) of a C*-algebra D.

Proposition 3.2. *Let D be a C*-algebra and let $(\pi_i)_{i\in I}$ be a net in $\mathrm{Rep}(D)$ which converges to $\pi \in \mathrm{Rep}(D)$.*

(1) *If $\rho \in \mathrm{Rep}(D)$ is such that $\ker\rho \supseteq \ker\pi$, then $\pi_i \to \rho$ in $\mathrm{Rep}(D)$.*

(2) *Suppose that $\rho \in \hat{D}$ is such that $\ker\rho \supseteq \ker\pi$, and let $\mathscr{D}_i \subseteq \hat{D}$ be such that $\ker\mathscr{D}_i = \ker\pi_i$ for all $i \in I$. Then there exists a subnet $(\pi_j)_{j\in J}$ and elements $\rho_j \in \mathscr{D}_j$ such that $\rho_j \to \rho$ in $\hat{D}$.*

Proof. The first assertion is just [8, Proposition 1.2], while the second is a consequence of [31, Theorem 2.2] (note that Schochetman proves his result only for separable D, but the same proof applies in the general case). $\square$

In the sequel we need an explicit description of $C^*(\Omega^N)$ in case where G is abelian.

Lemma 3.3. *Suppose that Ω is a locally compact space and $N : \Omega \to \mathfrak{K}(G); x \mapsto N_x$ is a continuous map for some locally compact abelian group G. Then the map*

$$M : \Omega \times \hat{G} \to C^*(\Omega^N)^{\wedge}; (x,\chi) \mapsto (x, \chi|_N)$$

defines a homeomorphism between $\Omega \times_{N^\perp} \hat{G}$ and $C^*(\Omega^N)^{\wedge}$. As a consequence, $C^*(\Omega^N)$ is isomorphic to $C_0(\Omega \times_{N^\perp} \hat{G})$.

Proof. It is clear that M defines a bijection between $\Omega \times_{N^\perp} \hat{G}$ and $C^*(\Omega^N)^{\wedge}$, so it is enough to show that M is continuous and open. The continuity is a direct consequence of continuity of restricting representations (see [2, Proposition 7]). For proving the openness, let $\{(x_i, \chi_i)\}_{i\in I}$ be a net in $\Omega \times G$ such that $\{(x_i, \chi_i|_{N_i})\}$ converges to $(x, \chi|_{N_x})$ in $C^*(\Omega^N)^{\wedge}$. Then $x_i \to x$ in Ω. By the continuity of induction [2, Proposition 6] we know that $\mathrm{Ind}_{N_{x_i}}^G(\chi_i|_{N_{x_i}})$ converges to $\mathrm{Ind}_{N_x}^G(\chi|_{N_x})$ in $\mathrm{Rep}(C^*(G))$. But it is well known that $\mathrm{Ind}_{N_{x_i}}^G(\chi_i|_{N_{x_i}})$ is weakly equivalent to the coset $\chi_i N_{x_i}^{\perp} \subseteq \hat{G}$ for each $i \in I$, and that $\mathrm{Ind}_{N_x}^G(\chi|_{N_x})$ is weakly equivalent to $\chi N_x^{\perp}$ (see, for example, [32, Lemma 5.1]). Thus it is a consequence of Proposition 3.2 that we may pass to a subnet in order to find elements $\mu_i \in N_{x_i}^{\perp}$ such that $(x_i, \chi_i\mu_i)$ converges to (x, χ) in $\Omega \times \hat{G}$. But this shows the openness of M. $\square$

We use this lemma to prove

Lemma 3.4. *Suppose that (Ω,R) is a G-invariant open regularization for the covariant system (A, G, α) and that $r : E \to \Omega$ is the corresponding* C*-*bundle such that $A = \Gamma_0(E)$. Assume that* $\mathfrak{t}$ *is a generalized Green twisting map for (A, G, α) with domain Ω^N. Then $A \rtimes_\alpha \Omega^N$ is isomorphic to $A \rtimes_{\mathrm{id}} \Omega^N$, the subgroup crossed product of A with Ω^N by the trivial action. If G is abelian, then this implies that $A \rtimes_\alpha \Omega^N$ is canonically isomorphic to the balanced tensor product $A \otimes_{C_0(\Omega)} C_0(\Omega \times_{N\perp} \hat{G})$. The corresponding homeomorphism of*

$$(A \otimes_{C_0(\Omega)} C_0(\Omega \times_{N\perp} \hat{G}))^\wedge = \{(\rho,(x,\chi)); R(\rho) = x\} \subseteq \hat{A} \times (\Omega \times_{N\perp} \hat{G})$$

onto $(A \rtimes_\alpha \Omega^N)^\wedge$ is given by the map

$$(\rho,(x,\chi)) \mapsto (x, \rho \times \chi(\rho \circ \mathfrak{t}^x)) , \tag{3.1}$$

where the latter term denotes the representation of $A \times_\alpha \Omega^N$ given by $\rho \times \chi(\rho \circ \mathfrak{t}^x)$ acting on the quotient $A_x \rtimes_{\alpha^x} N_x$.

Proof. Let $q : \Omega^N \to \Omega$ denote the projection. Then it follows by straightforward computations that the map

$$\Psi : \Gamma_c(q^*E) \to \Gamma_c(q^*E); \Psi(f)(x,n) = f(x,n)\mathfrak{t}^x_n$$

extends to an isomorphism from $A \rtimes_\alpha \Omega^N$ onto $A \rtimes_{\mathrm{id}} \Omega^N$. Suppose now that G is abelian. Consider the map $\Phi : A \odot C_c(\Omega^N) \to A \rtimes_{\mathrm{id}} \Omega^N$ which is defined by

$$\left(\Phi\left(\sum_{i=1}^n a_i \otimes f_i\right)\right)(x,n) = \sum_{i=1}^n a_i(x) f_i(x,n) .$$

It is easily seen that Φ extends to a surjective $*$-homomorphism from $A \otimes C^*(\Omega^N)$ onto $A \rtimes_{\mathrm{id}} \Omega^N$ such that the kernel of this map is exactly the intersection of all representations $(\rho,(x,\chi)) \in \hat{A} \times C^*(\Omega^N)^\wedge$ such that $R(\rho) = x$. This implies that Φ factors through an isomorphism between $A \otimes_{C_0(\Omega)} C^*(\Omega^N)$ and $A \rtimes_{\mathrm{id}} \Omega^N$. Thus we conclude from Lemma 3.3 that $A \rtimes_\alpha \Omega^N$ is canonically isomorphic to $A \otimes_{C_0(\Omega)} C_0(\Omega \times_{N\perp} \hat{G})$.

Finally, let $(\rho,(x,\chi)) \in (A \otimes_{C_0(\Omega)} C_0(\Omega \times_{N\perp} \hat{G}))^\wedge$, and let $a \in A$ and $f \in C_c(\Omega^N)$, the latter viewed as an element of $C_0(\Omega \times_{N\perp} \hat{G})$ via Fourier transform. Then we compute

$$\begin{aligned}(x, \rho \times \chi(\rho \circ \mathfrak{t}^x))(\Psi^{-1}\Phi(a \otimes f)) &= \rho \times \chi(\rho \circ \mathfrak{t}^x)((a(x)f(x,\cdot)(\mathfrak{t}^x)^*) \\ &= \int_{N_x} \rho(a(x)) f(x,n)\chi(n)\, d_{N_x}n = (\rho,(x,\chi))(a \otimes f) ,\end{aligned}$$

which finishes the proof. □

In the sequel, it will be necessary to use some continuity results for inducing and restricting representations, which are not explicitly stated in [2].

Proposition 3.5. *Suppose that (Ω,R) is a G-invariant open regularization of the system (A, G, α) and let $N : \Omega \to \mathfrak{K}(G)$ be a continuous map such that*

N_x is normal in G for all $x \in \Omega$. Then the maps

$$\mathrm{Ind} : \mathcal{S}(A \rtimes_\alpha \Omega^N) \to \mathrm{Rep}(A \rtimes_\alpha G); (x, \rho \times V) \mapsto \mathrm{Ind}_{N_x}^G(\rho \times V)$$

and

$$\mathrm{Res} : (A \rtimes_\alpha G)^\wedge \to \mathcal{S}(A \rtimes_\alpha \Omega^N); \pi \times U \mapsto (P(\pi \times U), \pi \times U|_{N_{P(\pi \times U)}})$$

are continuous, where $P : (A \rtimes_\alpha G)^\wedge \to \Omega$ denotes the canonical projection.

Proof. By Proposition 2.6 we may view $\mathcal{S}(A \rtimes_\alpha \Omega^N)$ topologically as a subset of $\mathcal{S}(\Omega^N, A, \alpha)$. But it is a direct consequence of [2, Corollary 2] that the extension of Ind to $\mathcal{S}(\Omega^N, A, \alpha)$ is continuous. Thus our map is continuous, too. The continuity of Res follows easily from Proposition 2.6 and [2, Proposition 7] by identifying $A \rtimes_\alpha G$ with $A \rtimes_\alpha \Omega^G$ as in Lemma 2.10. □

We now prove the openness of the canonical projection from $(A \rtimes_{\alpha,\mathfrak{t}} G)^\wedge$ onto Ω when $\mathfrak{t}$ has domain Ω^N. In fact we will prove a slightly more general result which we will need in the proof of Theorem 3.1.

Proposition 3.6. *Suppose that (Ω, R) is a G-invariant open regularization, $N : \Omega \to \mathfrak{K}(G)$ a continuous map, and $\mathfrak{t}$ a generalized Green twisting map for (A, G, α) with domain Ω^N. Suppose further that (Λ, P) is another G-invariant open regularization of (A, G, α) such that there exists an open continuous map $q : \Lambda \to \Omega$ satisfying $R = q \circ P$. Let $Q : (A \rtimes_{\alpha,\mathfrak{t}} G)^\wedge \to \Lambda$ denote the restriction to $(A \rtimes_{\alpha,\mathfrak{t}} G)^\wedge$ of the canonical projection from $(A \rtimes_\alpha G)^\wedge$ onto Λ. Then Q is open and surjective whenever G/N_x is amenable for all $x \in \Omega$.*

Proof. Recall that $Q(\pi \times U) = y$ if and only if $\ker \pi \supseteq P^{-1}(\{y\})$. This implies that $\pi \times U$ factors through the quotient $A_y \rtimes_{\alpha^y, \mathfrak{t}^y} G$, where α^y denotes the canonical action of G on $A_y = A/\ker P^{-1}(\{y\})$ and $\mathfrak{t}^y$ denotes the image of $\mathfrak{t}^{q(y)}$ under the canonical quotient map $A_{q(y)} \to A_y$ (which exists due to the fact that $\ker R^{-1}(\{q(y)\}) \subseteq P^{-1}(\{y\})$). Now let ρ be a faithful representation of A_y. Then we may produce a covariant representation $\rho \times V$ of $A_y \rtimes_{\alpha^y, \mathfrak{t}^y} N_{q(y)}$ by defining $V = \rho \circ \mathfrak{t}^y$. Then $\rho \times V$ preserves the twist $\mathfrak{t}^y$, and $\mathrm{Ind}_{N_{q(y)}}^G(\rho \times V)$ is a representation of $A_y \rtimes_{\alpha^y} G$ which preserves $\mathfrak{t}^y$ by [17, Corollary 5]. It follows from [17, Proposition 13] and the amenability of $G/N_{q(y)}$ that this induced representation defines a faithful representation of $A \rtimes_{\alpha^y, \mathfrak{t}^y} G$. Regarding $\mathrm{Ind}_{N_{q(y)}}^G(\rho \times V)$ as a representation of $A \rtimes_{\alpha,\mathfrak{t}} G$, we have $\ker\big(\mathrm{Ind}_{N_{q(y)}}^G(\rho \times V)\big) = \ker Q^{-1}(\{y\})$.

Now let $\{y_i\}_{i \in I}$ be a net in Λ such that $y_i \to y$ and let $\pi \times U \in Q^{-1}(\{y\})$. We have to show that there exists a subnet $\{y_j\}_{j \in J}$ of $\{y_i\}_{i \in I}$, and a net $\{\pi_j \times U^j\}_{j \in J} \subseteq (A \rtimes_{\alpha,\mathfrak{t}} G)^\wedge$, converging to $\pi \times U$, such that $Q(\pi_j \times U^j) = y_j$ for all $j \in J$.

For each $i \in I$, let $\rho_i \in \mathrm{Rep}(A)$ be such that $\ker \rho_i = \ker P^{-1}(\{y_i\})$, and let $\rho \in \mathrm{Rep}(A)$ be such that $\ker \rho = \ker R^{-1}(\{y\})$. Then the same arguments as used in the proof of [5, Proposition 11] show that $\rho_i \to \rho$ in $\mathrm{Rep}(A)$.

Looking at the subgroup crossed product $A \rtimes_{\mathrm{id}} \Omega^N$ by the trivial action of Ω^N on A, we observe that $\left(q(y_i), \rho_i \times 1_{\mathscr{H}_{\rho_i}}\right)$ converges to $(q(y), \rho \times 1_{\mathscr{H}_\rho})$ in $\mathscr{S}(A \rtimes_{\mathrm{id}} \Omega^N)$, where $1_{\mathscr{H}_{\rho_i}}$ denotes the trivial representation of $N_{q(y_i)}$ into $\mathscr{H}_{\rho_i}$ and $1_{\mathscr{H}_\rho}$ is defined similarly. If $\Phi : A \rtimes_\alpha \Omega^N \to A \rtimes_{\mathrm{id}} \Omega^N$ is given as in Lemma 3.4, then we see that the representations $\left(q(y_i), \rho_i \times 1_{\mathscr{H}_{\rho_i}}\right) \circ \Phi$ of $A \rtimes_\alpha \Omega^N$ are just given by the subgroup representations $\left(q(y_i), \rho_i \times V^i\right)$ with $V^i = \rho_i \circ \mathfrak{t}^{q(y_i)}$.

It follows that $(q(y_i), \rho_i \times V^i)$ converges to $(q(y), \rho \times V)$ in $\mathscr{S}(A \rtimes_\alpha \Omega^N)$. Now the continuity of induction implies that $\operatorname{Ind}_{N_{q(y_i)}}^G(\rho_i \times V^i)$ converges to $\operatorname{Ind}_{N_{q(y)}}^G(\rho \times V)$ in $\operatorname{Rep}(A \rtimes_\alpha G)$. But as above, we know that $\ker\left(\operatorname{Ind}_{N_{q(y_i)}}^G(\rho_i \times V^i)\right) = \ker Q^{-1}(\{y_i\})$ for each $i \in I$. Thus by Proposition 3.2 we may pass to a subnet in order to find elements $\pi_i \times U^i \in Q^{-1}(\{y_i\})$ such that $\pi_i \times U^i$ converges to $\pi \times U$ in $A \rtimes_{\alpha,\mathfrak{t}} G$. This finishes the proof. □

Corollary 3.7. *Suppose that* $\mathfrak{t}$ *is a generalized Green twisting map for* (A, G, α) *with domain* Ω^N *such that* G/N_x *is amenable for all* $x \in \Omega$. *Then* $A \rtimes_{\alpha,\mathfrak{t}} G$ *is isomorphic to the section algebra* $\Gamma_0(D)$ *of a* C*-*bundle* $q : D \to \Omega$ *with fibers* $A_x \rtimes_{\alpha^x,\mathfrak{t}^x} G$.

Proof. The Corollary follows from Lee's theorem [19, Theorem 4] and by taking $A = \Omega$ in Proposition 3.6. □

Our next proposition gives the implication (1) $\Rightarrow$ (2) in Theorem 3.1.

Proposition 3.8. *Suppose that* (Ω, R) *is an open* G-*invariant regularization for an abelian dynamical system* (A, G, α), *and that* $N : \Omega \to \mathfrak{K}(G)$ *is a continuous map such that there is a generalized Green twisting map* $\mathfrak{t}$ *with domain* Ω^N.

(1) *If we identify the spectrum of* $A \rtimes_\alpha \Omega^N$ *with* $\{(\rho, (x, \chi)) : R(\rho) = x\}$ (*via Lemma* 3.4), *then* $\tilde{R}(\rho, (x, \chi)) = (x, \chi)$ *defines an open* $\hat{G}$-*equivariant and* G-*invariant regularization of the iterated system* $(A \rtimes_\alpha \Omega^N, G, \gamma^N)$ *as defined in Example* 2.14.

(2) *If* $\pi \times U$ *is an irreducible representation of* $A_x \rtimes_\alpha G$, *then there is a unique* $(x, \chi) \in \Omega \times_{N^\perp} \hat{G}$ *with* $\ker(\pi \times U|_{N_x}) \supseteq \ker\left(\tilde{R}^{-1}(\{(x, \chi)\})\right)$. *In particular,* $\hat{R}(\pi \times U) = (x, \chi)$ *defines an open* σ-*trivial regularization for the dual system* $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$.

Proof. Let $(\gamma^N, \mathfrak{t}^N)$ denote the canonical generalized twisted action of G on $A \rtimes_\alpha \Omega^N$ as given in Example 2.14. We show that there is a continuous open $\hat{G}$-equivariant and G-invariant surjection $\tilde{R}$ from $(A \rtimes_\alpha \Omega^N)^\wedge$ onto $\Omega \times_{N^\perp} \hat{G}$. By Lemma 3.4 we know that $(A \rtimes_\alpha \Omega^N)^\wedge$ is homeomorphic to $\{(\rho, (x, \chi)); R(\rho) = x\} \subseteq \hat{A} \times (\Omega \times_{N^\perp} \hat{G})$ via the homeomorphism $(\rho, (x, \chi)) \mapsto (x, \rho \times \chi(\rho \circ \mathfrak{t}^x))$. We define $\tilde{R}((x, \rho \times \chi(\rho \circ \mathfrak{t}^x))) = (x, \chi)$. It follows directly from the openness and surjectivity of R that $\tilde{R}$ is open and surjective, too. It is also clear that $\tilde{R}$ is $\hat{G}$-equivariant, since on both spaces the action of $\hat{G}$ is given by pointwise multiplication of characters. In order to show that $\tilde{R}$ is

G-invariant note first that by the definition of a Green twisting map $\alpha_s^x \circ \mathfrak{t}^x = \mathfrak{t}^x$ for all $x \in \Omega$ and $s \in G$. Thus we conclude for $f \in \Gamma_c(q^*E)$:

$$\begin{aligned}(x, \rho \times \chi(\rho \circ \mathfrak{t}^x))\big(\gamma_s^N(f)\big) &= \int_{N_x} \rho\big(\alpha_s\big(f(x,n)\big)\mathfrak{t}_n^x\big)\chi(n)\, d_{N_x}n \\ &= \int_{N_x} \rho\big(\alpha_s\big(f(x,n)\mathfrak{t}_n^x\big)\big)\chi(n)\, d_{N_x}n = \big(x, \rho\circ\alpha_s \times \chi(\rho\circ\alpha_s\circ\mathfrak{t}^x)\big)(f)\,.\end{aligned}$$

Thus we see that $\big(x, \rho \times \chi(\rho \circ \mathfrak{t}^x)\big) \circ \gamma_s$ is equal to $\big(x, \rho\circ\alpha_s \times \chi(\rho\circ\alpha_s\circ\mathfrak{t}^x)\big)$ which has the same image under $\tilde{R}$.

We have seen above that $\tilde{R} : (A \rtimes_\alpha \Omega^N)^\wedge \to \Omega \times_{N^\perp} \hat{G}$ is an open G-invariant regularization for the system $(A \rtimes_\alpha \Omega^N, G, \gamma^N)$ such that the projection, say r, from $\Omega \times_{N^\perp} \hat{G}$ onto Ω satisfies the relation $r \circ \tilde{R} = P$, where P denotes the projection from $(A \rtimes_\alpha \Omega^N)^\wedge$ onto Ω. Thus it follows from Proposition 3.6 that there is a canonical open map Q from $((A \rtimes_\alpha \Omega^N) \rtimes_{\gamma^N, \mathfrak{t}^N} G)^\wedge$ onto $\Omega \times_{N^\perp} \hat{G}$, and we define $\hat{R}$ as the corresponding open map from $(A \rtimes_\alpha G)^\wedge$ onto $\Omega \times_{N^\perp} \hat{G}$ via the isomorphism Ψ between $A \rtimes_\alpha G$ and $(A \rtimes_\alpha \Omega^N) \rtimes_{\gamma^N, \mathfrak{t}^N} G$ constructed in Example 2.14.

Now let $\pi \times U \in (A \rtimes_\alpha G)^\wedge$. Then $\pi \times U$ factors through some quotient $A_x \rtimes_{\alpha^x} G$, and is transported to the representation $(\pi \times U|_{N_x}) \times U$ of the fiber $(A_x \rtimes_{\alpha^x} N_x) \rtimes_{\gamma^{N_x}, \mathfrak{t}^{N_x}} G$ of $(A \rtimes_\alpha \Omega^N) \rtimes_{\gamma^N, \mathfrak{t}^N} G$ via Ψ. Thus we see that $\hat{R}(\pi \times U) = (x, \mu)$ for some $(x, \mu) \in \Omega \times_{N^\perp} \hat{G}$ if and only if the kernel of the representation $(x, \pi \times U|_{N_x})$ of $A_x \rtimes_{\alpha^x} N_x$ contains $\ker \tilde{R}^{-1}(\{(x,\mu)\})$. Now the dual action of $\chi \in \hat{G}$ on the element $\pi \times U \in (A \rtimes_\alpha G)^\wedge$ is given by multiplication of χ with U, i.e., $(\pi \times U) \circ \hat{\alpha}_{\chi^{-1}} = \pi \times \chi U$. This representation corresponds to the representation $(\pi \times \chi U|_{N_x}) \times \chi U$ of the fiber $(A_x \rtimes_{\alpha^x} N_x) \rtimes_{\gamma^{N_x}, \mathfrak{t}^{N_x}} G$. Hence it follows from the $\hat{G}$-equivariance of $\tilde{R}$ that $\hat{R}(\pi \times \chi U) = (x, \chi\mu)$ if and only if $Q(\pi \times U) = (x, \mu)$. Thus we see that $(\Omega \times_{N^\perp} \hat{G}, \hat{R})$ is a σ-trivial open regularization for the dual system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$. □

Since G is abelian, the continuity of the map $N : \Omega \to \mathfrak{K}(G)$ implies that $N^\perp : \Omega \to \mathfrak{K}(\hat{G})$, defined by $N_x^\perp = (N_x)^\perp$, is also continuous [34]. The implication $(3) \Rightarrow (1)$ in Theorem 3.1 will follow from the next result.

Proposition 3.9. *Suppose that (A, G, α) is an abelian dynamical system. Assume that $q : E' \to \Omega$ is a C*-bundle such that $D = \Gamma_0(E')$ admits an action γ of $\Omega^{N^\perp}$ on D such that there is a covariant isomorphism from $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$ onto $(\mathrm{Ind}_{\Omega^{N^\perp}}^{\hat{G}} D, \hat{G}, \mathrm{Ind}\,\gamma)$. Then there is an open G-invariant regularization for (A, G, α) for which there is a generalized Green twisting map with domain Ω^N.*

The proof of the above proposition will depend heavily on the following two lemmas which may be of independent interest.

Lemma 3.10. *Suppose that $\alpha = \{\alpha^x\}_{x\in\Omega}$ is an action of Ω^N on a C*-algebra D, and that (A, G, β) is the induced system $(\mathrm{Ind}_{\Omega^N}^G D, G, \mathrm{Ind}\,\alpha)$. Then there is a $\hat{G}$-invariant open regularization $(\hat{P}, \Omega)$ of the dual system $(A \rtimes_\beta G, \hat{G}, \hat{\beta})$ with* *respect to which there is a generalized Green twisting map for* $(A \rtimes_\beta G, \hat{G}, \hat{\beta})$ *with domain* $\Omega^{N^\perp}$.

Proof. It follows from [5, Theorem 3] that there is an open G-invariant regularization (P, Ω) for (A, G, β). As a consequence of [35, Theorem 2.1], there is an open surjection $\hat{P} : \mathrm{Prim}(A \rtimes_\beta G) \to \Omega$, and this map is easily seen to be $\hat{G}$-invariant.

In particular, we may assume that $A = \Gamma_0(E')$ and that $D = \Gamma_0(E)$ where E and E' are C*-bundles over Ω. Furthermore, $A_x = \mathrm{Ind}_{N_x}^G D_x$. Thus if $\sigma \in N_x^\perp$, then we can define $\tilde{\mathrm{t}}_\sigma^x \in \mathcal{U}(A_x)$ by

$$(\tilde{\mathrm{t}}_\sigma^x f)(s) = \overline{\sigma(s)} f(s) = (f \tilde{\mathrm{t}}_\sigma^x)(s)\,.$$

We let t_σ^x be the image of $\tilde{\mathrm{t}}_\sigma^x$ under the natural inclusion of $\mathcal{U}(A_x)$ into $\mathcal{U}(A_x \rtimes_{\alpha^x} G)$. Thus, if we view elements of $C_c(G, A)$ as functions on $G \times \Omega \times G$, we have

$$\big(\mathrm{t}_\sigma^x f(s)(x)\big)(t) = \overline{\sigma(t)} f(s)(x)(t) \quad \text{and} \quad (f(s)(x)\mathrm{t}_\sigma^x)(t) = \overline{\sigma(s^{-1}t)} f(s)(x)(t)\,.$$

The first step will be to show that $(x, \sigma) \mapsto \mathrm{t}_\sigma^x f(\cdot)(x)$ and $(x, \sigma) \mapsto f(\cdot)(x)\mathrm{t}_\sigma^x$ are continuous from $\Omega^{N^\perp}$ into E for a suitable collection of $f \in C_c(G, A)$. Certainly we may assume that $f(s)(x)(t) = g(s)a(t)(x)$ for $g \in C_c(G)$ and $a \in A$. So it suffices to see that $(x, \sigma) \mapsto \tilde{\mathrm{t}}_\sigma^x a(x)$ is continuous. Using the definition of the topology on E (see [10, Proposition 1.6]), it will suffice to show that given any $(x_0, \sigma_0) \in \Omega^{N^\perp}$ and some $b \in A$ satisfying $b(\cdot)(x_0) = \tilde{\mathrm{t}}_{\sigma_0}^{x_0} a(\cdot)(x_0)$,

$$\|\tilde{\mathrm{t}}_\sigma^x a(\cdot)(x) - b(\cdot)(x)\|_{A_x}$$

goes to zero as (x, σ) approaches (x_0, σ_0). However,

$$\|\tilde{\mathrm{t}}_\sigma^x a(\cdot)(x) - b(\cdot)(x)\|_{A_x} = \sup_{t \in G} \|\overline{\sigma(t)} a(t)(x) - b(t)(x)\|_{D_x}\,.$$

Furthermore the map $(x, \sigma, t) \mapsto \|\overline{\sigma(t)} a(t)(x) - b(t)(x)\|_{D_x}$ is continuous from $\Omega^{N^\perp} \times G$ into $\mathbb{R}^+$. (Just consider the map from $\Omega \times \hat{G} \times G$ with any $a, b \in C_b(G, D)$.)

We may assume that a, and hence b, have compact support in $\Omega \times_N G$. Let $\kappa : \Omega \times G \to \Omega \times_N G$ be the quotient map. Observe that there is a compact set $C \subseteq \Omega \times G$ such that $\kappa(C) \supseteq \operatorname{supp} a \cup \operatorname{supp} b$ [33, Lemma 2.3]. Now choose $\phi \in C_c(\Omega \times G)$ such that $0 \leqq \phi \leqq 1$ and $\phi \equiv 1$ on C. Then define $\psi(x, \sigma, t)$ to be $\phi(x, t)\|\overline{\sigma(t)} a(t)(x) - b(t)(x)\|_{D_x}$. The point being that $\|\overline{\sigma(t)} a(t)(x) - b(t)(x)\|_{D_x}$ depends only on the class of t in G/N_x so that

$$\begin{aligned} \|\tilde{\mathrm{t}}_\sigma^x a(\cdot)(x) - a(\cdot)(x)\|_{A_x} &= \sup_{t \in G} \|\tilde{\mathrm{t}}_\sigma^x a(t)(x) - a(t)(x)\|_{D_x} \\ &= \sup_{t \in G} \psi(x, \sigma, t)\,. \end{aligned}$$

But we may assume that σ lies in a compact neighborhood of σ_0, and, hence, that ψ has compact support in $\Omega \times \hat{G} \times G$. Then, if $\psi_{(x,\sigma)} \in C_c(G)$ is defined by $\psi_{(x,\sigma)}(s) = \psi(x, \sigma, t)$, the map $(x, \sigma) \mapsto \psi_{(x,\sigma)}$ is continuous from $\Omega \times \hat{G}$ to $C_0(G)$. Since $\psi_{(x_0,\sigma_0)} = 0$, we have $\|\psi_{(x,\sigma)}\|_\infty < \varepsilon$ for (x, σ) near (x_0, σ_0) which is what we wanted.

It is now immediate that $\sigma \mapsto \mathfrak{t}^x_\sigma$ is strictly continuous on $N_x^\perp$, and it follows from straightforward calculations that $\mathfrak{t}^x_{\sigma\rho} = \mathfrak{t}^x_\sigma \mathfrak{t}^x_\rho$ if $\sigma, \rho \in N_x^\perp$, that if $\sigma \in N_x^\perp$, then $\operatorname{Ad} \mathfrak{t}^x_\sigma = \hat{\alpha}^x_\sigma$, and that $\hat{\alpha}^x_\rho(\mathfrak{t}^x_\sigma) = \mathfrak{t}^x_\sigma$ for all $\rho \in \hat{G}$ and $\sigma \in N_x^\perp$. Thus $\sigma \mapsto \mathfrak{t}^x_\sigma$ is an ordinary Green twisting map for $(A_x, \hat{G}, \hat{\alpha}^x)$. This completes the proof of the lemma. □

Lemma 3.11. *Suppose that (A, G, α) and (B, G, β) are Morita equivalent via (X, u). Let $h_X : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$ be the Rieffel homeomorphism. If (P, Ω) is an open G-invariant regularization of (B, G, β) such that there is a generalized Green twisting map with domain Ω^N, then $(P \circ h_X, \Omega)$ is an open regularization of (A, G, α) for which there exists a generalized Green twisting map with domain Ω^N.*

Proof. Let $\mathfrak{t}$ be a generalized Green twisting map for β. We may assume that $B = \Gamma_0(E)$ for a C*-bundle $p : E \to \Omega$ induced by the map P and that $A = \Gamma_0(E')$ for a C*-bundle E' induced by the map $P \circ h_X$. Then (A_x, G, α^x) is Morita equivalent to (B_x, G, β^x) via (X^x, u^x), where $X^x = X/X \cdot \ker(P^{-1}(\{x\}))$, and $n \mapsto \mathfrak{t}^x_n$ is, by definition, a Green twisting map for (B_x, G, β^x). Then [3, Proposition 2] implies that $\mathfrak{s}^x$, defined by $\mathfrak{s}^x_n \cdot \xi = u^x_n(\xi) \cdot \mathfrak{t}^x_n$, $\xi \in X^x$ defines a Green twisting map for α^x. Naturally, we claim that $\mathfrak{s}(x, n) = \mathfrak{s}^x_n$ defines a generalized Green twisting map for (A, G, α). At this point, we need only show that the maps $(x, n) \mapsto \mathfrak{s}^x_n a(x)$ and $(x, n) \mapsto a(x)\mathfrak{s}^x_n$ are continuous from Ω^N to E' for each $a \in A$.

Using [10, Propositions 1.6 and 1.7], realize X as the section space of a Banach (space) bundle $q : F \to \Omega$. We claim that the maps from $F * F$ to A and B, respectively, defined by $(a, b) \mapsto {}_{A_{q(a)}}\langle a, b\rangle$ and $(a, b) \mapsto \langle a, b\rangle_{B_{q(b)}}$ are continuous. To see this, consider a convergent net $(a_i, b_i) \to (a, b)$ in $F * F$. Suppose $q(a_i) = q(b_i) = x_i$. Choose sections $\xi, \eta \in X$ such that $\xi(x) = a$ and $\eta(x) = b$ (where, of course, $q(a) = x = q(b)$). Then $\|a_i - \xi(x_i)\| \to 0$ as does $\|b_i - \eta(x_i)\|$. Thus

$$ {}_{A_{x_i}}\langle a_i, b_i\rangle \to {}_{A_x}\langle \xi(x), \eta(x)\rangle = {}_{A_x}\langle a, b\rangle . $$

(And similarly for $\langle \cdot, \cdot\rangle_{B_x}$.)

The hypotheses that $(x, n) \mapsto \mathfrak{t}^x_n \cdot a(x)$ and $(x, n) \mapsto a(x) \cdot \mathfrak{t}^x_n$ are continuous into E implies that $(x, n) \mapsto \xi(x) \cdot \mathfrak{t}^x_n$ is continuous into F. (Simply expand $\|\xi(x) \cdot \mathfrak{t}^x_n - \eta(x)\|^2$ in terms of $\langle \cdot, \cdot\rangle_{B_x}$.)

Finally, we may check continuity for $a \in A$ of the form $a(x) = {}_{A_x}\langle \xi(x), \eta(x)\rangle$. But then

$$ \mathfrak{s}^x_n \cdot a(x) = {}_{A_x}\langle \mathfrak{s}^x_n \cdot \xi(x), \eta(x)\rangle = {}_{A_x}\langle u^x_n(\xi(x)) \cdot \mathfrak{t}^x_n, \eta(x)\rangle , $$

which is continuous in view of the above discussion. A similar argument applies to $a(x)\mathfrak{s}_n^x$ and completes the proof of the lemma. □

Remark 3.12. Suppose that X is an A–B-equivalence bimodule and that A and B are both section algebras of C*-bundles over Ω, say determined by continuous open maps $P : \mathrm{Prim}(B) \to \Omega$ and $P' : \mathrm{Prim}(A) \to \Omega$. There is, unfortunately, no reason to suspect that $P' = P \circ h_X$ in general. However, note that the actions of A and B on X define left and right $C_0(\Omega)$ module actions on X (see [24, p. 187]). Then $P' = P \circ h_X$ exactly when we have $\phi \cdot x = x \cdot \phi$ for all $x \in X$ and $\phi \in C_0(\Omega)$.

Proof of Proposition 3.9. More or less by assumption, there is a covariant $C_0(\Omega)$-isomorphism of the double-dual system $((A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}, G, \hat{\hat{\alpha}})$ onto $(\mathrm{Ind}_{\Omega^{N\perp}}^{\hat{G}} D \rtimes_{\mathrm{Ind}\,\gamma} \hat{G}, G, (\mathrm{Ind}\,\gamma)^\wedge)$. However, it follows from [3, Corollary 2] that the double-dual system is Morita equivalent to (A, G, α). Therefore, $(\mathrm{Ind}_{\Omega^{N\perp}}^{\hat{G}} D \rtimes_{\mathrm{Ind}\,\gamma} \hat{G}, G, (\mathrm{Ind}\,\gamma)^\wedge)$ is Morita equivalent to (A, G, α). The proposition now follows from Lemmas 3.10 and 3.11. □

Proof of Theorem 3.1. It is a consequence of Propositions 3.8 and 3.9 as well as [5, Theorem 3] that conditions (1), (2), and (3) are equivalent. Thus all that remains is the final assertion.

So suppose that $\mathfrak{t}$ is a generalized Green twisting map for (A, G, α) with domain Ω^N, and that $\tilde{R}$ and $\hat{R}$ are as in Proposition 3.8. Then [5, Theorem 3] implies that the algebra B in (3) is simply $(A \rtimes_\alpha G)/I$, where $I = \ker\big(\hat{R}^{-1}(\{(x, 1) \in \Omega \times_{N^\perp} G\})\big)$. A moments reflection shows that it will suffice to show that $I = I^{\mathfrak{t}}$.

If $(x, \pi \times U)$ is an irreducible representation of $A_x \rtimes_\alpha G$, then

$$I = \bigcap\{\ker(x, \pi \times U) : x \in \Omega \text{ and } \ker(x, \pi \times U|_{N_x}) \supseteq \ker\big(\tilde{R}^{-1}(\{(x, 1)\})\big)\}\,.$$

But $\pi \times U$ preserves $\mathfrak{t}^x$ if and only if $\pi \times U|_{N_x}$ does. Although $\pi \times U|_{N_x}$ may not be irreducible, $\pi \times U|_{N_x}$ preserves $\mathfrak{t}^x$ if and only if every irreducible representation $\rho \times V$ of $A_x \rtimes_\alpha N_x$ which satisfies $\ker(\rho \times V) \supseteq \ker(\pi \times U|_{N_x})$ preserves $\mathfrak{t}^x$. But if $\ker(\rho \times V) \supseteq \ker(\pi \times U|_{N_x})$, then $\tilde{R}(\rho \times V) = (x, 1)$. Then Lemma 3.4 implies that $V = \rho \circ \mathfrak{t}^x$; that is, $\rho \times V$ preserves $\mathfrak{t}^x$. It follows that $I = I^{\mathfrak{t}}$ as claimed. □

4 Main Theorem

In this section (A, G, α) will be a *separable* abelian dynamical system. One of the basic invariants employed in the study of crossed products is the Connes spectrum $\Gamma(\alpha)$. In our situation, Gootman has shown [13, Lemma 4.1] that

$$\Gamma(\alpha) = \bigcap\{\hat{S}_P : P \text{ is separated in } \mathrm{Prim}(A \rtimes_\alpha G)\}\,,$$

where $\hat{S}_P$ denotes the stabilizer of P in $\hat{G}$. (Recall that a point p in a topological space X is called separated if for all $q \in X \backslash \overline{\{p\}}$, p and q have disjoint neighborhoods.) Actually, we will be most interested in the subgroup $\tilde{\tilde{\Gamma}}(\alpha)$ as defined in [20, Sect. 5]. Note that

$$\tilde{\tilde{\Gamma}}(\alpha) = \bigcap\{\hat{S}_P : P \in \mathrm{Prim}(A \rtimes_\alpha G)\}\ .$$

It is the fact that one can have $\tilde{\tilde{\Gamma}}(\alpha) \subsetneq \Gamma(\alpha)$ even for G-simple systems which makes characterizing simple crossed products so subtle (see, for example, [20, Theorems 5.5 and 5.7]); it also makes it necessary to employ $\tilde{\tilde{\Gamma}}$ rather than Γ in our next theorem (but see Remark 4.2).

Theorem 4.1. *Suppose that (A, G, α) is a separable abelian dynamical system. Then, with respect to the dual $\hat{G}$-action, $\mathrm{Prim}(A \rtimes_\alpha G)$ is a σ-trivial $\hat{G}$-space if and only if the following three conditions are satisfied.*

(1) The quasi-orbit space $\Omega = \mathcal{Q}^G(\mathrm{Prim}(A))$ is Hausdorff.

(2) If α^x is the action on the quotient A_x corresponding to $x \in \Omega$ and if $C_x = \tilde{\tilde{\Gamma}}(\alpha^x)^\perp$, then $C : \Omega \to \mathfrak{K}(G)$ is continuous.

(3) There is a generalized Green twisting map for (A, G, α) with domain Ω^C.

In this event, $\mathrm{Prim}(A \rtimes_\alpha G)$ is $\hat{G}$-homeomorphic to $\Omega \times_{C^\perp} G$.

Proof of necessity. In the case that $\mathrm{Prim}(A \rtimes_\alpha G)$ is a σ-trivial $\hat{G}$-space, then certainly $\mathcal{Q}^{\hat{G}}(\mathrm{Prim}(A \rtimes_\alpha G)) = \mathrm{Prim}(A \rtimes_\alpha G)/\hat{G}$ is Hausdorff. But $\Omega = \mathcal{Q}^G(\mathrm{Prim}(A))$ is homeomorphic to $\mathcal{Q}^{\hat{G}}(\mathrm{Prim}(A \rtimes_\alpha G))$ by [12, Corollary 2.5]. This establishes the necessity of (1).

Now fix $x \in \Omega$, and consider the system (A_x, G, α^x). Since $\mathcal{Q}^G(\mathrm{Prim}(A_x)) = \{\mathrm{pt}\}$, we employ [12] again to conclude that $\mathcal{Q}^{\hat{G}}(\mathrm{Prim}(A_x \rtimes_{\alpha^x} G)) = \{\mathrm{pt}\}$. Since $\mathrm{Prim}(A_x \rtimes_{\alpha^x} G)$ is a closed irreducible subset of a σ-trivial space, it must consist of a single orbit. Therefore there is a single stabilizer group $\hat{S}_x = \hat{S}_P$ for all $P \in \mathrm{Prim}(A_x \rtimes_{\alpha^x} G)$. Since $P \mapsto \hat{S}_P$ is continuous by assumption, so is $x \mapsto \hat{S}_x^\perp = C_x$ [34]. Condition (2) follows.

Using (1) and (2) above, we now have a continuous $\hat{G}$-homeomorphism $R : \mathrm{Prim}(A \rtimes_\alpha G) \to \Omega \times_{C^\perp} \hat{G}$. Now the necessity of (3) is a corollary of (2) $\Rightarrow$ (1) in Theorem 3.1. $\square$

Remark 4.2. In the proof of necessity, we actually showed that $C_x = \Gamma(\alpha^x)^\perp$ rather than $\tilde{\tilde{\Gamma}}(\alpha^x)^\perp$. This follows because $\mathrm{Prim}(A \rtimes_\alpha G)$, and therefore $\mathrm{Prim}(A_x \rtimes_{\alpha^x} G)$, is Hausdorff; hence [13, Lemma 4.1] implies that $\Gamma(\alpha^x) = \bigcap_{P \in \mathrm{Prim}(A_x \rtimes_{\alpha^x} G)} \hat{S}_P$ in this case.

Proof of sufficiency. Let $q : \mathrm{Prim}(A) \to \mathcal{Q}^G(\mathrm{Prim}(A))$ be the quasi-orbit map. Recall that q is continuous and open by [17, p. 221]. Therefore, (q, Ω) is a G-invariant regularization of (A, G, α), and by assumption, there is a generalized Green twisting map t with domain Ω^C for (A, G, α). By Proposition 3.8 (or (1) $\Rightarrow$ (2) of Theorem 3.1), there is an open σ-trivial regularization $\hat{R} : \mathrm{Prim}(A \rtimes_\alpha G) \to \Omega \times_{C^\perp} \hat{G}$ such that the composition of R with the projection $\mathrm{pr} : \Omega \times_{C^\perp} \hat{G} \to \Omega$ is the open surjection $P : \mathrm{Prim}(A \rtimes_\alpha G) \to \Omega$ arising from q via [35] and [19].

It will suffice to prove that $\hat{R}$ is a homeomorphism – which amounts to showing that $\hat{R}$ is injective. Since

$$\begin{tikzcd} \mathrm{Prim}(A) \arrow[rr, "\hat{R}"] \arrow[dr, "P"'] & & \Omega \times_{C^\perp} \hat{G} \arrow[dl, "\mathrm{pr}"] \\ & \Omega & \end{tikzcd}$$

commutes, it will suffice to consider a single fiber. Thus we fix $x \in \Omega$ and consider $\hat{R}_x : \mathrm{Prim}(A_x \rtimes_{\alpha^x} G) \to \hat{G}/C_x^\perp \cong \hat{C}_x$. Since $\hat{R}_x$ is $\hat{G}$-equivariant and surjective, it will suffice to show that $\mathrm{Prim}(A_x \rtimes_{\alpha^x} G)$ is homeomorphic to $\hat{C}_x$ as a $\hat{G}$-space.

Since $\mathfrak{t}^x$ is an (ordinary) Green twisting map for (A_x, G, α^x) with domain $C_x^\perp$, we have from [20] or Theorem 3.1 that $A_x \rtimes_{\alpha^x} G$ is covariantly isomorphic to $\mathrm{Ind}_{C^\perp}^{\hat{G}}(A_x \rtimes_{\alpha^x, \mathfrak{t}^x} G)$, where the action on $A_x \rtimes_{\alpha^x, \mathfrak{t}^x} G$ is the dual action. But $A_x \rtimes_{\alpha^x, \mathfrak{t}^x} G$ is simple by [20, Theorem 5.7]. An application of [5, Proposition 10] completes the proof. □

Recall that (A, G, α) is called *regular* [17, p. 223] if each quasi-orbit $Q \in \mathcal{Q}^G(\mathrm{Prim}(A))$ is locally closed, and there is a $P \in Q$ such that the map R_P defined by $s \cdot S_P \mapsto \alpha_s(P)$ defines a homeomorphism of G/S_P onto Q, S_P denoting the stabilizer of P. (This implies that the quasi-orbit space coincides with the orbit space.) Also recall that if A is type I and $\pi \in \hat{A}$, then the symmetrizer subgroup $\Sigma_{\ker \pi} = \Sigma_\pi$ [14, 18, 13] is the image in G of the center of the extension of $S_\pi = S_{\ker \pi}$ by $\mathbb{T}$ determined by the Mackey obstruction at π.

Corollary 4.3. *If A is type I and (A, G, α) is regular, then condition* (2) *in Theorem* 4.1 *can be replaced by*

(2)′: *The symmetrizer map $P \mapsto \Sigma_P$ is continuous, constant on quasi-orbits, and defines a continuous map $C : \Omega \to \mathfrak{K}(G)$.*

Proof. Since A is type I, the symmetrizer map will be constant on quasi-orbits by [13, Theorem 2.3]. Hence we obtain a well defined map $\Sigma : \Omega \to \mathfrak{K}(G)$. Thus we only have to check that $\hat{S}_P = \Sigma_{\hat{R}(P)}^\perp$ for each $P \in \mathrm{Prim}(A \rtimes_\alpha G)$, where $\hat{R} : \mathrm{Prim}(A \rtimes_\alpha G) \to \Omega$ denotes the canonical map. By the regularity assumption we know that $\mathrm{Prim}(A_x)$ is homeomorphic to G/S_J via $sS_J \mapsto \alpha_s(J)$, where J denotes any element in the quasi-orbit x.

So let $J \in x = \hat{R}(P)$. Then [17, Theorem 17] implies that $A_x \rtimes_{\alpha^x} G$ is (strongly) Morita equivalent to $(A_x/J) \rtimes_{\alpha^x} S_J$. Thus a typical primitive ideal in $A_x \rtimes_{\alpha^x} G$ is the kernel of a representation of the form $\mathrm{Ind}_{S_J}^G(\pi \times U)$ where $\pi \times U$ is an irreducible representation of $(A_x/J) \rtimes_{\alpha^x} S_J$. Furthermore, if $\hat{\alpha}^x$ is the dual action of $\hat{G}$ on $A_x \rtimes_{\alpha^x} G$ and β is the dual action of $(S_J)^\wedge$ on $(A_x/J) \rtimes_{\alpha^x} S_J$, then [25, Lemma 2.3] implies that

$$\mathrm{Ind}_{S_J}^G(\pi \times U) \circ (\hat{\alpha}_\gamma^x)^{-1} = \mathrm{Ind}_{S_J}^G(\pi \times \gamma|_{S_J} U) = \mathrm{Ind}_{S_J}^G\left((\pi \times U) \circ \beta_{\gamma|_{S_J}}^{-1}\right).$$

It follows that the isotropy group $\hat{S}_P$ for the dual action at $P = \ker\left(\mathrm{Ind}_{S_J}^G(\pi \times U)\right)$ is

$$\{\gamma \in \hat{G} : \gamma|_{S_J} \in \hat{S}_{\ker(\pi\times U)}\}, \tag{4.1}$$

where $\hat{S}_{\ker(\pi\times U)}$ is the stabilizer group for the dual action of $(S_J)^\wedge$ on $(A_x/J) \rtimes_{\alpha^x} S_J$ at $\ker(\pi \times U)$. But, since A is type I, $(A_x/J) \cong \mathcal{K}$ and it is well known that in this situation $\mathrm{Prim}\big((A_x/J) \rtimes_{\alpha^x} S_J\big)$ is $(S_J)^\wedge$-homeomorphic to $(\Sigma_J)^\wedge$ (see for instance [18]). This implies that $\hat{S}_P = \Sigma_J^\perp = \Sigma_{R(P)}^\perp$ which completes the proof. □

Remark 4.4. In a forthcoming paper [1] the first author shows that for any separable abelian system (A, G, α) such that A and $A \rtimes_\alpha G$ are type I the $\hat{G}$-stabilizer of a primitive ideal P of $A \rtimes_\alpha G$ is always equal to the annihilator of the symmetrizer $\Sigma_{\mathrm{Res}\,P}$, where $\mathrm{Res} : \mathrm{Prim}(A \rtimes_\alpha G) \to \mathcal{Q}^G\big(\mathrm{Prim}(A)\big)$ is the canonical restriction map as given by [17, Corollary 19]. Using this we see that Corollary 4.3 holds under the assumption that A and $A \rtimes_\alpha G$ are type I without assuming that (A, G, α) is regular.

A rather special case of regular actions, but one which is still of considerable interest, is the case where the induced G-action is trivial on the spectrum. In the type I case, our theorem specializes as follows.

Corollary 4.5. *Suppose that (A, G, α) is a separable abelian dynamical system such that A is type I and the action of G on* $\mathrm{Prim}(A)$ *is trivial. Then* $\mathrm{Prim}(A \rtimes_\alpha G)$ *is a σ-trivial $\hat{G}$-space if and only if the following three conditions are satisfied.*

(1) $\Omega = \mathrm{Prim}(A)$ *is Hausdorff.*

(2) *The symmetrizer map $\Sigma : \Omega \to \mathfrak{K}(G); x \mapsto \Sigma_x$ is continuous.*

(3) *The restriction of α to Ω^Σ defines a unitary subgroup action of Ω^Σ on A.*

In particular, if $\hat{A}$ is Hausdorff, then $(A \rtimes_\alpha G)^\wedge \cong_{\hat{G}} \Omega \times \hat{G}$ if and only if α is unitary.

Proof. Since a generalized Green twisting map is unitary on its domain (see Remark 2.8), the necessity of (1)–(3) follows from Theorem 4.1 and Corollary 4.3. So assume that (1)–(3) hold. If $\alpha|_{\Omega^\Sigma}$ is implemented by u, then, to apply Theorem 4.1, we only need to see that $\mathfrak{t}^x = u(x,\cdot)$ is an (ordinary) Green twisting map for the fiber $A_x \rtimes_{\alpha^x} G$ with domain Σ_x. But this follows from [6, Lemma 6].

Finally, the "only if" direction of the last assertion is immediate: if α is unitary, then $A \rtimes_\alpha G \cong A \otimes C^*(G)$. On the other hand, if $\mathrm{Prim}(A \rtimes_\alpha G) \cong \hat{A} \times \hat{G}$, then $\Sigma_x^\perp = \{1\}$, and $\Omega^\Sigma = \Omega \times G$. It is straightforward to check that α must be unitary if the restriction of α to Ω^Σ is unitary. □

It is interesting to consider a local version of Corollary 4.5. Recall that if α is locally unitary, then $(A \rtimes_\alpha G)^\wedge$ is a locally trivial principal $\hat{G}$-bundle over $\hat{A}$ [23]. Conversely, if $(A \rtimes_\alpha G)^\wedge$ is a locally trivial principal $\hat{G}$-bundle over $\hat{A}$,

[1] There are two "Lemma 4"s in [6]; presumably this reference will change

then we may apply Corollary 4.5 to the local trivializations to conclude that α is locally unitary. In summary, we have the following corollary.

Corollary 4.6. *Suppose that* (A, G, α) *is a separable abelian dynamical system with* $\hat{A}$ *Hausdorff. Then* $(A \rtimes_\alpha G)^\wedge$ *is a locally trivial principal* $\hat{G}$*-bundle over* $\hat{A}$ *if and only if* α *is locally unitary.*

More generally, we can consider the case where $\mathrm{Prim}(A \rtimes_\alpha G)$ is any principal bundle. Our result generalizes [27, Theorem 7.2]. Here we do not assume that A is type I. We omit the proof.

Corollary 4.7. *Suppose that* (A, G, α) *is a separable abelian dynamical system, and that* C *is a closed subgroup of* G. *Then, with respect to the dual action,* $\mathrm{Prim}(A \rtimes_\alpha G)$ *is a trivial* $\hat{C}$*-bundle (that is,* $\mathrm{Prim}(A \rtimes_\alpha G) \cong_{\hat{G}} \Omega \times \hat{C}$*) if and only if the following three conditions are satisfied.*

(1) *The quasi-orbit space* $\Omega = \mathcal{Q}^G(\mathrm{Prim}(A))$ *is Hausdorff.*

(2) *For all* $x \in \Omega$, $C = \tilde{\Gamma}(\alpha^x)$.

(3) *There is an (ordinary) Green twisting map* τ *for* (A, G, α) *with domain* C.

Let us finally remark that our results here have analogues in the case of abelian twisted covariant systems (A, G, α, τ) (i.e. G/N_τ is abelian). These can be obtained by passing to Morita equivalent ordinary actions by applying [3, Theorem 1]. To see that this works well, one only has to use the results about Morita equivalent twisted actions as presented in [3]. We omit the straightforward details.

Acknowledgement. The majority of the research for this article was conducted while the second author was a visitor at the University of Paderborn. The second author would like to thank the first author as well as Eberhard Kaniuth and his research group for their warm hospitality.

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