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Crossed Products by Actions Which Are Locally Unitary on the Stabilisers*

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Let (A, G, α) be a C^*-dynamical system with G abelian and $\hat{A}$ Hausdorff. We investigate the ideal structure of the crossed product $A \rtimes G$ under the hypothesis that the stabiliser subgroups for the action of G on $\hat{A}$ vary continuously. We discuss a new notion of locally trivial G-space for such actions, and, dually, actions α which are locally unitarily implemented on the stabiliser groups. Our main result asserts that, when α is locally unitary in this sense and $\hat{A}$ is a locally trivial G-space, $(A \rtimes G)\hat{}$ is a locally trivial $\hat{G}$-space.

1. INTRODUCTION

Let α be a strongly continuous action of a locally compact abelian group G on a C^*-algebra A. Our object here is to study the topologies on the spectrum and primitive ideal space of the crossed product C^*-algebra $A \rtimes_\alpha G$. When A is commutative, and hence isomorphic to $C_0(X)$ for some locally compact Hausdorff space X, $C_0(X) \rtimes_\alpha G$ is the transformation group C^*-algebra $C^*(G, X)$, and a complete description of the primitive ideal space $\mathrm{Prim}(C^*(G, X))$ has been given in [32]: roughly speaking, it is proved that Green's version of the Mackey machine [10] also describes the topology (see Proposition 4.8). In principle, this machine gives at least a set-wise description of $\mathrm{Prim}(A \rtimes_\alpha G)$, but there can be substantial com-

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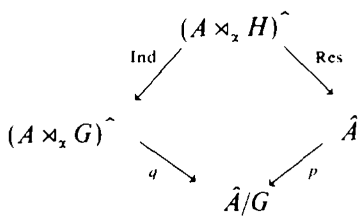
plications, even for the relatively innocuous looking algebra $C_0(X, \mathcal{K})$ of continuous functions from X to the compact operators (see, for example, [13, 22]).

The basic ingredients in the Mackey machine are the isotropy groups G_x for the action of G on $X = \text{Prim}(A)$, a family of Borel cocycles $\omega_x \in Z^2(G_x, \mathbb{T})$, called the Mackey obstructions, and the induction of ideals from $A \rtimes_\alpha G_x$ to $A \rtimes_\alpha G$. It was originally hoped that, at least when all the Mackey obstructions are cohomologically trivial, this procedure would also determine $\text{Prim}(A \rtimes_\alpha G)$ as a topological space—this is precisely what happens when A is commutative, and it also works when G acts freely on X [10, Theorem 24]. A particularly interesting example is the case $G = \mathbb{R}$ because $H^2(S, \mathbb{T}) = 0$ for all subgroups S of $\mathbb{R}$ [28]. However, the situation is more subtle even when the isotropy groups are constant and the Mackey obstructions vanish. The main results of [22] concern this case: when G is compactly generated and $\hat{A} \to \hat{A}/G$ is a locally trivial G/H-bundle over a reasonable space, there is a commutative diagram



(*)

where the southeast arrows are principal $\hat{H}$-bundles and the southwest arrows are principal G/H-bundles [22, Theorem 2.2]. The complication is that the bundles involved can be non-trivial—indeed, any $\hat{H}$-bundle can arise as the bottom left-hand arrow. When $G = \mathbb{R}$ and $H = \mathbb{Z}$, the diagram (*) consists of principal $\mathbb{T}$-bundles, and the possible non-triviality of q gave counterexamples to the conjecture in [28].

The above result does, however, give a two-step description of $(A \rtimes_\alpha G)^\wedge$ a the orbit space for an action of G on a principal bundle $(A \rtimes_\alpha H)^\wedge$ over $\hat{A}$. Our intention here is to extend the results of [22 Sect. 2] to actions where the isotropy groups G_x vary, and thus obtain a similar two-step description of $(A \rtimes_\alpha G)^\wedge$. Our results concern the case where the map $x \mapsto G_x$ is continuous from X to the space Σ_G of closed subgroups of G [5]. The crossed product $A \rtimes_\alpha H$ in (*) is replaced by a stabiliser algebra $A \rtimes_\alpha \mathcal{S}$, similar to the subgroup algebra of Fell [4] and the "algèbre de stabilisateurs" of Sauvageot [30]. As in [22, Sect. 2], we first prove a general version concerning primitive ideal spaces, under minimal hypotheses on the action (Theorem 3.10). We then formulate appropriate local triviality hypotheses, and prove that when they hold the diagram consists of bundles which are locally trivial in the required sense.

Our first task, then, is to prove existence and commutativity of the diamond in maximal generality. We have to construct restriction and induction maps on $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P})$, which reduce to the standard ones in the case of constant isotropy, and which have properties like those of the standard ones used in the proof of [22, Proposition 2.1]. We begin in Section 2 by discussing our stabiliser algebra $A \rtimes_\alpha \mathscr{P}$, constructing the restriction map $\mathrm{Res}\colon \mathrm{Prim}(A \rtimes_\alpha \mathscr{P}) \to \mathrm{Prim}(A)$, and establishing its main properties (Theorem 2.2). In Section 3, we discuss the induction of representations and ideals from $A \rtimes_\alpha \mathscr{P}$ to $A \rtimes_\alpha G$. This is done via a left $A \rtimes_\alpha G$- right $A \rtimes_\alpha \mathscr{P}$-rigged bimodule, in a way consistent with the usual induction of representations from crossed products by stabilisers; the main result, Theorem 3.1, has been phrased entirely in terms of the usual procedure. At the end of Section 3, we give the general version of our commutative diamond. In fact, our result is a bit stronger that the version in [22], even in the case of constant isotropy.

In Section 4, we discuss our notion of locally trivial space for non-free actions: a good motivating example is the case of a constant isotropy group H, where it makes perfectly good sense to describe the G-space X as a locally trivial G/H-bundle. We begin by looking at an analogue of proper actions of locally compact groups, where the isotropy map $x \mapsto G_x$ is assumed continuous, but where the groups G_x are typically not compact. The idea comes from [33], where it is shown that $C^*(G, X)$ has continuous trace if and only if the action is proper in this sense. A G-space X is called locally trivial if it is locally (over X/G) G-isomorphic to the quotient of $(X/G) \times G$ by the relation

$$(G \cdot x, s) \sim (G \cdot y, t) \Leftrightarrow G \cdot x = G \cdot y \quad \text{and} \quad st^{-1} \in G_x.$$

It turns out that a proper G-space is locally trivial exactly when $X \to X/G$ has local sections, so this does seem to be consistent extension of the usual notion, and we give a variety of examples as evidence that it is useful and interesting.

Ordinary locally trivial principal G-bundles can be dually realized in C^*-algebra theory as locally unitary actions of $\hat{G}$ [20]; in Section 5 we give a definition of "locally unitary on the stabilisers" which we intend to be dual to our more general locally trivial spaces. In Section 6, we show that if α is locally unitary on the stabilisers and $\hat{A} \to \hat{A}/G$ is locally trivial, then the diamond of Theorem 3.10 consists entirely of locally trivial spaces. We go on to discuss examples and special cases; in particular, we analyse in some detail what happens for actions of $\mathbb{R}$. As in [22], we actually obtain new information about transformation group C^*-algebras: if Y is a locally trivial $\mathbb{R}$-space, then $C^*(\mathbb{R}, Y)$ is a continuous trace algebra whose Dixmier–Douady class $\delta(C^*(\mathbb{R}, Y))$ vanishes if and only if Y is globally trivial.

2. The Stabiliser Algebra

Throughout this paper (A, G, α) will be a C^*-dynamical system with $\operatorname{Prim}(A)$ Hausdorff. In later sections we will usually want to require G to be abelian; however, in this section, unless stated otherwise, G can be any locally compact group. For convenience, we will put $X = \operatorname{Prim}(A)$ and view A as the collection of continuous sections vanishing at infinity, $\Gamma_0(\xi)$, of a C^*-bundle ξ over X. If $F \subseteq X$, then I_F will denote the ideal in A given by

$$\{a \in A: a(x) = 0 \text{ for all } x \in F\}.$$

When $F = \{x\}$, we will write I_x in place of $I_{\{x\}}$.

Recall that the action of G on A induces an action of G on X, defined by

$$s \cdot x = \{\alpha_s(a): a \in x\},$$

which makes (G, X) a topological transformation group [6, Lemma 1.3]. We will always assume that the stabiliser groups,

$$G_x = \{s \in G: s \cdot x = x\},$$

vary continuously; in other words, the map $\sigma: X \to \Sigma$ defined by $\sigma(x) = G_x$ is continuous, where Σ is the compact Hausdorff space of closed subgroups of G [5]. As in [33, p. 44], choose a Haar measure λ_H on each $H \in \Sigma$ so that for each $f \in C_0(G)$, the map

$$H \mapsto \int_G f(s)\, d\lambda_H(s)$$

is continuous (see also [6, p. 907]). For convenience, we will write λ_x instead of λ_{G_x}, and Δ_x will denote the modular function on G_x.

As a consequence of the continuity of σ,

$$\mathscr{P} = \{(s, x) \in G \times X: s \in G_x\}$$

is locally compact Hausdorff. Let $p: \mathscr{P} \to X$ be the projection on the second factor. Then we may form the associated pull-back C^*-algebra $p^*(A) = C_0(\mathscr{P}) \otimes_{C(X)} A$ ([23]), and by [23, Proposition 1.3] we have

$$p^*(A) \cong \Gamma_0(p^*\xi), \tag{2.1}$$

where $p^*\xi$ is the usual bundle pull-back. We now collect some technical observations which will be useful in the sequel.

Lemma 2.1. *Suppose that $\phi \in C_c(\mathscr{P})$ and that $a \in \Gamma_c(\xi)$. Then*

(i) $w(t, x) = \phi(t, x)\, a(x)$ *defines an element of* $\Gamma_c(p^*\xi)$ *and such elements span a dense subspace of* $\Gamma_c(p^*\xi)$ *with respect to the inductive limit topology,*

(ii) $r(t, x) = \phi(t, x)\, \alpha_t(a)(x)$ *defines an element of* $\Gamma_c(p^*\xi)$, *and*

(iii) *if* $c \in \Gamma_c(p^*\xi)$, *then* $b(x) = \int_{G_x} c(s, x)\, d\lambda_x(s)$ *defines an element of* $\Gamma_c(\xi)$.

Proof. The proof of (i) is standard and is essentially contained in [23, Proposition 1.3]. Parts (ii) and (iii) follow from straightforward approximation arguments. ∎

Notice that if $(s, x) \in \mathscr{S}$, then α_s induces an automorphism $\alpha(s, x)$ of the fibre A_x. We construct our stabiliser algebra following the lines of [30; 4; 33, Sect. 2]. If $f, g \in \Gamma_c(p^*\xi)$, then define

$$f * g(t, x) = \int_{G_x} f(v, x)\, \alpha(v, x)[g(v^{-1}t, x)]\, d\lambda_x(v),$$

$$f^*(t, x) = \Delta_x(t^{-1})\, \alpha(t, x)[f(t^{-1}, x)^*],$$

and

$$\|f\|_I = \sup_{x \in X} \left\{ \int_{G_x} \|f(t, x)\|\, d\lambda_x(t) \right\}.$$

Using Lemma 2.1 and [33, Lemma 2.5], it follows that $f * g$ and f^* belong to $\Gamma_c(p^*\xi)$. Furthermore, it is straightforward to check that $\Gamma_c(p^*\xi)$ has an approximate identity for the $\|\ \|_I$-norm; in fact, since $\{\lambda_x(K)\}_{x \in X}$ is bounded for any compact set $K \subseteq G$, it will suffice to produce an approximate identity in $\Gamma_c(p^*\xi)$ for the inductive limit topology. For this, notice that if N is a neighborhood of e in G, and K is compact in X, then there is a self-adjoint $\phi \in C_c^+(\mathscr{S})$ with $\{s \in G : \phi(s, x) \neq 0 \text{ for some } x\} \subseteq N$, and such that

$$\int_{G_x} \phi(s, x)\, d\lambda_x(s) = 1$$

for all $x \in K$ (cf. Lemma 3.3 in the next section). Thus, if $\{a_\gamma\}$ is a bounded approximate identity for $\Gamma_0(\xi)$, the collection of

$$\Phi_{(N, K, \gamma)}(s, x) = \phi(s, x)\, a_\gamma(x),$$

indexed by decreasing N and increasing K and γ, is a bounded approximate identity for $\Gamma_c(p^*\xi)$. (By Lemma 2.1, it suffices to check that $\Phi_{(N, K, \gamma)} * f$ converges to f for $f(s, x) = \psi(s, x)\, a(x)$, and one can do this by applying the usual compactness arguments, as in [23, Lemma 2.4].) The *stabiliser algebra*, $A \rtimes_\alpha \mathscr{P}$, is defined to be the enveloping C^*-algebra of the $\|\cdot\|_I$-norm completion of $\Gamma_c(p^*\xi)$, $L^I(\mathscr{P}, A)$. (Alternatively, it is evident that $\mathscr{P}$ is a locally compact groupoid [24] with unit space X and both range and source maps equal to p, and Haar system $\{\lambda_x\}_{x \in X}$. Then $A \rtimes_\alpha \mathscr{P}$ is the groupoid crossed-product of A by $\mathscr{P}$ as defined in [25].)

To state our main result on $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P})$ we shall need some further notation, which will be used throughout the paper. For a closed subset F of X, we set $A_F = A/I_F \cong \Gamma_0(\xi|_F)$, so that $\mathrm{Prim}(A_F)$ can be naturally identified with F. Let $\mathscr{P}_F = \{(s, x) \in \mathscr{P}: x \in F\}$. Then exactly the same procedure as we have just followed starting with $\Gamma_c(\xi|_F)$ gives us a C^*-algebra $A_F \rtimes_\alpha \mathscr{P}_F$. The restriction map $\pi_F: \Gamma_c(p^*\xi) \to \Gamma_c(p^*(\xi|_F))$ is $\|\cdot\|_I$-decreasing, and so extends to the completion $L^I(\mathscr{P}, A)$; composing a representation of $L^I(\mathscr{P}_F, A_F)$ with π_F gives a representation of $L^I(\mathscr{P}, A)$, and π_F is therefore decreasing for the C^*-norm too. Since π_F maps onto $\Gamma_c(p^*(\xi|_F))$, the extension, also denoted π_F, maps $A \rtimes_\alpha \mathscr{P}$ onto $A_F \rtimes_\alpha \mathscr{P}_F$. When $F = \{x\}$ consists of a single point, we write A_x for A/I_x, the fibre of ξ over x, and π_x for the canonical surjection of $A \rtimes_\alpha \mathscr{P}$ onto $A_x \rtimes_\alpha G_x$.

When G is abelian, the stabiliser algebra $A \rtimes_\alpha \mathscr{P}$ and all the analogously defined algebras $A_F \rtimes_\alpha \mathscr{P}_F$ carry a canonical *dual action* of $\hat{G}$. This is defined on $\Gamma_c(p^*\xi)$ by

$$\hat{\alpha}_\gamma(g)(s, x) = \overline{\gamma(s)}\, g(s, x);$$

each $\hat{\alpha}_\gamma$ is $\|\cdot\|_I$-isometric, and hence extends to an automorphism of $L^I(\mathscr{P}, A)$ and $A \rtimes_\alpha \mathscr{P}$. It is routine to check that $\hat{\alpha}$ then gives a strongly continuous action of $\hat{G}$ on $A \rtimes_\alpha \mathscr{P}$.

THEOREM 2.2. *Let (A, G, α) be a C^*-dyamical system with G abelian, $X = \mathrm{Prim}(A)$ Hausdorff, and continuous stabiliser map $x \mapsto G_x$. For $x \in X$, let ε_x denote the quotient map of A onto A_x. Then every primitive ideal of $A \rtimes_\alpha \mathscr{P}$ has the form* $\ker((\pi \times U) \circ \pi_x)$ *for some $x \in X$ and some irreducible representation $\pi \times U$ of $A_x \rtimes_\alpha G_x$, and*

$$\mathrm{Res}(\ker(\pi \times U) \circ \pi_x) = \ker(\pi \circ \varepsilon_x) = I_x \tag{2.2}$$

defines a continous open surjection Res *from* $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P})$ *to* $\mathrm{Prim}(A)$, *which is invariant under the dual action of $\hat{G}$. If A is type I,* Res *induces a homeomorphism of* $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P})/\hat{G}$ *onto $\hat{A}$.*

The proof of this theorem will occupy the rest of the section. For the moment, we do *not* assume that G is abelian. We shall construct the *restriction map* Res by defining a homomorphism R of A into the multiplier algebra $\mathscr{M}(A \rtimes_\alpha \mathscr{P})$, taking the induced map R^* on ideals, and showing that it has the required properties.

For $a \in A = \Gamma_0(\xi)$ and $g \in \Gamma_c(p^*\xi)$ we define

$$(R(a)g)(t, x) = a(x)g(t, x). \tag{2.3}$$

Since we then have $\|R(a)g\|_l \leqslant \|a\| \|g\|_l$, and $L^l(\mathscr{P}, A)$ has a $\|\cdot\|_l$-bounded approximate identity, we can extend (2.3) to g in the enveloping C^*-algebra $A \rtimes_\alpha \mathscr{P}$. We can then verify that (2.3) gives a $*$-homomorphism R of A into $\mathscr{M}(A \rtimes_\alpha \mathscr{P})$. (The right multiplication is defined by $(gR(a))(t, x) = g(t, x)\alpha_t(a)(x)$.) In the same manner we can define a natural $*$-homomorphism V of $C_0(X)$ into the center of $\mathscr{M}(A \rtimes_\alpha \mathscr{P})$, and the two maps are related by $R(\phi a) = V(\phi)R(a)$ for $\phi \in C_0(X)$, and $a \in A$.

LEMMA 2.3. *For $x \in X$, let J_x denote the ideal $\{f: f(x) = 0\}$ in $C_0(X)$. Then the kernel of $\pi_x: A \rtimes_\alpha \mathscr{P} \to A_x \rtimes_\alpha G_x$ coincides with*

$$V(J_x) \cdot (A \rtimes_\alpha \mathscr{P}) = \overline{\operatorname{span}}\{V(\phi)f: \phi \in J_x, f \in \Gamma_c(p^*\xi)\}.$$

Proof. It is clear that $K = V(J_x) \cdot (A \rtimes_\alpha \mathscr{P})$ is contained in $\ker(\pi_x)$. We shall prove the converse by showing that, if τ is a representation of $A \rtimes_\alpha \mathscr{P}$ with $K = \ker(\tau)$, then τ factors through π_x. So suppose that τ is such a representation. By a standard approximation argument, we can see that K contains any sections in $\Gamma_c(p^*\xi)$ which vanish on $p^{-1}(x)$. Thus if $f, g \in \Gamma_c(p^*\xi)$ satisfy $\pi_x(f) = \pi_x(g)$ we have $f - g \in K$ and hence $\tau(f) = \tau(g)$. Since $\pi_x: \Gamma_c(p^*\xi) \to C_c(G_x, A_x)$ is surjective, this means we can define a representation $\tilde{\tau}$ of $C_c(G_x, A_x)$ by $\tilde{\tau}(\pi_x(f)) = \tau(f)$. Further, for any $\phi \in C_c(X)$ with $\phi(x) = 1$ we have

$$\|\tilde{\tau}(\pi_x(f))\| = \|\tau(f)\| = \|\tau(\phi f)\| \leqslant \|\phi f\|$$

$$\leqslant \sup_{y \in X} |\phi(y)| \int_{G_y} \|f(t, y)\| \, d\lambda_y(t).$$

By [33, Lemma 2.5], the right-hand side is a continuous function of y for any $\phi \in C_c(X)$, so by letting the support of ϕ shrink to a small neighborhood of x we obtain

$$\|\tilde{\tau}(\pi_x(f))\| \leqslant \int_{G_x} \|f(t, x)\| \, d\lambda_x(t).$$

Thus $\tilde{\tau}$ is bounded in the L^l-norm on $C_c(G_x, A_x)$, and therefore extends to a representation of $A_x \rtimes_\alpha G_x$ satisfying $\tau = \tilde{\tau} \circ \pi_x$. Since $K = \ker(\tau)$, this implies $K \supseteq \ker(\pi_x)$. ∎

Now let ρ be an irreducible representation of $A \rtimes_\alpha \mathscr{P}$; we also use the same letter for the canonical extension of ρ to $\mathscr{M}(A \rtimes_\alpha \mathscr{P})$. Observe that V is non-degenerate in the sense that if $\{e_i\}$ is an approximate identity for

$C_0(X)$, then $V(e_i)$ converges to 1 strictly, and that the range of V is contained in the center of $\mathcal{M}(A \rtimes_\alpha \mathcal{P})$. Thus $\rho \circ V$ is a non-trivial homomorphism with values in $\mathbb{C}1$, and hence is given by evaluation at some point x of X. It follows from the lemma and the Cohen factorization theorem that every element of $\ker(\pi_x)$ has the form $V(\phi)f$ for some $\phi \in J_x$, so $\ker(\rho) \supseteq \ker(\pi_x)$ and ρ has the form $(\pi \times U) \circ \pi_x$ as claimed.

To see that (2.2) gives a well-defined continuous map Res on ideals, we consider the map $R^*: \mathcal{I}(A \rtimes_\alpha \mathcal{P}) \to \mathcal{I}(A)$ defined by

$$R^*(J) = \{a \in A: R(a) \cdot (A \rtimes_\alpha \mathcal{P}) \subseteq J\},$$

which is continuous by [10, Proposition 9]. With the notation of the previous paragraph, we have

$$\begin{aligned} R^*(\ker(\rho)) &= \{a \in A: (\pi \times U) \circ \pi_x(R(a)f) = 0 \text{ for } f \in A \rtimes_\alpha \mathcal{P}\} \\ &= \{a \in A: \pi(a(x))((\pi \times U) \circ \pi_x)(f) = 0 \text{ for } f \in A \rtimes_\alpha \mathcal{P}\} \\ &= \ker(\pi \circ \varepsilon_x); \end{aligned}$$

note that because $\mathrm{Prim}(A)$ is Hausdorff, $\ker(\pi \circ \varepsilon_x) = I_x$ is primitive. In particular, our restriction map Res is well defined and continuous, with values in $\mathrm{Prim}(A)$. It is clearly surjective, and when G is abelian, a routine computation shows that

$$(\pi \times U) \circ \pi_x \circ \hat{\alpha}_\gamma^{-1} = (\pi \times \chi|_{G_x} U) \circ \pi_x,$$

so it is $\hat{G}$-invariant as well.

Next we shall prove that Res is open. Our proof of this will depend on the construction of a unified induction process from ideals of A to ideals of $A \rtimes_\alpha \mathcal{P}$; this will be done using a left $A \rtimes_\alpha \mathcal{P}$- right A-rigged bimodule, so that it is automatically continuous. We begin with some general results on inducing ideals via bimodules, which will also be required in the next section.

Suppose that Y is a A-rigged space [26, Definition 2.1]. Let $\mathcal{K}(Y)$ denote the imprimitivity algebra associated to Y so that Y is a $\mathcal{K}(Y)$-A-imprimitivity bimodule. Hence there is a lattice preserving homeomorphism h of $\mathcal{I}(A)$ onto $\mathcal{I}(\mathcal{K}(Y))$ [27, Theorem 3.1]. By [10, Lemma 16], $\mathcal{M}(\mathcal{K}(Y))$ is isomorphic to the algebra $\mathcal{L}(Y)$ of A-linear, adjointable operators on Y. If in addition, Y is a left D-module in such a way that d^* acts as the adjoint of $d \in D$, then there is a $*$-homomorphism S of D into $\mathcal{M}(\mathcal{K}(Y))$, and hence a continuous intersection preserving map S^* of $\mathcal{I}(\mathcal{K}(Y))$ into $\mathcal{I}(D)$. We will call such a Y a left D- right A-rigged bimodule and write $\mathfrak{I}_Y$ for the continuous intersection preserving map $S^* \circ h$ of $\mathcal{I}(A)$ into $\mathcal{I}(D)$.

LEMMA 2.4. *Let Y be an E-A-imprimitivity bimodule and let $h: \mathcal{I}(A) \to \mathcal{I}(E)$ be Rieffel's lattice preserving homeomorphism. Then for $J \in \mathcal{I}(A)$ we have*

$$h(J) = \{e \in E: \langle e \cdot x, y \rangle_A \in J \text{ for all } x, y \in Y\}.$$

Proof. Suppose that $e \in h(J)$ and that $x, y \in Y$. The submodule corresponding to $h(J)$ is by definition Y_J, and equals $h(J) \cdot Y$ by [27, Lemma 3.1], so $e \cdot x \in Y_J$ and $\langle e \cdot x, y \rangle_A \in J$ by [27, Theorem 3.1].

Conversely, suppose that $\langle e \cdot x, y \rangle_A \in J$ for all $x, y \in Y$. Then we have

$$\langle z, e \cdot x \rangle_E \cdot y = z \cdot \langle e \cdot x, y \rangle_A \in Y \cdot J$$

for all $x, y, z \in Y$. Thus,

$$\langle z, e \cdot x \rangle_E \langle y, w \rangle_E = \langle \langle z, e \cdot x \rangle_E \cdot y, w \rangle_E \in h(J)$$

for all $x, y, z, w \in Y$, which implies

$$\langle z, e \cdot x \rangle_E f \in h(J)$$

for all $z, x \in Y$ and $f \in E$. Letting f run through an approximate identity, we see this can happen if and only if

$$e \langle x, z \rangle_E = (\langle z, e \cdot x \rangle_E)^* \in h(J)$$

for all $x, z \in X$, and hence if and only if $ef \in h(J)$ for all $f \in E$. Thus, $e \in h(J)$. ∎

COROLLARY 2.5. *Let Y be a left D- right A-rigged bimodule and $\mathcal{I}_Y: (A) \to \mathcal{I}(D)$ the map defined above. Then for $K \in \mathcal{I}(A)$ we have*

$$\mathfrak{I}_Y(K) = \{d \in D: \langle d \cdot x, y \rangle_A \in K \quad \text{for all } x, y \in Y\}.$$

Proof. By definition,

$$\mathfrak{I}_Y(K) = S^* \circ h(K) = \{d \in D: S(d) e \in h(K) \text{ for all } e \in \mathcal{K}(Y)\}.$$

Thus, by the lemma (with $E = \mathcal{K}(Y)$),

$$\begin{aligned} \mathfrak{I}_Y(K) &= \{d \in D: \langle S(d) e \cdot z, y \rangle_A \in K \quad \text{for all } e \in \mathcal{K}(Y), z, y \in Y\} \\ &= \{d \in D: \langle d \cdot x, y \rangle_A \in K \quad \text{for all } x, y \in Y\}, \end{aligned}$$

since $E \cdot Y$ spans Y. ∎

Let Y_0 be $\Gamma_c(p^*\xi)$ (without the $*$-algebraic structure). Using Lemma 2.1, we see that

$$\langle f, g \rangle_A (x) = \int_{G_x} \alpha(s, x) [f(s^{-1}, x)^* g(s^{-1}, x)] \, d\lambda_x(s)$$

defines an A-valued sesqui-linear form on Y_0. Similarly, if $a \in A$ and $g \in Y_0$, then

$$g \cdot a(t, x) = g(t, x)\, \alpha(t, x)[a(x)]$$

defines a right A-action on Y_0 which satisfies

$$\langle f, g \cdot a \rangle_A = \langle f, g \rangle_A\, a.$$

LEMMA 2.6. *With the above definitions, Y_0 is a right A-rigged space.*

Proof. It is clear that

$$\langle f, g \rangle_A^* = \langle g, f \rangle_A,$$

so it will suffice to show that $\langle Y_0, Y_0 \rangle_A$ is dense in A, and that $\langle f, f \rangle_A \geqslant 0$. However,

$$\langle f, g \rangle_A (x) = \langle f_x, f_x \rangle_{A_x},$$

where $\langle \cdot, \cdot \rangle_{A_x}$ is the A_x-valued inner-product defined on the left $C_c(C_x, A_x)$- right A_x-rigged bimodule $C_c(G_x, A_x)$ defined in [10, Sect. 2], and f_x, g_x are the obvious restrictions of f and g in $C_c(G_x, A_x)$. Thus for each $x \in X$,

$$\langle f, f \rangle_A (x) \geqslant 0 \qquad \text{and} \qquad \langle Y_0, Y_0 \rangle_A (x) \neq \{0\}.$$

Therefore, $\langle f, f \rangle_A \geqslant 0$, and since $\langle Y_0, Y_0 \rangle_A$ is an ideal in A, $\langle Y_0, Y_0 \rangle_A$ is dense. ∎

Furthermore, there is a left action of $\Gamma_c(p^*\xi)$ on Y_0 defined by convolution:

$$f \cdot g = f * g.$$

Again, $f \cdot g(t, x) = f_x * g_x(t)$, and since every irreducible representation of $A_x \rtimes_\alpha G_x$ lifts to $A \rtimes_\alpha \mathscr{P}$, we have

$$\begin{aligned} \langle f \cdot g, f \cdot g \rangle_A (x) &= \langle f_x * g_x, f_x * g_x \rangle_{A_x} \\ &\leqslant \|f_x\|^2_{A_x \rtimes_\alpha G_x} \langle g_x, g_x \rangle_{A_x} \\ &\leqslant \|f\|^2_{A \rtimes_\alpha \mathscr{P}} \langle g, g \rangle_A (x). \end{aligned}$$

Since this action is clearly A-linear and adjointable, we obtain a $*$-homomorphism

$$W: A \rtimes_\alpha \mathscr{P} \to \mathscr{L}_A(Y),$$

where Y is the completion of Y_0 (cf. [26, Sect. 3]). In other words, Y is a left $A \rtimes_\alpha \mathscr{P}$- right A-rigged bimodule. We shall be interested in the inducing map $\mathfrak{I}_Y$ defined by Y.

LEMMA 2.7. *Let $x \in X$ and suppose that G_x is amenable. Then with the above notation, $\mathfrak{I}_Y(I_x) = \ker(\pi_x)$.*

Proof. We first observe that if $f \in \Gamma_c(p^*\xi)$ and $\pi_x(f) = 0$, then

$$\langle f \cdot y, z \rangle_A (x) = 0, \qquad \text{for all} \quad y, z \in Y_0.$$

It therefore follows from Corollary 2.5 and Lemma 2.3 that $\ker(\pi_x) \subseteq \mathfrak{I}_Y(I_x)$. To prove the converse, we relate Y to the A_x-rigged bimodule V_x used for inducing from A_x to $A_x \rtimes_\alpha G_x$ [10, p. 200]. Define $r_x: Y_0 \to C_c(G_x, A_x)$ by $r_x(y)(s) = y(s, x)$; then it is easy to check that for $f \in \Gamma_c(p^*\xi)$, $y, z \in Y_0$ we have

$$\langle f \cdot y, z \rangle_A (x) = \langle \pi_x(f) \cdot r_x(y), r_x(z) \rangle_{A_x}. \tag{2.4}$$

The map r_x is continuous from $(Y_0, \|\cdot\|_A)$ to $(V_x, \|\cdot\|_{A_x})$, and has dense range, so it extends to continuous map of Y onto V_x. The module actions are continuous, and we can therefore extend (2.4) to $f \in A \rtimes_\alpha \mathscr{P}$ and $y, z \in Y$. In particular, this shows that for $f \in \mathfrak{I}_Y(I_x)$, $\pi_x(f)$ belongs to the ideal $\mathrm{Ind}_e^{G_x}\{0\}$ in $A_x \rtimes_\alpha G_x$ induced in the usual way from the zero ideal of A_x. Since G_x is amenable, $\mathrm{Ind}\{0\} = \{0\}$ [10, Proposition 13], and we have proved that $\mathfrak{I}_Y(I_x) \subseteq \ker(\pi_x)$. ∎

PROPOSITION 2.8. *Suppose that (A, G, α) is a C^*-dynamical system with* $\mathrm{Prim}(A)$ *Hausdorff and such that $x \mapsto G_x$ is continuous. If for each $x \in X$, G_x is amenable, then* Res *is a continuous open map of* $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P})$ *onto* $\mathrm{Prim}(A)$.

Proof. At this point, we have only to prove that Res: $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P}) \to X$ is open. Let K be an ideal in $A \rtimes_\alpha \mathscr{P}$, so that

$$\mathscr{O} = \{P \in \mathrm{Prim}(A \rtimes_\alpha \mathscr{P}) : P \not\supseteq K\}$$

is a typical basic open set in $A \rtimes_\alpha \mathscr{P}$, and suppose we have a convergent net $x_i \to x$ such that $x_i \notin \mathrm{Res}\,\mathscr{O}$ for all i. If $x \in \mathrm{Res}\,\mathscr{O}$, then we can find a primitive ideal $M \in \mathrm{Res}^{-1}(x)$ such that $M \in \mathscr{O}$. Now $M \supseteq \ker(\pi_x)$, and then we must have $\ker(\pi_x) \not\supseteq K$. By Lemma 2.7, this implies $\mathfrak{I}_Y(I_x) \not\supseteq K$. Since $\mathfrak{I}_Y$ is continuous and $x_i \to x$, we eventually have $\ker(\pi_{x_i}) = \mathfrak{I}_Y(I_{x_i}) \not\supseteq K$. But then there is a primitive ideal P with $P \not\supseteq K$ and with $P \supseteq \ker(\pi_{x_i})$—in other words, $P \in \mathscr{O}$ and Res $P = x_i$. This is a contradiction, so we must have $x \notin \mathrm{Res}\,\mathscr{O}$ and $\mathrm{Res}\,\mathscr{O}$ is open. ∎

Since when G is abelian, Res is $\hat{G}$-invariant, it follows from the above proposition that Res defines a continuous open surjection of $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P})/\hat{G}$ onto $\mathrm{Prim}(A)$. If in addition, A is type I, $A_x \cong \mathscr{K}$ and $A_x \rtimes_\alpha G_x$ is stably isomorphic to a twisted group algebra $C^*(G_x, \omega_x)$ via an isomorphism which preserves the dual actions of $\hat{G}$ [10, Theorem 18]. It follows from [10, Proposition 34] that $\hat{G}$ acts transitively on $\mathrm{Prim}(C^*(G_x, \omega_x))$, and hence also on $\mathrm{Prim}(A_x \rtimes_\alpha G_x)$. Thus Res induces a bijection of the orbit space $\mathrm{Prim}(A \rtimes_\alpha \mathscr{P})/\hat{G}$ onto $\hat{A}$, which is a homeomorphism because Res is both continuous and open. This completes the proof of Theorem 2.2. ∎

Remark 2.9. It is not clear if amenability of the G_x is required for Proposition 2.8 (although Lemma 2.7 is false without assuming amenability, as can be seen from the case $A = \mathbb{C}$ and G any non-amenable group). At this writing, we know of no counterexamples.

3. Inducing Representations from the Stabiliser Algebra

We will retain our notation as well as our assumptions from the previous section. In particular, given $s \in G$ and $x \in X$, there is a $*$-isomorphism of A_x onto $A_{s \cdot x}$ defined, given $a \in \Gamma_0(\xi)$, by $a(x) \mapsto \alpha_s(a)(s \cdot x)$. Writing $\alpha(s, s \cdot x)$ for this isomorphism, we have

$$\alpha_s(a)(x) = \alpha(s, x)[a(s^{-1} \cdot x)].$$

If G is abelian, so that $G_x = G_{s^{-1} \cdot x}$, then, for each $f \in \Gamma_c(p^*\xi)$, we can define

$$\beta_s(f)(t, x) = \alpha(s, x)[f(t, s^{-1} \cdot x)].$$

Using Lemma 2.1(i), it follows that $(A \rtimes_\alpha \mathscr{P}, G, \beta)$ is a C^*-dynamical system.

Let ε_x and π_x be the natural projections of A and $A \rtimes_\alpha \mathscr{P}$ onto A_x and $A_x \rtimes_\alpha G_x$, respectively. Our object in this section is to prove:

THEOREM 3.1. *Let A be a separable C^*-algebra with Hausdorff primitive ideal space X and α an action of a second countable locally compact abelian group G on A with continuously varying stabiliser groups. Then every irreducible representation of $A \rtimes_\alpha \mathscr{P}$ has the form $(\pi \times U) \circ \pi_x$ for some $x \in X$ and $\pi \times U \in (A_x \rtimes_\alpha G_x)^\wedge$. Moreover, the map*

$$\ker((\pi \times U) \circ \pi_x) \mapsto \ker(\mathrm{Ind}_{G_x}^G((\pi \circ \varepsilon_x) \times U))$$

defines a homeomorphism of the quasi-orbit space $\mathcal{Q}(\operatorname{Prim}(A \rtimes_\alpha \mathscr{P}), G, \beta)$ *onto the primitive ideal space of* $A \rtimes_\alpha G$, *which is equivariant for the dual actions of* $\hat{G}$.

Our strategy for proving this theorem will be as follows. We know the result is true when the stabiliser map is *constant* [22, Proposition 2.1]; the inverse in this case is Green's restriction map [10; Proposition 9]. In general, the stabiliser map is continuous on a Hausdorff space and constant on orbits, hence constant on orbit closures. We will construct continuous induction and restriction maps between $A \rtimes_\alpha \mathscr{P}$ and $A \rtimes_\alpha G$, which are consistent with the usual ones on the quotients of A corresponding to orbit closures, and then deduce the desired properties from known facts about these quotients.

We begin by recalling what is known when the stabilisers are constant. Throughout this section we shall assume that (A, G, α) satisfies the hypotheses of the theorem.

LEMMA 3.2. *Suppose that the stabilisers of points in* X *are all equal to a fixed closed subgroup* H *in* G. *Then induction gives a continuous map* Ind *of* $\operatorname{Prim}(A \rtimes_\alpha H)$ *onto* $\operatorname{Prim}(A \rtimes_\alpha G)$ *which defines a homeomorphism of the quasi-orbit space* $\mathcal{Q}(\operatorname{Prim}(A \rtimes_\alpha H))$ *onto* $\operatorname{Prim}(A \rtimes_\alpha G)$; *the inverse is given by restriction, and we have*

$$\operatorname{Res}(\operatorname{Ind}(M)) = \bigcap_{s \in G} s \cdot M$$

for all $M \in \operatorname{Prim}(A \rtimes_\alpha H)$.

Proof. This follows by applying Theorem 24 of [10] to the essentially free system $(G, A \rtimes_\alpha H, \tau^H)$, as in the second paragraph of the proof of [22, Proposition 2.1]. As remarked there, practically all that is needed to make this work is Effros–Hahn regularity of $(G, A \rtimes_\alpha H, \tau^H)$, which is automatic since G is amenable [8]. ∎

We now want to define a continuous map Ind from $\mathscr{I}(A \rtimes_\alpha \mathscr{P})$ to $\mathscr{I}(A \rtimes_\alpha G)$. As in the previous section we will do this by first constructing a left $A \rtimes_\alpha G$- right $A \rtimes_\alpha \mathscr{P}$-rigged bimodule Y. As usual, we start with dense subspaces of the algebras and modules in question, and pass to completions later.

Let $B_0 = \Gamma_c(p^*\xi) \subseteq A \rtimes_\alpha \mathscr{P}$. View $G \times \xi$ as a bundle over $G \times X$ and let $Y_0 = \Gamma_c(G \times \xi)$. The module action of B_0 on Y_0 and the B_0-valued inner product will be given by

$$f \cdot b(s, x) = \int_{G_x} f(st^{-1}, x)\, \alpha(st^{-1}, x)[b(t, s^{-1} \cdot x)]\, d\lambda_x(t) \tag{3.1}$$

$$\langle f, g \rangle_B (t, x) = \int_G \alpha(s, x)[f(s^{-1}, s^{-1} \cdot x)^* g(s^{-1}t, s^{-1} \cdot x)]\, ds. \tag{3.2}$$

Since it is not immediately clear that these formulas define continuous sections, we shall outline a proof that $f \cdot b$ belongs to $\Gamma_c(G \times \xi)$, and merely observe that similar arguments show that $\langle f, g \rangle_B \in B_0$.

Suppose that $f \in Y_0$ and $b \in B_0$. Both sections have compact support, and hence if f_1 and b_1 uniformly approximate f and b then $f_1 \cdot b_1$ will uniformly approximate $f \cdot b$. Because the formula is bilinear, we may therefore suppose that $f = \phi \otimes a$, $b = \psi \otimes c$ for some $\phi \in C_c(G)$, $\psi \in C_c(\mathscr{P})$, and $a, c \in \Gamma_c(\xi)$. Then

$$f \cdot b(s, x) = \int_{G_x} \phi(st^{-1}, x)\, \psi(t, s^{-1} \cdot x)\, a(x)\, \alpha(st^{-1}, x)[c(s^{-1} \cdot x)]\, d\lambda_x(t).$$

The function Ξ, defined by

$$(s, t, x) \mapsto \phi(st^{-1}, x)\, \psi(t, s^{-1} \cdot x),$$

is continuous and compactly supported on $G \times \mathscr{P}$, and hence uniformly approximable by a finite tensor in $C_c(G) \otimes C_c(\mathscr{P})$. Thus we can replace Ξ by an elementary tensor $\zeta \otimes \eta \in C_c(G) \otimes C_c(\mathscr{P})$, and then we have

$$\begin{aligned} f \cdot b &= \zeta(s)\, a(x) \int_{G_x} \eta(t, x)\, \alpha_{st^{-1}}(c)(x)\, d\lambda_x(t) \\ &= \zeta(s)\, a(x)\, \alpha_s \left(\int_{G_x} \eta(t, x)\, \alpha_{t^{-1}}(c)(x)\, d\lambda_x(t) \right). \end{aligned}$$

By Lemma 2.1, the last integral defines an element d in $\Gamma_c(\xi)$, and the formula

$$f \cdot b(s, x) = \zeta(s)\, a(x)\, \alpha_s(d)(x)$$

defines a continuous function from G to $\Gamma_c(\xi)$—in other words, a continuous section of $G \times \xi$ as required.

It is now straightforward to verify that the formula for $f \cdot b$ does define a module action of B_0 on Y_0, and that

$$\langle f, g \cdot b \rangle_B = \langle f, g \rangle_B * b.$$

We still need to check the positivity of the inner products, and that the range of the inner products spans a dense subspace of $A \rtimes_\alpha \mathscr{P}$. For the first of these, it will be enough to prove that

$$\pi_x(\langle f, f \rangle_B) \geq 0$$

in $A_x \rtimes_\alpha G_x$ for every $f \in Y_0$ and $x \in X$. Let $F = \overline{G \cdot x}$, let $\langle \cdot, \cdot \rangle_F$ denote the $C_c(G_x, A_F)$-valued inner product on $C_c(G, A_F)$ constructed in [10, p. 200], and define $r_F: Y_0 \to C_c(G, A_F)$ by $r_F(f)(s) = \pi_F(f(s))$. Then it is easy to verify that

$$\pi_F(\langle f, f \rangle_B) = \langle r_F(f), r_F(f) \rangle_F,$$

which is positive in $A_F \times G_x$ by [10, p. 202]. Since π_x factors through π_F, this proves that $\pi_x(\langle f, f \rangle_B)$ is positive in $A_x \rtimes_\alpha G_x$, and hence that $\langle f, f \rangle_B$ itself is positive.

The density will follow in the standard way from the existence of a suitable approximate identity:

LEMMA 3.3. *There is an approximate identity for $A \rtimes_\alpha \mathscr{P}$ consisting of elements of the form $\langle f, f \rangle_B$ with $f \in Y_0$.*

Proof. By virtue of the remarks following Lemma 2.1, it will suffice to produce, for each relatively compact neighborhood U of e in G and each compact set $C \subseteq X$, a non-negative function $F \in C_c(\mathscr{P})$ such that

$$\phi_{(U,C)}(t, x) = \int_G F(s^{-1}, s^{-1} \cdot x)\, F(s^{-1}t, s^{-1} \cdot x)\, ds$$

vanishes for t outside U, and satisfies

$$\int_{G_x} \phi_{(U,C)}(t, x)\, d\lambda_x(t) = 1$$

for $x \in C$. Once we have constructed F, then if $\{a_\gamma\}$ is an approximate identity for $\Gamma_0(\xi)$ and we define $f(s, x) = \phi_{(U,C)}(s, x)\, a_\gamma^{1/2}(x)$, it follows that $\Psi_{(U,C,\gamma)} = \langle f, f \rangle_B = \phi_{(U,C)} a_\gamma$ will be an approximate identity for the inductive limit topology, and hence for $A \rtimes_\alpha \mathscr{P}$ as well.

To construct F, we choose a symmetric neighborhood V of e with $V^2 \subseteq V$ and $g \in C_c(G)$ such that $g \geq 0$, $\operatorname{supp}(g) \subseteq V$, and $g \neq 0$. If in addition, $\psi \in C_c(X)$ is such that $\psi \equiv 1$ on $U \cdot C$, then

$$F(s, x) = \psi(x)^{1/2}\, g(s) \left(\int_{G_x} \int_G g(r^{-1})\, g(r^{-1}t)\, dr\, d\lambda_x(t) \right)^{-1/2}$$

has the required properties. This proves the lemma. ∎

We have now proved:

PROPOSITION 3.4. *Let Y denote the completion of Y_0 in the (semi-) norm defined by the inner product* (3.2). *Then with the module action given by* (3.1), *and the inner product* (3.2), *Y is a right $A \rtimes_\alpha \mathscr{P}$-rigged space.*

The next step is to define a left action of $A \rtimes_\alpha G$ on Y. For $f \in C_c(G, A)$ and $g \in Y_0$, we define

$$f \cdot g(t, x) = \int_G f(s, x)\, \alpha(s, x)[g(s^{-1}t, s^{-1} \cdot x)]\, ds. \tag{3.3}$$

This is the usual formula for multiplication in $C_c(G, A)$, so it certainly defines an element of Y_0, and gives a module action of $C_c(G, A)$ on Y_0.

PROPOSITION 3.5. (1) *The formula* (3.3) *defines a left action of* $A \rtimes_\alpha G$ *as adjointable* $A \rtimes_\alpha \mathscr{P}$*-linear operators on the rigged space* Y.

(2) *Suppose that* F *is a closed* G*-invariant subset of* X *on which all stabilisers are equal to* H *(i.e.,* $G_x = H$ *for all* $x \in F$*), and let* Y^F *be the (complete) left* $A_F \rtimes_\alpha G$*- right* $A_F \rtimes_\alpha H$*-rigged space constructed in* [10, Sect. 2]. *Let* ρ_F *and* π_F *denote the natural surjections of* $A \rtimes_\alpha G$ *and* $A \rtimes_\alpha \mathscr{P}$ *onto* $A_F \rtimes_\alpha G$ *and* $A_F \rtimes_\alpha H$. *Then the restriction of sections from* $\mathscr{P}$ *to* $p^{-1}(F)$ *extends to a surjection* r_F *of* Y *onto* Y^F *such that*

(a) $r_F(f \cdot b) = r_F(f) \cdot \pi_F(b)$, *for* $f \in Y$ *and* $b \in A \rtimes_\alpha \mathscr{P}$;

(b) $\pi_F(\langle f, g \rangle_B) = \langle r_F(f), r_F(g) \rangle_{A_F \rtimes_\alpha H}$, *for* $f, g \in Y$;

(c) $r_F(f \cdot g) = \rho_F(f) \cdot r_F(g)$, *for* $f \in A \rtimes_\alpha G$ *and* $g \in Y$.

Proof. The usual boring calculations show that

$$\langle f \cdot g, h \rangle_{A \rtimes_\alpha \mathscr{P}} = \langle g, f^* \cdot h \rangle_{A \rtimes_\alpha \mathscr{P}}$$

for $g, h \in Y_0$ and $f \in C_c(G, A)$, that

$$(f \cdot g) \cdot b = f \cdot (g \cdot b)$$

for $f \in C_c(G, A)$, $g \in Y_0$, and $b \in B_0$, and that the three properties (a), (b), and (c) hold, at least for continuous sections of compact support. To establish (1) it remains only to check that the left action satisfies

$$\langle f \cdot g, f \cdot g \rangle_{A \rtimes_\alpha \mathscr{P}} \leqslant \|f\|^2 \langle g, g \rangle_{A \rtimes_\alpha \mathscr{P}}. \tag{3.4}$$

Let $f \in C_c(G, A)$, $g \in Y_0$, $x \in X$, and $F = \overline{G \cdot x}$. Write H for the constant stabiliser of points in F. We know that $A_F \rtimes_\alpha G$ acts as adjointable operators on Y_F, so, using (b) and (c), we have

$$\begin{aligned}\pi_F(\langle f \cdot g, f \cdot g \rangle_{A \rtimes_\alpha \mathscr{P}}) &= \langle \rho_F(f)\, r_F(g), \rho_F(f)\, r_F(g) \rangle_{A \rtimes_\alpha H} \\ &\leqslant \|\rho_F(f)\|^2 \langle r_F(g), r_F(g) \rangle_{A_F \rtimes_\alpha H} \\ &\leqslant \|f\|^2 \pi_F(\langle g, g \rangle_{A \rtimes_\alpha \mathscr{P}}).\end{aligned}$$

Since every irreducible representation of $A \rtimes_\alpha \mathscr{P}$ factors through some π_x, and hence through some π_F, this implies (3.4) and (1) is proved.

Equations (a) and (b) show that r_F is norm-decreasing on Y_0, hence extends to the completion, and that r_F then vanishes on the submodule $Y_{\ker(\pi_F)}$ corresponding to the ideal $\ker(\pi_F) = I_F \rtimes_\alpha G$ in $A \rtimes_\alpha G$. Let $\tilde{r}_F$ denote the induced homomorphism of $Y/Y_{\ker(\pi_F)}$ into Y^F: by part (a), $\tilde{r}_F$ is an isometry of $A_F \rtimes_\alpha G$-rigged spaces. Every elementary tensor $\phi \otimes \alpha$ in $C_c(G) \otimes \Gamma_c(\xi|_F)$ is in the range of r_F, so $\tilde{r}_F$ has dense range and is therefore surjective. We can now extend (a), (b), and (c) to the completions by continuity. ∎

Definition 3.6. Let $\mathrm{Ind}: \mathscr{I}(A \rtimes_\alpha \mathscr{P}) \to \mathscr{I}(A \rtimes_\alpha G)$ denote the continuous map $\mathfrak{I}_Y$ defined by the left $A \rtimes_\alpha G$- right $A \rtimes_\alpha \mathscr{P}$-rigged bimodule Y. Similarly, if F is a closed G-invariant subset of X with constant stabiliser H, then we denote by Ind_F the continuous map from $\mathscr{I}(A_F \rtimes_\alpha H)$ to $\mathscr{I}(A_F \rtimes_\alpha G)$ defined by the bimodule Y^F.

Proposition 3.7. (1) *If* $\ker(\pi \times U)$ *is a primitive ideal in* $A_x \rtimes_\alpha H$, *then*

$$\mathrm{Ind}(\pi_x^*(\ker(\pi \times U))) = \ker(\mathrm{Ind}_{G_x}^G((\pi \circ \varepsilon_x) \times U)).$$

(2) *If F is a closed G-invariant subset of X with constant stabiliser H, then the diagram*

$$\begin{array}{ccc} \mathscr{I}(A \rtimes_\alpha \mathscr{P}) & \xrightarrow{\ \mathrm{Ind}\ } & \mathscr{I}(A \rtimes_\alpha G) \\ \Big\uparrow \pi_F^* & & \Big\uparrow \rho_F^* \\ \mathscr{I}(A_F \rtimes_\alpha H) & \xrightarrow{\ \mathrm{Ind}_F\ } & \mathscr{I}(A_F \rtimes_\alpha G) \end{array}$$

commutes.

Proof. By Corollary 2.5 we have

$$\begin{aligned} &\mathrm{Ind}(\pi_x^*(\ker(\pi \times U))) \\ &\quad = \{f \in A \rtimes_\alpha G : (\pi \times U) \circ \pi_x(\langle f \cdot z, y \rangle_B) = 0 \text{ for all } y, z \in Y_0\}. \end{aligned}$$

Inducing representations from $A \rtimes_\alpha G_x$ to $A \rtimes_\alpha G$ is done using $C_c(G, A)$, viewed as a $C_c(G_x, A)$-rigged space with inner product given by

$$\langle z, y \rangle_{A \rtimes_\alpha G_x}(s) = \int_G \alpha_t(z(t^{-1})^* y(t^{-1}s))\, dt.$$

It is easy to check that

$$\pi_x(\langle z, y \rangle_B)(s) = \langle z, y \rangle_B(s, x) = \varepsilon_x(\langle z, y \rangle_{A \rtimes_\alpha G_x}(s)),$$

and hence

$$\operatorname{Ind}(\pi_x^*(\ker(\pi \times U))) = \{f \in A \rtimes_\alpha G \colon (\pi \otimes \varepsilon_x) \times U(\langle f \cdot z, y\rangle_{A \rtimes_\alpha G_x}) = 0 \text{ for all } y, z \in Y_0\}.$$

The last ideal is the kernel of $\operatorname{Ind}_{G_x}^G((\pi \otimes \varepsilon_x) \times U)$, and (1) is proved.

For an ideal $J \subseteq A_F \rtimes_\alpha H$, Corollary 2.5 implies

$$\begin{aligned}\rho_F^*(\operatorname{Ind}_F(J)) &= \{c \in A \rtimes_\alpha G \colon \langle \rho_F(c) \cdot x, y\rangle_{A_F \rtimes_\alpha H} \in J \text{ for all } x, y \in Y^F\}\\ &= \{c \in A \rtimes_\alpha G \colon \langle \rho_F(c) \cdot r_F(x), r_F(y)\rangle_{A_F \rtimes_\alpha H} \in J \text{ for all } x, y \in Y\}\\ &= \{c \in A \rtimes_\alpha G \colon \pi_F(\langle c \cdot x, y\rangle_B) \in J \text{ for all } x, y \in Y\}.\end{aligned}$$

Another application of Corollary 2.5 shows that the latter is $\operatorname{Ind}_{G_x}^G(\pi_F^*(J))$. ∎

We shall now build a homomorphism R' of $A \rtimes_\alpha \mathscr{P}$ into $\mathscr{M}(A \rtimes_\alpha G)$, and define our restriction map RES from $\mathscr{I}(A \rtimes_\alpha G)$ to $\mathscr{I}(A \rtimes_\alpha \mathscr{P})$ to be R'^* [10, Proposition 9]. For $f \in \Gamma_c(p^*\xi) \subseteq A \rtimes_\alpha \mathscr{P}$ and $g \in C_c(G, A)$ we define

$$(R'(f)g)(s)(x) = \int_{G_x} f(t, x)\, \alpha(t, x)[g(t^{-1}s)(x)]\, d\lambda_x(t)$$

$$(gR'(f))(s)(x) = \int_{G_x} g(st^{-1})(x)\, \alpha(st^{-1}, x)[f(t, s^{-1} \cdot x)]\, d\lambda_x(t).$$

The usual arguments involving approximations of f and g by finite tensors show that $R'(f)g$ and $gR'(f)$ belong to $C_c(G, A)$, and a quick calculation shows that

$$h(R'(f)g) = (hR'(f))g \tag{3.5}$$

for $h, g \in C_c(G, A)$ and $f \in \Gamma_c(p^*\xi)$.

PROPOSITION 3.8. (1) *For each $f \in \Gamma_c(p^*\xi)$, $R'(f)$ extends to a multiplier of $A \rtimes_\alpha G$, and R' extends to a homomorphism of $A \rtimes_\alpha \mathscr{P}$ into $\mathscr{M}(A \rtimes_\alpha G)$.*

(2) *Suppose that F is a closed G-invariant set with constant stabiliser group H, and that R'_F denotes the usual embedding of $A_F \rtimes_\alpha H$ in $\mathscr{M}(A_F \rtimes_\alpha G)$, so that $R_F'^*$ is the usual restriction map $\mathrm{RES}_F \colon \mathscr{I}(A_F \rtimes_\alpha G) \to \mathscr{I}(A_F \rtimes_\alpha H)$* [10, p. 209]. *Then we have a commutative diagram*

$$\begin{array}{ccc}\mathscr{I}(A \rtimes_\alpha G) & \xrightarrow{\ \mathrm{RES}\ } & \mathscr{I}(A \rtimes_\alpha \mathscr{P})\\ \rho_F^* \uparrow & & \uparrow \pi_F^*\\ \mathscr{I}(A_F \rtimes_\alpha G) & \xrightarrow{\ \mathrm{RES}_F\ } & \mathscr{I}(A_F \rtimes_\alpha H)\end{array}$$

Proof. We have seen that the pair of maps R' has the correct algebraic property (3.5), and we just need to check continuity to ensure that R' extends to a multiplier. However, a simple calculation shows that with F as in part (2),

$$R'_F(\pi_F(f))\rho_F(g) = \rho_F(R'(f)\,g). \tag{3.6}$$

This enables us to exploit the known properties of R'_F:

$$\begin{aligned}\|R'(f)\,g\|_{A\rtimes_\alpha G} &= \sup_{x\in X} \|\rho_{\overline{G\cdot x}}(R'(f)\,g)\|_{A_{\overline{G\cdot x}}\rtimes_\alpha G}\\ &\leqslant \sup_{x\in X} \|R'_{\overline{G\cdot x}}(\pi_{\overline{G\cdot x}}(f))\rho_{\overline{G\cdot x}}(g)\|\\ &\leqslant \sup_{x\in X} \|\pi_{\overline{G\cdot x}}(f)\|_{A_{\overline{G\cdot x}}\rtimes_\alpha G_x} \|\rho_{\overline{G\cdot x}}(g)\|_{A_{\overline{G\cdot x}}\rtimes_\alpha G}\\ &\leqslant \|f\|_{A\rtimes_\alpha \mathscr{P}} \|g\|_{A\rtimes_\alpha G}.\end{aligned}$$

A similar calculation works on the other side. Hence $R'(f)$ extends to a multiplier of norm less than or equal to $\|f\|$, and we obtain a homomorphism R' as claimed. Now Eq. (3.6) implies that $R'_F \circ \pi_F = \rho_F \circ R'$. The homomorphism ρ_F is surjective, and it is straightforward to verify that the image under R'_F of an approximate identity in $A_F \rtimes_\alpha H$ converges to the identity of $\mathscr{M}(A_F \rtimes_\alpha G)$ in the strict topology. The next lemma completes the proof. ∎

LEMMA 3.9. *Suppose that $T\colon C \to \mathscr{M}(D)$ and $S\colon D \to \mathscr{M}(E)$ are homomorphisms, and that there is an approximate ientity d_j for D such that $S(d_j)$ converges to the identity of $\mathscr{M}(E)$ in the strict topology. Then S extends uniquely to a strictly continuous homomorphism of $\mathscr{M}(D)$ into $\mathscr{M}(E)$, and $(S \circ T)^* = T^* \circ S^*$.*

Proof. That S extends uniquely is well known (cf. [14, Lemma 1.1]). Then, given an ideal J in E,

$$T^* \circ S^*(J) = \{c \in C\colon S(T(c)d)e \in J \text{ for all } d \in D \text{ and } e \in E\}.$$

But the hypothesis implies that elements of the form $S(d)e$ are dense in E, so the above is equal to $(S \circ T)^*(J)$. ∎

Proof of Theorem 3.1. We have already seen that every irreducible representation of $A \rtimes_\alpha \mathscr{P}$ factors through some π_x, and it is easy to check that Ind is $\hat{G}$-equivariant. By part (1) of Propositon 3.7, it will be enough for us to show that the map Ind of Definition 3.6 induces a homeomorphism of $\mathscr{Q}(\mathrm{Prim}(A \rtimes_\alpha \mathscr{P}))$ onto $\mathrm{Prim}(A \rtimes_\alpha G)$. In fact, we claim it suffices to show that

(1) If $K \in \mathrm{Prim}(A \rtimes_\alpha G)$, then $\mathrm{Ind}(\mathrm{RES}(K)) = K$;

(2) If $J \in \operatorname{Prim}(A \rtimes_\alpha \mathscr{P})$, then $\operatorname{Ind}(J)$ is a primitive ideal in $A \rtimes_\alpha G$, and

$$\operatorname{RES}(\operatorname{Ind}(J)) = \bigcap_{s \in G} \beta_s(J).$$

We now prove the claim, and then prove (1) and (2). It follows from (1) and (2) that Ind is constant on quasi-orbits, and hence induces a map h from $\mathscr{Q}(\operatorname{Prim}(A \rtimes_\alpha \mathscr{P}))$ into $\operatorname{Prim}(A \rtimes_\alpha G)$; this map is continuous since Ind is. By the Gootman–Rosenberg theorem [8] the system (G, A, α) is Effros-Hahn regular; by Proposition 3.7, this means every primitive ideal of $A \rtimes_\alpha G$ is induced from a primitive ideal of the form $\ker((\pi \circ \varepsilon_x) \times U)$, and Ind is surjective. As it stands, RES is a continuous map of $\operatorname{Prim}(A \rtimes_\alpha G)$ into $\mathscr{I}(A \rtimes_\alpha \mathscr{P})$, but the map sending a quasi-orbit in $\mathscr{Q}(A \rtimes_\alpha \mathscr{P})$ to its kernel is a homeomorphism onto its range in $\mathscr{I}(A \rtimes_\alpha \mathscr{P})$ [10, Lemma, p. 221], and (2) shows that RES maps primitive ideals to kernels of quasi-orbits, so RES defines a continuous map k of $\operatorname{Prim}(A \rtimes_\alpha \mathscr{P})$ into $Q(\operatorname{Prim}(A \rtimes_\alpha \mathscr{P}))$. Now (1) says that $h \circ k$ is the identity, (2) says that $k \circ h$ is the identity, and we have proved the claim.

We have already observed that Ind is surjective, so every $K \in \operatorname{Prim}(A \rtimes_\alpha G)$ has the form $\operatorname{Ind}(\pi_x^*(J))$ for some $x \in X$ and $J \in \operatorname{Prim}(A_x \rtimes_\alpha G_x)$. Let F denote the G-invariant set $\overline{G \cdot x}$, and $H = G_x$ the common stabiliser. The map π_x factors through π_F, so $\pi_x^*(J) = \pi_F^*(L)$ for some primitive ideal L of $A_F \rtimes_\alpha G_x$, and by Proposition 3.7 we have

$$K = \operatorname{Ind}(\pi_x^*(J)) = \operatorname{Ind}(\pi_F^*(L)) = \rho_F^*(\operatorname{Ind}_F(L)).$$

By Proposition 3.8, therefore,

$$\operatorname{RES}(K) = \operatorname{RES}(\rho_F^*(\operatorname{Ind}_F(L))) = \pi_F^*(\operatorname{RES}_F(\operatorname{Ind}_F(L))).$$

Thus, by Proposition 3.7(2),

$$\operatorname{Ind}(\operatorname{RES}(K)) = \operatorname{Ind}(\pi_F^*(\operatorname{RES}_F(\operatorname{Ind}_F(L)))) = \rho_F^*(\operatorname{Ind}_F(\operatorname{RES}_F(\operatorname{Ind}_F(L)))).$$

Now RES_F and Ind_F are the usual restriction and induction maps for $A_F \rtimes_\alpha G$, so $\operatorname{Ind}_F(\operatorname{RES}_F)$ is the identity by Lemma 3.2, and (1) follows.

Finally, suppose that $J \in \operatorname{Prim}(A \rtimes_\alpha \mathscr{P})$. Then, as above, $J = \pi_F^*(L)$ for some $x \in X$, $F = \overline{G \cdot x}$, and $L \in \operatorname{Prim}(A_F \rtimes_\alpha G_x)$. Then

$$\begin{aligned}
\operatorname{RES}(\operatorname{Ind}(J)) &= \operatorname{RES}(\operatorname{Ind}(\pi_F^*(L))) \\
&= \operatorname{RES}(\rho_F^*(\operatorname{Ind}_F(L))) \\
&= \pi_F^*(\operatorname{RES}_F(\operatorname{Ind}_F(L))) \\
&= \pi_F^*\Big(\bigcap_{s \in G} \beta_s^F(L)\Big),
\end{aligned}$$

by Lemma 3.2. But π_F intertwines β^F and β, so it follows that

$$\mathrm{RES}(\mathrm{Ind}(J)) = \bigcap_{s \in G} \beta_s(\pi_F^*(L)) = \bigcap_{s \in G} \beta_s(J).$$

This completes the proof of Theorem 3.1. ∎

We can now combine our two main results to give a version of [22, Proposition 2.1] describing the primitive ideal space of $A \rtimes_\alpha G$ when the stabilisers vary continuously. In fact our result is slightly stronger than [22, Proposition 2.1] even when the stabilisers are constant: we have also established openness of the restriction map.

THEOREM 3.10. *Let (A, G, α) be a separable C^*-dynamical system with G abelian,* Prim(A) *Hausdorff, and with continuously varying stabilisers. Then we have a commutative diagram of continuous open surjections*

$$\begin{array}{ccccc} & & \mathrm{Prim}(A \rtimes_\alpha \mathscr{P}) & & \\ & \swarrow^{\mathrm{Ind}} & & \searrow^{\mathrm{Res}} & \\ \mathrm{Prim}(A \rtimes_\alpha G) & & & & \mathrm{Prim}(A) \\ & \searrow_{q} & & \swarrow_{p} & \\ & & \mathscr{Q}(\mathrm{Prim}(A), G) & & \end{array}$$

where Ind *is the inducing map of Theorem* 3.1, Res *is the restriction map of Theorem* 2.2, *q assigns to each primitive ideal P the quasi-orbit $q(P)$ on which it lives (cf.* [10, p. 221]), *and p is the canonical map of* Prim(A) *onto the quasi-orbit space* $\mathscr{Q}(\mathrm{Prim}(A), G)$ *for the action of G.*

Proof. The properties of Res and Ind are established in the cited theorems. That p is a continuous open surjection follows from the lemma on page 221 of [10], and the same properties of q have been established by Gootman and Lazar [7, Theorem 2.4]. So it remains only to show that the diagram commutes. If $\pi \times U \in (A_x \rtimes_\alpha G_x)\hat{}$, so that $J = \ker((\pi \times U) \circ \pi_x)$ is a typical element of Prim$(A \rtimes_\alpha \mathscr{P})$, then $\mathrm{Res}\, J = I_x$ by Theorem 2.2, and $p(\mathrm{Res}\, J)$ is the quasi-orbit $(G \cdot x)^{\sim}$ of x. The argument in the second paragraph of [22, Proposition 2.1] can be used essentially verbatim (replacing H by G_x) to show that

$$\ker(\mathrm{Ind}_{G_x}^G((\pi \circ \varepsilon_x) \times U)|_A) = \bigcap_{s \in G} \alpha_s(\ker(\pi \circ \varepsilon_x)).$$

Thus $q \circ \mathrm{Ind}(J)$ is also the quasi-orbit $(G \cdot x)^{\sim}$ (see the definition of q on p. 221 of [10]). ∎

4. Locally Trivial G-Spaces

Let (G, X) be a locally compact Hausdorff transformation group with continuous stabiliser map $\sigma: X \to \Sigma_G$ defined by $\sigma(x) = G_x$ (here, G need not be abelian). Define an equivalence relation on $X \times G$ by

$$(x, s) \sim (y, t), \qquad \text{if and only if} \quad x = y \quad \text{and} \quad st^{-1} \in G_x.$$

Then the continuity of σ implies that the quotient topological space $X \times G/\sim$ is locally compact Hausdorff, and that the quotient map is open [33, Lemma 2.3].

Definition 4.1. Let (G, X) be as above. We say the G-action is proper relative to the stabilisers, or just σ-proper, if the map $(x, s) \mapsto (x, s \cdot x)$ of $X \times G/\sim$ into $X \times X$ is proper.

As a good motivating example, consider the case where the action has constant stabiliser H, so that $\sigma(x) = H$ for all x. Then $X \times G/\sim$ is naturally homeomorphic to $X \times G/H$, and X is a σ-proper G-space if and only if X is a proper G/H-space in the usual sense. It is reasonably easy to show that G acts σ-properly on X if and only if, given any compact subset K of X, the image in $X \times G/\sim$ of

$$\{(x, s) \in X \times G: x \in K \text{ and } s \cdot x \in K\} \tag{4.1}$$

is relatively compact. Any set K for which the image of (4.1) is relatively compact is called G-wandering. This terminology does not agree with that of Definition 2.4 of [33], and is in fact logically distinct, as we shall show by example in an appendix. Unfortunately, the argument in the first paragraph of the proof of [33, Lemma 4.3] requires the above definition of G-wandering. Fortunately, the above definition suffices for *all* the results in [33, 17], so that they remain true if we use this definition in place of [33, Definition 2.4].

Our definition has been motivated by recent characterizations of properness for second countable pairs (G, X) involving the transformation group C^*-algebras $C^*(G, X)$. When G acts freely, our definition reduces to the usual one, and Green [9, Theorems 14 and 17] has shown that the action is proper if and only if $C^*(G, X)$ has continuous trace. In the general case of a continuous stabiliser map $\sigma: X \to \Sigma_G$, [33, Theorem 5.1] asserts that the action is σ-proper if and only if $C^*(G, X)$ has continuous trace. Thus we feel confident that we have made an appropriate and useful generalization of the usual notion. We shall now discuss the analogous version of local triviality when G is abelian.

DEFINITION 4.2. Suppose that (G, X) is a locally compact abelian transformation group with continuous stabiliser map $\sigma: X \to \Sigma_G$. We shall say that X is a locally σ-trivial space if X/G is Hausdorff and if every $x \in X$ has a G-invariant neighborhood U which is G-homeomorphic to $U/G \times G/\sim_1$, where

$$(G \cdot x, s) \sim_1 (G \cdot y, t) \qquad \text{if and only if} \quad G \cdot x = G \cdot y \quad \text{and} \quad st^{-1} \in G_x,$$

and G acts on $U/G \times G/\sim_1$ by translation in the second factor.

Extending Definition 4.2 to the case when G is non-abelian is complicated by the fact that the stabilisers vary along orbits. Since our definition of a locally σ-trivial space implies that there are local sections for the orbit map $p: X \to X/G$, Definition 4.2 can be modified by requiring the existence of a continuous section $q: U/G \to U$ and defining

$$(G \cdot x, s) \sim_1 (G \cdot y, t) \qquad \text{if and only if} \quad G \cdot x = G \cdot y \quad \text{and} \quad st^{-1} \in G_{q(G \cdot x)}.$$

However, as the applications we have in mind here involve only abelian groups, we have chosen to restrict ourselves to that case for ease of exposition, even though some of our results—such as the next proposition—go through for non-abelian groups.

A free and proper action of G on X is locally trivial exactly when the orbit map $X \to X/G$ has local cross-sections, and we shall now show that σ-properness and local σ-triviality are related in the same way. Again, for motivation we note that, in the case of a constant stabiliser group H, the following proposition implies that X is locally σ-trivial if and only if it is locally trivial as a G/H-space.

PROPOSITION 4.3. *Let X be a locally compact Hausdorff G-space with continuous stabiliser map $\sigma: X \to \Sigma_G$.*

(1) *If X is locally σ-trivial, then the G-action is σ-proper.*

(2) *If the G-action is σ-proper, then the orbit space X/G is Hausdorff.*

(3) *If the G-action is σ-proper and the orbit map $X \to X/G$ has local sections, then X is locally σ-trivial.*

Proof. To prove (1), let K be a compact set in X. Also suppose that $\{(x_i, s_i)\}$ is a net in $X \times G$ with $x_i, s_i \cdot x_i \in K$ for all i. Since it will suffice to find a convergent subnet, we may assume that $x_i \to x$ and $s_i \cdot x_i \to y$. Moreover, we can even assume that $K \subseteq U$, where U is the G-invariant neighborhood of x specified in Definition 4.2. In particular, we can reduce to the case where $X = X/G \times G/\sim_1$. Notice that the natural map of $X/G \times G$ onto X will be open by [33, Lemma 2.3]. Thus we may write K as the image of $C_1 \times C_2 \subseteq X/G \times G$ with C_i compact. We put $x_i = [G \cdot x_i, t_i]$ and

$s_i \cdot x_i = [G \cdot x_i, s_i t_i]$ for $G \cdot x_i \in C_1$ and $t_i \in C_2$. But $[G \cdot x_i, t_i] \to [G \cdot x, t]$ while $s_i \cdot x_i$ converges to some $[G \cdot x, s]$. Using the openness of the natural map again, we can assume that $t_i \to t$ and that there are $v_i \in G_{x_i}$ so that $v_i s_i t_i$ converges to s. However, all this means that

$$(x_i, s_i v_i) = ([G \cdot x_i, t_i], s_i v_i) = ([G \cdot x_i, t_i], (s_i v_i t_i) t_i^{-1})$$

converges to (x, st^{-1}) in $X \times G$. This implies $[x_i, s_i]$ converges in the quotient.

Statement (2) is [33, Proposition 2.17].

For (3), it will suffice to show that if $\eta: X \to X/G$ has a global (continuous) section $c: X/G \to X$, then X is G-homeomorphic to $X/G \times G/\sim_1$. Define $f: X/G \times G/\sim_1 \to X$ and $g: X \to X/G \times G/\sim_1$ by

$$f([G \cdot x, s]) = s \cdot c(G \cdot x)$$

and

$$g(x) = [G \cdot x, s], \qquad \text{where} \quad x = s \cdot c(G \cdot x).$$

It is easy to see that both f and g are well-defined G-equivariant maps, and that f is continuous. Furthermore, both $f \circ g$ and $g \circ f$ are the identity map; hence, it will suffice to show that g is continuous.

Suppose that x_i converges to x in X. It will suffice to show that every subnet of $\{g(x_i)\}$ has a subnet which converges to $g(x)$. Put $x_i = s_i \cdot c(G \cdot x_i)$ and $x = s \cdot c(G \cdot x)$. Since c is continuous, the σ-properness implies that $\{[x_i, s_i]\}$ (has a subnet which) converges to some $[x, r]$ in $X \times G/\sim$. Since the quotient map of $X \times G$ onto $X \times G/\sim$ is open [33, Lemma 2.3], we can assume that there are $t_i \in G_{x_i}$ such that $s_i t_i$ converges to r. Therefore, $x_i = s_i \cdot c(G \cdot x_i) = s_i t_i \cdot c(G \cdot x_i)$ converges to $r \cdot c(G \cdot x)$. In particular, $g(x) = [G \cdot x, r]$. ▮

We believe this proposition provides convincing evidence that we have the correct notion of local triviality to go with our definition of σ-properness. However, there is one slight problem. For free actions, locally trivial G-spaces can be recovered from the local trivializations using transition functions, and this allows us to use the machinery of sheaf cohomology. In our more general setting, these transition functions need not exist, and we are forced to impose an extra condition on the stabiliser map σ to make sheaf-theoretic techniques available. (We give examples following Proposition 4.5 to show that the extra hypothesis is not redundant.) We shall distinguish between the two situations by reserving the word "bundle" for the case where transition functions exist.

DEFINITION 4.4. Let Z be a locally compact Hausdorff space, G a locally compact abelian group, and $\sigma: Z \to \Sigma_G$ a continuous map.

(1) We say that σ is locally liftable if every continuous section $c: Z \to Z \times G/\sim_1$ is given locally by a continuous map $s: Z \to G$ (i.e., $c(z) = [z, s(z)]$ for $z \in Z$).

(2) We say a locally compact G-space X is a locally σ-trivial G-bundle over Z, if $Z = X/G$, if σ is locally liftable, and if X is a locally σ-trivial G-space as in Definition 4.2 (when σ is viewed as a map on X which is constant on orbits).

PROPOSITION 4.5. *Let G be a locally compact abelian group and $\sigma: Z \to \Sigma_G$ a locally liftable map from a paracompact space Z. Let $\mathfrak{G}$ denote the sheaf of germs of continuous G-valued functions on Z, and $\mathfrak{P}$ the subsheaf whose sections over U are continuous functions $s: U \to G$ such that $s(t) \in \sigma(t)$ for $t \in U$. Then the set of isomorphism classes of locally σ-trivial G-bundles over Z is in one-to-one correspondence with the cohomology group $H^1(Z, \mathfrak{G}/\mathfrak{P})$.*

Proof. Suppose $p: X \to Z$ is a locally σ-trivial G-bundle, and let

$$h_i: U_i \times G/\sim \to p^{-1}(U_i)$$

be equivariant homeomorphisms. Then for each pair i, j, the map $z \mapsto h_i^{-1} \circ h_j([z, e])$ is a section of $U_{ij} \times G/\sim$, and hence is given locally by a continuous map s into G. By the argument of [2, 10.7.11], we can refine the cover $\{U_i\}$ and assume there are continuous maps $s_{ij}: U_{ij} \to G$ such that

$$h_i([z, s_{ij}(z)t]) = h_j([z, t]), \qquad \text{for} \quad z \in U_{ij}, \quad t \in G. \tag{4.2}$$

By comparing $(h_i^{-1} \circ h_j) \circ (h_j^{-1} \circ h_k)$ and $h_i^{-1} \circ h_k$ on triple overlaps, we find that

$$s_{ij}(z)\, s_{jk}(z)\, s_{ik}(z)^{-1} \in G_z = \sigma(z), \qquad \text{for} \quad z \in U_{ijk}, \tag{4.3}$$

so that the s_{ij} define a 1-cocycle $\{U_i, s_{ij}\}$ with values in the quotient sheaf $\mathfrak{G}/\mathfrak{P}$. (We shall refer to the s_{ij} as *transition functions* for the bundle.)

Although Eq. (4.2) does not determine the functions s_{ij} completely, the corresponding section of $\mathfrak{G}/\mathfrak{P}$ is uniquely determined, and of course refining the cover will not affect the class of $\{U_i, s_{ij}\}$ in $H^1(Z, \mathfrak{G}/\mathfrak{P})$. Suppose we had picked different trivializations g_i over U_i. By the liftability of σ we can refine the cover to ensure there are continuous maps $k_i: U_i \to G$ satisfying

$$h_i([z, k_i(z)t]) = g_i([z, t]), \qquad \text{for} \quad z \in U_i, \quad t \in G.$$

The corresponding transition functions s_{ij} and r_{ij} are then related by

$$k_i(z)^{-1} s_{ij}(z)\, k_j(z)\, r_{ij}(z)^{-1} \in G_z, \qquad \text{for} \quad z \in U_{ij},$$

so that the cocycles $\{U_i, s_{ij}\}$, $\{U_i, r_{ij}\}$ are cohomologous. Similar arguments show that the class of $\{U_i, s_{ij}\}$ depends only on the isomorphism class of the G-space $X \to Z$.

Conversely, if we are given a 1-cocycle with values in $\mathfrak{G}/\mathfrak{P}$, we can by refining the cover assume that it is given by continuous $s_{ij}: U_{ij} \to G$ satisfying (4.3), and by the usual method construct a bundle with transition functions s_{ij}. As usual, cohomologous cocycles give isomorphic bundles, and the result follows. ∎

Of course, in the freely acting case, the locally trivial spaces are automatically bundles, but in general they need not be. To see this, it suffices to consider the case where H is a subgroup of G and $\sigma: Z \to \Sigma_G$ is defined by $\sigma(z) = H$ for all z. Then $Z \times G/\sim\, = Z \times G/H$, and sections are just functions from Z to G/H, which always lift locally if and only if $G \to G/H$ has local cross sections. Although it is automatic for Lie groups [19], there are pairs (G, H) for which this is not the case. For example, let $G = \prod_{n=1}^{\infty} \mathbb{T}$ and $H = \prod_{n=1}^{\infty} \{1, -1\}$. Our next example shows, however, that even for actions of $\mathbb{R}$, there can be a difference between our locally trivial spaces and bundles when the stabilisers vary.

EXAMPLE 4.6. Choose a complex line bundle $p: L \to Y$, and give it a Hermitian structure. Then we can define an action of $\mathbb{R}$ on L by

$$r \cdot z = \begin{cases} e^{2\pi i r/|z|} \cdot z, & \text{if } |z| \neq 0, \\ z, & \text{if } |z| = 0. \end{cases} \tag{4.4}$$

Then we have

$$\sigma(z) = G_z = \begin{cases} |z|\, \mathbb{Z}, & \text{if } |z| \neq 0, \\ \mathbb{R}, & \text{if } |z| = 0, \end{cases} \tag{4.5}$$

so the stabilisers vary continuously. The argument of [33, Example 5.4] (which is the special case where Y is a point) shows that the action is σ-proper, and it is easy to verify that the continuous map $q: L \to Y \times [0, \infty)$ defined by $q(z) = (p(z), |z|)$ induces a homeomorphism of the orbit space $L/\mathbb{R}$ onto $Y \times [0, \infty)$. The local triviality of L as a line bundle implies that there are local cross sections of q, so that L is a locally σ-trivial space for the map $\sigma: Y \times [0, \infty) \to \Sigma$ defined by $\sigma(y, r) = r\mathbb{Z}$ $(r \neq 0)$, and $\sigma(y, 0) = \mathbb{R}$. In general, L is not globally σ-trivial—in fact, it is easy to see that it is globally σ-trivial exactly when it is trivial as a line bundle. We claim that L is not a locally σ-trivial *bundle*; the map σ is not locally liftable.

To see this, it is enough to consider the case where Y is a point, so that $L = \mathbb{C}$. We define $c: [0, \infty) \to [0, \infty) \times \mathbb{R}/\sim$ by

$$c(r) = \begin{cases} [r, 1/r], & \text{for } r \neq 0, \\ [0, 0], & \text{for } r = 0. \end{cases}$$

Away from 0, c is continuous because $r \mapsto 1/r$ is; thus, to prove that c is continuous on $[0, \infty)$, it will be enough to show that $c(r_n) \to c(0)$ whenever $r_n \searrow 0$. For each n we can choose an integer m_n such that $|1/r_n - r_n m_n| < r_n$. Then since $\sigma(r_n) = r_n\mathbb{Z}$, we have

$$\left[r_n, \frac{1}{r_n}\right] = \left[r_n, \frac{1}{r_n} - r_n m_n\right] \to [0, 0] \quad \text{in} \quad [0, \infty) \times \mathbb{R}/\sim,$$

and c is continuous. If $s: (0, \varepsilon) \to \mathbb{R}$ is a continuous function such that $c(r) = [r, s(r)]$, then we have $1/r - s(r) \in r\mathbb{Z}$ for all $r \neq 0$. Then the function $r \mapsto 1/r^2 - s(r)/r$ is continuous on $(0, \varepsilon)$ and integer valued, so it must be constant, say equal to N. But this implies that $s(r) = -rN + 1/r$ for $r \neq 0$, which cannot possibly extend to be continuous at 0. Thus the section c does not have a local lifting near 0, and σ is not locally liftable.

EXAMPLE 4.7. To show that a proper space need not be locally trivial let $\mathbb{R}$ act on $\mathbb{C}^2$ by the formula (4.4). As in Example 4.6, this action is σ-proper for the stabiliser map σ given by (4.5). To see that $\mathbb{C}^2$ is not locally trival, let $\rho: S^3 \to S^2$ denote the Hopf fibration. Let $|\cdot|$ denote the usual norm in $\mathbb{C}^2$. Then, identifying the unit sphere in $\mathbb{C}^2$ with S^3 and the unit ball in $\mathbb{R}^3$ with S^2, we can define $q: \mathbb{C}^2 \to \mathbb{R}^3$ by

$$q(z) = \begin{cases} |z|\, \rho(z/|z|), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Then q is continuous and open, and therefore induces a homeomorphism of $\mathbb{C}^2/\mathbb{R}$ onto $\mathbb{R}^3$. If there was a local section s near 0, then for sufficiently small $\delta > 0$ we would have a continuous map $s: \delta S^2 \to \delta S^3$ such that $q(s(\delta x)) = \delta x$ for all $x \in S^2$, which in turn would give a continuous section $x \mapsto (1/\delta)\, s(\delta x)$ for the Hopf fibration. Since no such section exists, we deduce from Proposition 4.3 that $\mathbb{C}^2$ is not a locally σ-trivial $\mathbb{R}$-space.

PROPOSITION 4.8. *Suppose (G, X) is a second countable locally compact abelian transformation group which is proper relative to the continuous stabiliser map $\sigma: X \to \Sigma_G$. Then, with respect to the dual action of $\hat{G}$, the spectrum of the transformation group C^*-algebra $C^*(G, X)$ has stabiliser map $\hat{\sigma}: X \to G_x^\perp$, and $C^*(G, X)^\wedge$ is a (globally) $\hat{\sigma}$-trivial $\hat{G}$-space over X/G.*

Proof. To see this, we recall from [32, Theorem 5.3] that the map

$$(x, \gamma) \mapsto \operatorname{Ind}_{G_x}^G(\varepsilon_x \times \gamma|_{G_x})$$

induces a homeomorphism of $X \times \hat{G}/\sim_1$ onto $\operatorname{Prim}(C^*(G, X)) = C^*(G, X)^\wedge$, where

$$(x, \gamma) \sim_1 (y, \chi) \Leftrightarrow \overline{G \cdot x} = \overline{G \cdot y} \quad \text{and} \quad \gamma\chi^{-1} \in G_x^\perp = G_y^\perp.$$

It follows from [34] that $C^*(G,X)^\wedge$ is Hausdorff. Because G acts σ-properly, the orbits are all closed, X/G is Hausdorff (Proposition 4.5), and we can identify $X \times \hat{G}/\sim_1$ with $(X/G) \times \hat{G}/\sim$. If $\chi \in \hat{G}$ and $\hat{\alpha}$ denotes the dual action of $\hat{G}$ on $C^*(G, X)$, then by [22, Lemma 2.3] we have

$$\mathrm{Ind}_{G_x}^G(\varepsilon_x \times \gamma|_{G_x}) \circ \hat{\alpha}_\chi^{-1} \sim \mathrm{Ind}_{G_x}^G(\varepsilon_x \otimes \chi\gamma|_{G_x}),$$

This implies both that the stabiliser of $\mathrm{Ind}(\varepsilon_x \times \gamma)$ is $G_x^\perp$, and that the above homeomorphism is $\hat{G}$-equivariant. Notice that the continuity of $x \mapsto G_x^\perp$ follows from the continuity of $x \mapsto G_x$ and that of the map $H \mapsto H^\perp$ from $\Sigma_G \to \Sigma_{\hat{G}}$ [34]. ∎

Some particularly well-known and interesting subsets of $\Sigma_{\mathbb{R}^n}$ are the Grassmanian manifolds $\mathrm{Gr}(n, k)$ of k-dimensional subspaces. We shall now study locally σ-trivial $\mathbb{R}^n$-bundles where σ takes values in a component of the Grassmanian. Our main result says that, in this case, the only σ-proper $\mathbb{R}^n$-actions are the trivial ones on $Z \times \mathbb{R}^n/\sim$. This is, of course, well known for free actions, but the general result is a little surprising in view of the highly non-trivial topology of $\mathrm{Gr}(n, k)$. We shall, however, give some examples to show that even these trivial actions can arise in interesting ways, and we show how they can be modified to give locally trivial actions of larger groups which are not globally trivial. The basic idea here is that locally we can adjust such actions to ensure the stabilisers are constant. More formally, we make a definition:

DEFINITION 4.9. A continuous map $\sigma: Z \to \Sigma_G$ is locally constant if for each $z_0 \in Z$ there are a neighborhood U of z_0 and a map $c: U \to \mathrm{Aut}(G)$, continuous in the compact open topology on G, such that $c(z)(\sigma(z_0)) = \sigma(z)$ for $z \in U$.

PROPOSITION 4.10. (1) *Suppose $\sigma: Z \to \Sigma_G$ is locally constant, and that each map $G \to G/G_z$ has local cross sections. Then σ is locally liftable.*

(2) *Suppose σ is locally constant, and that each quotient G/G_z is a Lie group. Then every σ-proper space is locally σ-trivial.*

Remark. Both hypotheses on G/G_z are automatically satisfied if G is a Lie group.

Proof. (1) Given $z_0 \in Z$, we choose U and c as in Definition 4.9, and define a homeomorphism h of $U \times G/G_{z_0}$ onto $p^{-1}(U) = U \times G/\sim$ by

$$h(z, tG_{z_0}) = [z, c(z)(t)].$$

Thus any continuous section s of $p^{-1}(U)$ is given by a continuous function of U into G/G_{z_0}, which by shrinking U we can lift to a map r of U into G. Then the map $z \mapsto c(z)(r(z))$ provides the required lifting.

(2) Let X be a σ-proper G-space with $X/G = Z$. We are only interested in local properties, so without loss of generality there is a point $x_0 \in X$ and a continuous map $c\colon X/G \to \mathrm{Aut}(G)$ such that $c(x)(G_{x_0}) = G_x$ for all x. We now define a new action of G on X by $s \circ x = c(x)(s) \cdot x$; this is an action because $c(x)$ is a homomorphism and c is constant on cosets, and continuous because it is a composition of continuous maps:

$$(x, s) \mapsto (s, c(x), x) \mapsto (c(x)(s), x) \mapsto c(x)(s) \cdot x.$$

We have $G \circ x = G \cdot x$ because $c(x)$ is onto, so the orbit spaces for the two actions coincide; furthermore,

$$\{t\colon t \circ x = x\} = \{t\colon c(x)(t) \in G_x\} = G_{x_0},$$

so $\circ$ induces a free action of G/G_{x_0} on X. We claim it is also proper.

The map $h\colon X \times G \to X \times G$ defined by $h(x, s) = (x, c(x)^{-1}(s))$ is a homeomorphism which carries $\{(x, s)\colon x \in K,\ s \cdot x \in K\}$ into $\{(x, t)\colon x \in K,\ t \circ x \in K\}$. If we define an equivalence relation $\approx$ on $X \times G$ by $(x, r) \approx (x, t)$ if and only if $rt^{-1} \in G_{x_0}$, then we have

$$(x, s) \sim (x, t) \Leftrightarrow h(x, s) \approx h(x, t),$$

and h therefore induces a homeomorphism of $X \times G/\sim$ onto $X \times G/\approx\, = X \times G/G_{x_0}$. Now if K is compact in X, then $\{(x, s)\colon x \in K,\ s \cdot x \in K\}$ has relatively compact image in $X \times G/\sim$, and its image $\{(x, t)\colon x \in X,\ t \circ x \in K\}$ under h has relatively compact image in $X \times G/\approx$. Thus the action $\circ$ of G on X induces a proper free action of G/G_{x_0}. Since G/G_{x_0} is a Lie group, Palais' slice theorem [19] implies that there is a local cross section for this action. But X/G is the same for either action, so there are local cross sections for the original action too. ∎

LEMMA 4.11. *Suppose $\sigma\colon Z \to \sum_{\mathbb{R}_n}$ is a continuous map which takes values in one component* $\mathrm{Gr}(n, k)$ *of the Grassmanian. Then σ is locally constant.*

Proof. For $z_0 \in Z$ the natural map $\rho\colon GL_n(\mathbb{R}) \to \mathrm{Gr}(n, k)$ defined by $\rho(T) = T(\sigma(z_0))$ is a smooth surjection and therefore has local cross sections. Composing with one of these gives the required lifting of σ into $GL_n(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^n)$. ∎

EXAMPLE 4.12. If $Z = \mathbb{R}P = \mathbb{R}P^1$ and $\sigma\colon Z \to \mathbb{R}P = \mathrm{Gr}(2, 1)$ is the identity, then σ is not globally constant; that is, we cannot take $U = Z$ in Definition 4.9. To see this, let q denote the natural map from S^1 to $\mathbb{R}P$, let $R\colon S^1 \to SO_2(\mathbb{R})$ be the isomorphism which sends $e^{2\pi i\theta}$ to rotation through θ, define $\rho(T)$ to be the line through $T(1, 0)$, and let $k\colon GL_2(\mathbb{R}) \to SO_2(\mathbb{R})$ be the continuous map which sends the invertible matrix with Iwasawa decomposition KAN to K (see [11, p. 270]). Then we have a commutative diagram

$$\begin{array}{ccccccc} & & S^1 & \xrightarrow{R} & SO_2(\mathbb{R}) & \underset{k}{\overset{\subset}{\rightleftarrows}} & GL_2(\mathbb{R}) \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Z = \mathbb{R}P & \xrightarrow{\sigma = id} & \mathbb{R}P & \xrightarrow{id} & \mathbb{R}P & \xrightarrow{id} & \mathbb{R}P. \end{array}$$

Since q does not have a global section, there can be no global lifting of σ with values in $SO_2(\mathbb{R})$. However, a global lift $f: Z \to GL_2(\mathbb{R})$ for σ would give such a lifting $k \circ f: Z \to SO_2(\mathbb{R})$, which justifies the claim.

PROPOSITION 4.13. *If $\sigma: Z \to \mathrm{Gr}(n, k)$ is continuous and Z is paracompact, then every σ-proper $\mathbb{R}^n$-space with orbit space Z is isomorphic to $Z \times \mathbb{R}^n/\sim$.*

Proof. By Lemma 4.11 and Proposition 4.10, every σ-proper $\mathbb{R}^n$-space X is a locally σ-trivial bundle. According to Proposition 4.5, the isomorphism class of X is determined by the class in $H^1(Z, \mathfrak{G}/\mathfrak{P})$ of the cocycle defined by the transition functions of X. Here, $\mathfrak{G}$ is the fine sheaf of germs of $\mathbb{R}^n$-valued functions, so that $H^p(Z, \mathfrak{G}) = 0$ for all $p \geqslant 1$, and the long exact sequence of sheaf cohomology implies that $H^1(Z, \mathfrak{G}/\mathfrak{P}) \cong H^2(Z, \mathfrak{P})$. We shall prove that $H^2(Z, \mathfrak{P}) = 0$; this suffices by Proposition 4.5.

Suppose $\{U_i, \lambda_{ijk}\}$ is a 2-cocycle with values in $\mathfrak{P}$, so that in particular $\lambda_{ijk}(z) \in \sigma(z)$ for all $z \in U_{ijk}$. Viewed as a cocycle with values in $\mathfrak{G}$, it is trivial, so by refining the cover we can find $\mu_{ij}: U_{ij} \to \mathbb{R}^n$ such that

$$\lambda_{ijk}(z) = \mu_{ij}(z) - \mu_{ik}(z) + \mu_{jk}(z), \qquad \text{for} \quad z \in U_{ijk}. \tag{4.6}$$

If $P(z)$ denotes the orthogonal projection of $\mathbb{R}^n$ onto $\sigma(z)$, then $P: Z \to \mathrm{Aut}(\mathbb{R}^n)$ is continuous, and hence so is the map $P \times \mu_{ij}: Z \to \mathbb{R}^n$ which sends z to $P(z)(\mu_{ij}(z))$ for each i, j. But each $P(z)$ is also linear, so (4.6) implies

$$\lambda_{ijk}(z) = P(z)(\lambda_{ijk}(z)) = P \times \mu_{ij}(z) - P \times \mu_{ik}(z) + P \times \mu_{jk}(z).$$

Each $P \times \mu_{ij}$ is a section of $\mathfrak{P}$, so this shows that $\{\lambda_{ijk}\}$ is trivial in $H^2(Z, \mathfrak{P})$, and establishes the claim. ∎

EXAMPLE 4.14. Let E be the real sub-bundle of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ orthogonal to the canonical line bundle L over $\mathbb{R}P^n$, and for $l \in \mathbb{R}P^n$, let P_l denote the orthogonal projection of $\mathbb{R}^{n+1}$ onto the fibre $E_l = l^\perp$. Then we can define an action of $\mathbb{R}^{n+1}$ on E by

$$u \cdot (l, v) = (l, v + P_l(u)).$$

The stabiliser map σ is the inclusion of $\mathbb{R}P^n$ in $\Sigma_{\mathbb{R}^n}$, and, although it is not immediately apparent, E is a globally σ-trivial $\mathbb{R}^{n+1}$ space– in fact, the map h from $\mathbb{R}P^n \times \mathbb{R}^{n-1}$ to itself defined by $h(l, v) = (l, P_l(v))$ induces an equivariant homeomorphism of $\mathbb{R}P^n \times \mathbb{R}^{n+1}/\sim$ onto E. As an n-dimensional vector bundle, however, E is always non-trivial [16, Example 3, p. 43].

When $n = 2$, we can disguise this trivial space even more: $\mathbb{R}P$ can be identified with $[-\pi/2, \pi/2]/\{-\pi/2, \pi/2\}$, and E with the Möbius band $M = [-\pi/2, \pi/2] \times \mathbb{R}/\sim_2$, where $\sim_2$ identifies $(\pi/2, x)$ with $(-\pi/2, -x)$. The corresponding action of $\mathbb{R}^2$ on M is given by

$$(r, s) \cdot [\theta, x] = [\theta, x - r\sin(\theta) + s\cos(\theta)].$$

Obviously one can perform similar constructions for other components of the Grassmanian.

EXAMPLE 4.15. Let H be a locally compact abelian group, $q \colon Y \to Z$ a locally trivial principal H-bundle, and $\tau \colon Z \to \mathbb{R}P^n$ a continuous map. Then we claim that, for the diagonal action and with σ defined by $\sigma(z) = \{e\} \times \tau(z)$, $X = Y \times \mathbb{R}^{n+1}/\sim$ is a locally σ-trivial $(H \times \mathbb{R}^{n+1})$-bundle over Z, which is globally trivial only if Y is trivial.

It is easy to check that the orbit space $X/(H \times \mathbb{R}^{n+1})$ is homeomorphic to Z, that σ is as asserted, and that σ is continous. If Y is H-isomorphic to $Z \times H$, then $X \cong Z \times (H \times \mathbb{R}^{n+1})/\sim$ is σ-trivial. The map τ is locally constant by Lemma 4.11 and Proposition 4.10, and hence σ is too. Finally, if f denotes the continuous map $[y, v] \mapsto y$ from X to Y, then a continuous section s for X would give a continuous section $f \circ s$ for Y; thus, X can be globally trivial only if Y is.

For an interesting concrete example of such a space, we can take $H = \mathbb{Z}_2$, q to be the canonical map of S^n onto $\mathbb{R}P^n$, and τ to be the identity. Then, as in the preceding example, $(w, v) \mapsto (w, P_w(v))$ induces an isomorphism of $S^n \times \mathbb{R}^{n+1}/\sim$ onto the orthogonal complement E of the real line bundle q^*L over S^n. Since q^*L is the normal bundle, E is the tangent bundle $T(S^n)$, and we obtain an action of $\mathbb{Z}_2 \times \mathbb{R}^{n+1}$ on $T(S^n)$ which is a locally σ-trivial bundle over $\mathbb{R}P^n$ for the map $\sigma \colon l \to \{e\} \times l$, but which is not globally σ-trivial.

5. ACTIONS WHICH ARE LOCALLY UNITARY ON THE STABILISERS

Locally unitary actions of an abelian group G where introduced in [20] as a C^*-algebraic analogue of (locally trivial) principal $\hat{G}$-bundles. It was proved there that when A is type I and α is locally unitary, then $(A \rtimes_\alpha G)^\wedge$

is a principal $\hat{G}$-bundle over $\hat{A}$ [20, Theorem 2.2], and, conversely, that if X is a principal $\hat{G}$-bundle, then the dual action of $G = \hat{\hat{G}}$ on $C^*(\hat{G}, X)$ is locally unitary [20, Theorem 3.1]. In [22], the main results concerned actions which were locally unitary on a common stabiliser group H, and we shall be interested here in actions which have a similar property with respect to continuously varying stabilisers. We think of this property as a dual analogue of the locally trivial spaces we studied in Section 4, and justify this by giving versions of [20, Theorems 3.1 and 2.2] (see Propositions 5.9 and 6.1).

We resume the notation of Section 2, so that $A = \Gamma_0(\xi)$ is a C^*-algebra with $X = \mathrm{Prim}(A)$ Hausdorff, and we have automorphisms $\alpha(s, x)$ of the fibres A_x for each $(s, x) \in \mathscr{P}$. Also, recall that we can view a multiplier b of $\Gamma_0(p^*\xi)$ as a (strictly continuous) field of multipliers $\{b(s, x)\} \in \prod_{(s,x)\in\mathscr{P}} \mathscr{M}(A_x)$ ([15, Theorem 2] or [1, Theorem 3.3]). Note that the following definition works perfectly well for non-abelian groups.

DEFINITION 5.1. Suppose that $A = \Gamma_0(\xi)$ is a C^*-algebra with $X = \mathrm{Prim}(A)$ Hausdorff, and that (A, G, α) is a C^*-dynamical system with continuous stabiliser map $x \mapsto G_x$. We say that α is unitary on the stabilisers if there is a unitary $u \in \mathscr{M}(\Gamma_0(p^*\xi))$ such that for all $a \in A$ and $(t, x), (s, x) \in \mathscr{P}$ we have

$$\begin{aligned} u(st, x) &= u(s, x)u(t, x) \\ \alpha(s, x)(a(x)) &= u(s, x)a(x)u(s, x)^*. \end{aligned} \tag{5.1}$$

Similarly, we say that α is locally unitary on the stabilisers if for each point x_0 of X there are a neighborhood N of x_0 and a $u \in \mathscr{M}(\Gamma_0(p^*\xi))$ such that $u(t, x)$ is unitary for each $(t, x) \in p^{-1}(N)$ and (5.1) holds on $p^{-1}(N)$.

It may be illuminating to reformulate the above definition in terms of groupoids: first let $\mathrm{Aut}(\xi)$ denote the groupoid consisting of pairs (x, κ), where $x \in X$ and $\kappa \in \mathrm{Aut}(A_x)$ (both the range and the source maps are the projection onto the first factor). Then there is an associated groupoid homomorphism α' from $\mathscr{P}$ to $\mathrm{Aut}(\xi)$ defined by $\alpha'_{(s,x)} = (x, \alpha(s, x))$, which is continuous in the sense that $(s, x) \mapsto \alpha(s, x)[a(x)]$ is a section of $p^*\xi$ for each $a \in \Gamma_0(\xi)$. In our terminology, α is unitary on the stablisers exactly when α' is implemented by a unitary in $\mathscr{M}(\Gamma_0(p^*\xi))$.

Other than the trivial example of free actions, it may not be immediately obvious that many actions are locally unitary on the stabilisers; hence, we will provide a number of classes of examples. Our first observation points out that Definition 5.1 extends that of [20]; the proof is straightforward.

LEMMA 5.2. *Let $\alpha: G \to \mathrm{Aut}(\Gamma_0(\xi))$ be a locally compact abelian automorphism group with $G_x = H$ for all $x \in X$. Then $\alpha|_H$ is locally unitary if and only if α is locally unitary on the stabilisers.*

For convenience in the sequel, we will sometimes shorten the phrase "locally unitary on the stabilisers" to simply "locally unitary." This should cause no confusion as the previous lemma says that these notions coincide whenever it makes sense for them to do so.

We will find it useful to consider actions where the stabiliser map is locally constant on the orbit space X/G as described in Definition 4.9. For simplicity, we assume that we can take $U = X/G$ in Definition 4.9. In other words, we have a map $c: X \to \mathrm{Aut}(G)$ which is constant on G-orbits, continuous in the compact open topology on G, such that there is a subgroup $H \subseteq G$ satisfying $c(x)(H) = G_x$ for all $x \in X$. It follows that the maps

$$(s, x) \mapsto c(x)(s) \qquad \text{and} \qquad (s, x) \mapsto c(x)^{-1}(x)$$

are continuous from $G \times X$ to X. Furthermore, by the uniqueness of Haar measure on G, for each $x \in X$ there is a $m(x) \in \mathbb{R}^+$ so that

$$\int_G f(s)\, ds = m(x) \int_G f(c(x)(s))\, ds. \tag{5.2}$$

It is easy to see that $m: X \to \mathbb{R}^+$ is continuous [12, Sect. 26.21].

LEMMA 5.3. *Suppose that $\alpha: G \to \mathrm{Aut}(\Gamma_0(\xi))$ is an abelian automorphism group and that $c: X \to \mathrm{Aut}(G)$ is as above. Then if $s \in G$, $a \in \Gamma_0(\xi)$, and $x \in X$,*

$$\beta_s(a)(x) = \alpha_{c(x)(s)}(a)(x)$$

defines an automorphism group of $\Gamma_0(\xi)$ with all stabiliser groups equal to H. In addition,

$$A \rtimes_\alpha G \cong A \rtimes_\beta G.$$

Proof. This is a straightforward extension of [22, Lemma 4.16]. One defines Φ from $C_c(G, \Gamma_c(\xi))$ to $C_c(G, \Gamma_c(\xi))$ by

$$\Phi(h)(s)(x) = m(x)\, h(c(x)(s))(x).$$

The only subtlety lies in calculating elements of the form $\alpha_s(\Phi(g)(r))(x)$. First of all, we observe that

$$[\Phi(g)(r) - m(x)\, g(c(x)(r))](s^{-1} \cdot x) = 0.$$

Then, by definition of the induced action on X, we have $a(s^{-1} \cdot x) = 0$ if and only if $\alpha_s(a)(x) = 0$, so this implies

$$\alpha_s(\Phi(g)(r))(x) = m(x)\, \alpha_s(g(c(x)(r)))(x).$$

Using this formula, it is a matter of direct calculation to check that Φ is a $*$-homomorphism (see the proof of [22, Lemma 4.16]); as Φ preserves the L^1-norm, it passes to the C^*-completion. It is an isomorphism because we can define its inverse in a similar fashion. ∎

COROLLARY 5.4. *With the same notation as above, if $\beta|_H$ is locally unitary, then α is locally unitary (on the stabilisers).*

We will say that $\alpha: G \to \mathrm{Aut}(\Gamma_0(\xi))$ is pointwise unitary on the stabilisers (or simply pointwise unitary), if given $x \in X$ there is a representation u of G_x such that $\alpha_s(a)(x) = u_s a(x) u_s^*$ for all $s \in G_x$ and $a \in \Gamma_0(\xi)$. If A is type I, this is equivalent to saying the Mackey obstructions vanish.

PROPOSITION 5.5. *Let $A = \Gamma_0(\xi)$ be a separable continuous trace C^*-algebra. Suppose that $\alpha: G \to \mathrm{Aut}(A)$ is a compactly generated abelian automorphism group with locally constant stabilisers. Then, if α is pointwise unitary (on the stabilisers), α is locally unitary.*

Proof. Fix a $x \in X$. Then there is a representation u of G_x so that $\alpha_s(a)(x) = u_s a(x) u_s^*$ for all $a \in A$. Thus,

$$\beta_s(a)(x) = u_{c(x)(s)} a(x) u^*_{c(x)(s)}$$

for all $s \in H$. In other words, $\beta|_H$ is pointwise unitary. It follows from [29, Corollary 2.2] that $\beta|_H$ is locally unitary. Now apply Corollary 5.4. ∎

Our next family of examples involves the notion of pull-backs of C^*-algebras (see [23]). For this, let ρ denote the induced action of G on $C_0(X)$ and assume that X/G is Hausdorff. Then if B is a C^*-algebra with spectrum X/G we can define the pull-back

$$q^*B = C_0(X) \otimes_{C(X|G)} B$$

of B via the quotient map $q: X \to X/G$. Furthermore, if γ is an automorphism of B, then there is a well-defined automorphism of the pull-back $q^*\gamma = \rho \otimes_{C(X/G)} \gamma$ as defined following the proof of Theorem 2.2 in [23].

LEMMA 5.6. *If $\gamma: G \to \mathrm{Aut}_{C_0(X/G)}(B)$ is a locally unitary automorphism group and*

$$\alpha = \rho \otimes_{C(X/G)} \gamma,$$

then α is locally unitary (on the stabilisers).

Proof. As this is a local result, there is no loss in generality in assuming that γ is inner; thus, $\gamma = \mathrm{ad}\, u$, where $u: G \to U(\mathcal{M}(B))$ is a strictly continuous homomorphism. Then $v(s, x) = 1 \otimes_{C(X)} 1 \otimes_{C(X \cdot G)} u_s(G \cdot x)$ defines a $v \in U(\mathcal{M}(\Gamma_0(q^*\xi)))$ which implements α over $\mathscr{P}$ as in Definition 5.1. ▮

Remark 5.7. Even when an action α on a pull back q^*B does not have the form $q^*\gamma$, we can sometimes reduce to this case by comparing α with the canonical action $\tau = q^*id$ by translation. Provided $A = q^*B$ is separable, we obtain a 1-cocycle $\beta = \alpha \circ \tau^{-1}$ with values in the Polish G-module $\mathrm{Aut}_{C(X)} A$, where $s \in G$ acts by conjugation by τ_s. If β takes values in $\mathrm{Inn}(A)$, there is an obstruction in the Moore cohomology group $H^2(G, C(X, \mathbb{T}))$ which vanishes if and only if α is exterior equivalent to τ [22, Theorem 0.11], in which case we have

$$A \rtimes_\alpha G \cong q^*B \rtimes_\alpha G \cong q^*B \rtimes_{q^*id} G \cong C^*(G, X) \otimes_{C(X \cdot G)} B.$$

When B (and hence also A by [22, Lemma 1.2]) has continuous trace and the Čech group $\check{H}^2(X, \mathbb{Z})$ is countable, then $\mathrm{Inn}(A)$ is open and closed in $\mathrm{Aut}_{C(X)} A$ [22, Theorem 0.8]. Thus, for connected G, the cocycle β often automatically lies in $\mathrm{Inn}(A)$. This will be particularly interesting when $G = \mathbb{R}$, for then the cohomology group $H^2(\mathbb{R}, C(X, \mathbb{T}))$ is trivial [22, Theorem 4.1].

Another class of examples arises as the natural generalization of [20, Theorem 3.1]. As we stressed in Section 4, a locally σ-trivial G-space X is the (non-constant fibre) analogue of a locally trivial principal G-bundle. Therefore it is not surprising that it is a consequence of Proposition 4.5 and [33, Theorem 5.1] that $C^*(G, X)$ has continuous trace. Furthermore following Proposition 4.8, the map

$$(G \cdot x, \gamma) \mapsto \mathrm{Ind}_{G_x}^{G}(\varepsilon_x \times \gamma)$$

induces a homeomorphism of $X/G \times \hat{G}/\sim$ onto $C^*(G, X)\hat{}$. Furthermore, the map $[G \cdot x, \gamma] \mapsto G \cdot x$ is a well-defined open, continuous surjection by virtue of the remarks following [32, Theorem 5.3]. It follows from [31, Theorem 8.3] that $C^*(G, X)$ may be realized as the sections of a C^*-bundle over X/G with fibre over x isomorphic to $C^*(G, G \cdot x)$, which is in turn isomorphic to $C^*(G, G/G_x)$ since $s \mapsto s \cdot x$ induces a homeomorphism of G/G_x wih $G \cdot x$ (which certainly follows from [3], since (G, X) is second countable, or from [33, Proposition 2.17]).

PROPOSITION 5.8. *Suppose that X is a locally σ-trivial G-space as in Definition 4.2. Then the dual action on $C^*(G, X)$ is locally unitary on the stabilisers.*

Proof. We are guaranteed local sections for $X \to X/G$; thus, we may as well assume that there is a global section $c: X/G \to X$ and prove that $\hat{\alpha}$ is unitary on $C^*(G, X)$.

As above, we realize $C^*(G, X)$ as sections of C^*-bundle E over X/G, with fibre $C^*(G, G/G_x)$ over $G \cdot x$. In fact, if $z \in C_c(G \times X)$, then the associated section, $\pi(z)$, is given by

$$\pi(z)[G \cdot x](s, tG_x) = z(s, tc(G \cdot x)).$$

Now given $G \cdot x \in X/G$ and $\gamma \in G_x^{\perp}$ we define a multiplier $w_{(G \cdot x, \gamma)}$ of the fibre over $G \cdot x$ (i.e., $C^*(G, G/G_x)$) by

$$(w_{(G \cdot x, \gamma)} y)(s, tG_x) = \gamma(t)\, y(s, tG_x)$$

and

$$(y w_{(G \cdot x, \gamma)})(s, tG_x) = y(s, tG_x)\, \gamma(s^{-1}t),$$

where $y \in C_c(G \times G/G_x)$. It follows from [22, Lemma 2.4] that, for fixed $G \cdot x$, $\gamma \mapsto w_{(G \cdot x, \gamma)}$ defines a strictly continuous homomorphism of $G_x^{\perp}$ into $U(\mathcal{M}(C^*(G, G/G_x)))$ which implements the dual action of $\hat{G}$.

On the other hand Lemma 2.3 of [22] implies that

$$\mathrm{Ind}_{G_x}^{G}(\varepsilon_x \times \gamma) \circ \hat{\alpha}_\chi^{-1} \cong \mathrm{Ind}_{G_x}^{G}(\varepsilon \times \gamma\chi\,|_{G_x}).$$

Therefore in this example,

$$\mathscr{P} = \{([G \cdot x, \chi], \gamma) \in (X/G \times \hat{G}/\sim) \times \hat{G} : \gamma \in G_x^{\perp}\}.$$

Our object is to use w to define an element of $U(\mathcal{M}(p^*(C^*(G, X))))$. Now $p^*(C^*(G, X))$ has spectrum $\mathscr{P}$ [23, Lemma 1.1] and may be viewed as sections over $\mathscr{P}$ with fibre over $([G \cdot x, \chi], \gamma)$ equal to $C^*(G, G/G_x)$. Hence if $b \in p^*(C^*(G, X))$, then we can define

$$wb([G \cdot x, \chi], \gamma) = w_{(G \cdot x, \gamma)} b([G \cdot x, \chi], \gamma).$$

It will then suffice to show that wb is in $p^*(C^*(G, X))$ ([15] or [1]). Since $p^*(C^*(G, X))$ is spanned by sections of the form $b = \phi \otimes \pi(z)$ with $\phi \in C_c(\mathscr{P})$ and $z \in C_c(G \times X)$, it will be enough to show that

$$([G \cdot x, \chi], \gamma) \mapsto \phi([G \cdot x, \chi], \gamma)\, w_{(G \cdot x, \gamma)} \pi(z)[G \cdot x] \tag{5.3}$$

defines a section on $p^*(C^*(G, X))$. Since the appearance of ϕ in (5.3) guarantees that the section vanishes at infinity, it will suffice to show the following: given $\varepsilon > 0$, $G \cdot x_0 \in X/G$, and $\gamma_0 \in G_{x_0}^{\perp}$, there is a $y \in C_c(G, X)$ and a neighborhood N of $(G \cdot x_0, \gamma_0) \in \mathscr{P}' = \{(\tau, \chi) \in X/G \times \hat{G} : \chi \in G_\tau^{\perp}\}$ such that

$$\|w_{(\tau, \gamma)} \pi_\tau(z) - \pi_\tau(y)\| < \varepsilon$$

whenever $(\tau, \gamma) \in N$.

Let $K_1 \subseteq G$ and $K_2 \subseteq X$ be compact sets with $\mathrm{supp}(z) \subseteq K_1 \times K_2$. Because the action of G is σ-proper (Proposition 4.5) and the quotient map

$\delta: X \times G \to X \times G/\sim$ is open [33, Lemma 2.3], we can find compact sets $L \subseteq X$ and $M \subseteq G$ such that

$$\delta(c(K_2/G) \cup K_2) \subseteq L \times M/\sim.$$

Now choose $y \in C_c(G \times X)$ such that $\operatorname{supp} y \subseteq \operatorname{supp} z$ and

$$y(s, tc(G \cdot x_0)) = \gamma_0(t) z(s, tc(G \cdot x_0))$$

for all $s, t \in G$. Choose a neighborhood N of $(G \cdot x_0, \gamma_0)$ such that, whenever $(\tau, \gamma) \in N$, we have $\tau \in K_2/G$ and

$$\sup_{\substack{t \in M \\ s \in G}} \{|y(s, tc(\tau)) - y(s, tc(G \cdot x_0))|\} < \frac{\varepsilon}{3\mu(K_1)} \tag{5.4}$$

$$\sup_{\substack{t \in M \\ s \in G}} \{|z(s, tc(\tau)) - z(s, tc(G \cdot x_0))|\} < \frac{\varepsilon}{3\mu(K_1)} \tag{5.5}$$

$$\sup_{t \in M} \{|\gamma(t) - \gamma_0(t)|\} < \frac{\varepsilon}{3\, \|\pi_{G \cdot x_0}(z)\|_1}. \tag{5.6}$$

Now fix $(\tau, \chi) \in N$ and compute that

$$\|w_{(\tau,\chi)} \pi_\tau(z) - \pi_\tau(y)\|_{C^*(G, G/G_x)} \leqslant \int_G \|w_{(\tau,\chi)} \pi_\tau(z)(s) - \pi_\tau(y)(s)\|_{C_0(G/G_x)}\, ds$$

$$\leqslant \int_G \sup_{t \in M} |\chi(t) z(s, tc(\tau)) - y(s, tc(\tau))|\, ds,$$

and, using the fact that $y = \gamma_0 \cdot z$ on $G \times G \cdot x_0$, this is

$$\leqslant \int_G \sup_{t \in M} |\chi(t)\, z(s, tc(\tau)) - \chi(t)\, z(s, tc(G \cdot x_0))|\, ds$$

$$+ \int_G \sup_{t \in M} \{|\chi(t) - \gamma_0(t)|\, z(s, tc(G \cdot x_0))\}\, ds$$

$$+ \int_G \sup_{t \in M} |y(s, tc(G \cdot x_0)) - y(s, tc(\tau))|\, ds.$$

It follows from (5.4)-(5.6) that the last sum is bounded by ε; this completes the proof. ∎

6. The Spectrum of $A \rtimes_\alpha G$ When α Is Locally Unitary on the Stabilisers

In this section we assume that A is type I, and we shall show that when α is locally unitary on the stabilisers and $\hat{A} \to \hat{A}/G$ is locally trivial, then the diamond in Theorem 3.10 consists of locally trivial spaces. We shall then discuss some interesting examples in detail.

We begin by considering the $\hat{G}$-space $\mathrm{Res}: (A \rtimes_\alpha \mathscr{P})^\wedge \to \hat{A}$ of Theorem 2.2; we assume $\hat{A}$ is Hausdorff. Recall that the dual action of $\hat{G}$ on $(A \rtimes_\alpha \mathscr{P})^\wedge$ is given by

$$\gamma \cdot ((\pi \times U) \circ \pi_x) = ((\pi \times U) \circ \pi_x) \circ \hat{\alpha}_\gamma^{-1} = (\pi \times \gamma|_{G_x} U) \circ \pi_x.$$

When α is pointwise unitary, the action $s \mapsto \alpha(s, x)$ of G_x on A_x is unitary, and it follows from, for example, [20, Proposition 2.1] that $\pi \times \gamma|_{G_x} U$ is equivalent to $\pi \times U$ exactly when $\gamma \in G_x^\perp$. Thus the stabiliser map for the action of $\hat{G}$ on $(A \rtimes_\alpha \mathscr{P})^\wedge$ is the composition $\hat{\sigma}$ of σ with the map $H \mapsto H^\perp$ of Σ_G onto $\Sigma_{\hat{G}}$; this is continuous when σ is [34].

If $x \in X = \hat{A}$, we shall write ε_x for the representation $a \mapsto a(x)$ of A, and if $u: G \to U\mathscr{M}(A)$ implements $\alpha(\cdot, x)$ in the representation ε_x, we denote by $\varepsilon_x \times u$ the representation of $A \rtimes_\alpha \mathscr{P}$ such that

$$\varepsilon_x \times u(f) = \int_{G_x} f(s, x)\, u_s\, d\lambda_x(s), \qquad \text{for} \quad f \in \Gamma_c(p^*\xi).$$

Our next result is a direct generalization of [20, Theorem 2.3 and Proposition 2.5].

PROPOSITION 6.1. *Suppose that A is a type I C^*-algebra with Hausdorff spectrum X and $\alpha: G \to \mathrm{Aut}(A)$ is locally unitary on the stabilisers. Then $A \rtimes_\alpha \mathscr{P}$ is type I and $\mathrm{Res}: (A \rtimes_\alpha \mathscr{P})^\wedge \to X$ is locally $\hat{\sigma}$-trivial space with respect to the dual action of $\hat{G}$; if $u \in \mathscr{M}(p^*A)$ implements α over N, then*

$$h(x, \gamma) = \varepsilon_x \times \gamma|_{G_x} u(\cdot, x)$$

induces a homeomorphism h of $(N \times \hat{G})/\sim$ onto $\mathrm{Res}^{-1}(N)$. If $\hat{\sigma}$ is locally liftable, then Res *is a $\hat{\sigma}$-trivial bundle if and only if α is unitary on the stabilisers.*

The idea of the proof is to localize to an ideal of A where α is unitarily implemented, and then prove the spectrum is $\hat{\sigma}$-trivial. For this we need to know that if F is a closed subset of X, I_F is the corresponding ideal in A and $\mathscr{P}_{X \setminus F} = \{(s, x) \in \mathscr{P}: x \notin F\}$, then $I_F \rtimes_\alpha \mathscr{P}_{X \setminus F}$ embeds as an ideal in $A \rtimes_\alpha \mathscr{P}$. Technically, we have not yet defined $I_F \rtimes_\alpha \mathscr{P}_{X \setminus F}$, but we can use the construction of $A \rtimes_\alpha \mathscr{P}$ essentially verbatim. As usual, we write $A = \Gamma_0(\xi)$.

LEMMA 6.2. *Let F be a closed subset of X. Then the map $i: \Gamma_c(p^*\xi|_{X \setminus F}) \to \Gamma_c(p^*\xi)$ defined by*

$$i(f)(s, x) = \begin{cases} f(s, x), & \text{if} \quad (s, x) \in \mathscr{P}_{X \setminus F}, \\ 0, & \text{if} \quad (s, x) \in \mathscr{P}_F. \end{cases}$$

extends to a $*$-homomorphism of $I_F \rtimes_\alpha \mathscr{P}_{X \setminus F}$ into $A \rtimes_\alpha \mathscr{P}$, and we have a short exact sequence

$$0 \to I_F \rtimes_\alpha \mathscr{P}_{X \setminus F} \xrightarrow{i} A \rtimes_\alpha \mathscr{P} \xrightarrow{\pi_F} A_F \rtimes_\alpha \mathscr{P}_F \to 0.$$

In particular, $I_F \rtimes_\alpha \mathscr{P}_{X \setminus F} = \bigcap \{\ker(\pi_x): x \in F\}$.

Proof. It is clear that i is a $*$-homomorphism, and it is isometric for the C^*-norms because every irreducible representation of $A \rtimes_\alpha \mathscr{P}$ factors through some π_x, and $\pi_x \circ i = 0$ unless $x \in F$. For $x \notin F$, we have $(I_F)_x = A_x$, so the map $\pi_x \circ i: I_F \rtimes_\alpha \mathscr{P}_{X \setminus F} \to A_x \rtimes_\alpha G_x$ is surjective, and it follows that the range of i is not contained in $\ker(\rho \circ \pi_x)$ for any $\rho \in (A_x \rtimes_\alpha G_x)^\wedge$. Since the range of i is an ideal and every ideal is the intersection of the primitive ideals containing it, this shows that

$$\text{range } i = \bigcap \{\ker(\pi_x): x \in F\}.$$

But this is the kernel of π_F, again because every irreducible representation of $A_F \rtimes_\alpha \mathscr{P}_F$ factors through some π_x for $x \in F$. ∎

Proof of Proposition 6.1. For each $x \in X$, the action $\alpha(\cdot, x)$ of G_x on A_x is actually unitary, so

$$A_x \rtimes_\alpha G_x \cong A_x \otimes C^*(G_x) \cong A_x \otimes C_0(\hat{G}_x)$$

is of type I. Since every irreducible representation of $A \rtimes_\alpha \mathscr{P}$ factors through some π_x, it must be type I too. Now suppose that u implements α on $\mathscr{P}_N$. It follows from Lemma 6.2 that $(I_{X \setminus N} \rtimes_\alpha \mathscr{P}_N)^\wedge$ can be naturally identified with the open subset $\text{Res}^{-1}(N)$ of $(A \rtimes_\alpha \mathscr{P})^\wedge$. The dual action of $\hat{G}$ leaves the ideal $I_{X \setminus N} \rtimes_\alpha \mathscr{P}_N$ invariant, and this identification therefore preserves the actions of $\hat{G}$. So we may as well assume that α is implemented by u over all of X and prove that $(A \rtimes_\alpha \mathscr{P})^\wedge$ is $\hat{\sigma}$-trivial. (This reduction is more subtle than it may look at first sight, as we no longer have an action of G on $I_{X \setminus N}$ which restricts to the action of $\mathscr{P}_N$! However, this is not important for what follows because the action of G on A only enters the argument through the fibre automorphisms $\alpha(r, x)$ for $(r, x) \in \mathscr{P}$.)

We next consider the case $A = C_0(X)$, where $C^*(\mathscr{P}) = A \rtimes_\alpha \mathscr{P}$ is abelian. We define $\phi: X \times \hat{G} \to C^*(\mathscr{P})^\wedge$ by $\phi(x, \gamma) = \varepsilon_x \times \gamma$; note that ϕ is onto by Theorem 2.2. It can be easily verified that ϕ induces a $\hat{G}$-equivariant continous bijection of $X \times \hat{G}/\sim$ onto $C^*(\mathscr{P})^\wedge$; as in the final three paragraphs of [17, Lemma 2.6] (where $X = \Sigma_G$), the openness of ϕ follows from the continuity of inducing representations.

Now we suppose that A is as in the statement of the proposition. If we let D be the pull-back C^*-algebra $C^*(\mathscr{P}) \otimes_{C(X)} A$, then it follows from [23, Lemma 1.1] and the preceding paragraph that $(x, \gamma) \mapsto (\varepsilon_x \times \gamma) \otimes id$ induces a homeomorphism ϕ_1 of $(X \times \hat{G})/\sim$ onto $\hat{D}$. We can view D as the completion of $\Gamma_c(p^*\xi)$ with the $*$-algebraic structure defined as for the subalgebra of $A \rtimes_\alpha \mathscr{P}$, but with α trivial. Then the map $\Psi: \Gamma_c(p^*\xi) \to \Gamma_c(p^*\xi)$ defined by $\Psi(f) = fu$ is a $*$-homomorphism of a dense subalgebra of $A \rtimes_\alpha \mathscr{P}$ onto a dense subalgebra of D. Further, Ψ is isometric for the $\|\cdot\|_\Gamma$-norms, and extends to an isomorphism of $A \rtimes_\alpha \mathscr{P}$ onto D. A quick calculation shows that

$$((\varepsilon_x \times \gamma) \otimes id) \circ \Psi^{-1} = \varepsilon_x \times \gamma|_{G_x} u(\cdot, x),$$

so that the homeomorphism $\Psi^* \circ \phi_1$ of $(X \times G)/\sim$ onto $(A \rtimes_\alpha \mathscr{P})^\wedge$ is as described. It is clearly $\hat{G}$-equivariant.

Now suppose that $\hat{\sigma}$ is locally liftable, that $u_i \in \mathscr{M}(p^*A)$ implements α over N_i, and that h_i are the corresponding local trivializations. Then as in the proof of Proposition 4.5, we can assume there are transition functions $\gamma_{ij}: N_{ij} \to G$ such that

$$h_i([x, \gamma_{ij}(x)\gamma]) = h_j([x, \gamma]), \qquad \text{for} \quad x \in N_{ij}, \quad \gamma \in \hat{G}.$$

By Proposition 4.5, saying Res is $\hat{\sigma}$-trivial implies (shrinking the N_i again if necessary) that there are maps $\chi_i: N_i \to \hat{G}$ such that

$$\chi_i(x)^{-1} \gamma_{ij}(x) \chi_j(x) \in G_x^\perp, \qquad \text{for} \quad x \in N_{ij}.$$

Then

$$h_i([x, \chi_i(x)\gamma]) = h_j([x, \chi_j(x)\gamma]), \qquad \text{for} \quad x \in N_{ij}, \quad \gamma \in \hat{G},$$

which implies that

$$\chi_i(x)|_{G_x} u_i(\cdot, x) = \chi_j(x)|_{G_x} u_j(\cdot, x) \qquad \text{in } \hat{G}_x.$$

Thus $u(s, x) = \chi_i(x) u_i(s, x)$ for $x \in N_i$ defines an element of $\mathscr{M}(\Gamma_0(p^*\xi))$, $s \mapsto u(s, x)$ is a homomorphism on G_x for all x, and u implements α over all of X. ∎

We now come to our main theorem. The key point here is that there are commuting actions of G and $\hat{G}$ on the stabiliser algebra $A \rtimes_\alpha \mathscr{P}$: the dual action $\hat{\alpha}$ of $\hat{G}$, and the action β of G discussed in Section 3. The space $(A \rtimes_\alpha \mathscr{P})^\wedge$ is locally trivial for both these actions, and we can use Res and Ind to identify the orbit spaces $(A \rtimes_\alpha \mathscr{P})^\wedge/\hat{G}$ and $(A \rtimes_\alpha \mathscr{P})^\wedge/G$ with $\hat{A}$ and $(A \rtimes_\alpha G)^\wedge$.

THEOREM 6.3. *Suppose that (A, G, α) is a separable C^*-dynamical system with G abelian, $\hat{A}$ Hausdorff, and continuous stabiliser map $\sigma: \hat{A} \to \Sigma_G$. If α is locally unitary on the stabilisers and $\hat{A}$ is a locally σ-trivial*

G-space, then $A \rtimes_\alpha \mathscr{P}$ *and* $A \rtimes_\alpha G$ *are type I and the commutative diagram of Theorem* 3.10

$$\begin{array}{ccccc} & & (A \rtimes_\alpha \mathscr{P})\hat{} & & \\ & \swarrow^{\text{Ind}} & & \searrow^{\text{Res}} & \\ (A \rtimes_\alpha G)\hat{} & & & & \hat{A} \\ & \searrow_{q} & & \swarrow_{p} & \\ & & \hat{A}/G & & \end{array}$$

consists of locally trivial spaces: the southeast arrows are locally $\hat{\sigma}$*-trivial* $\hat{G}$*-spaces and the southwest arrows are locally* σ*-trivial G-spaces.*

Proof. We have already seen that $A \rtimes_\alpha \mathscr{P}$ is type I, and because the orbits in $\hat{A}/G$ are closed, the corresponding fact for $A \rtimes_\alpha G$ follows easily from Green's version of the Mackey machine (see, for example, the first paragraph of the proof of [22, Theorem 2.2]). That Res is locally $\hat{\sigma}$-trivial was proved in Proposition 6.1, and in particular $(A \rtimes_\alpha \mathscr{P})\hat{}$ is Hausdorff: we shall next prove it is a σ-proper G space. To this end, suppose that $\{(y_n, s_n)\}$ is a sequence in $(A \rtimes_\alpha \mathscr{P})\hat{} \times G$ such that $s_n \cdot y_n \to w$ and $y_n \to y$. It will suffice to show that there is a subsequence $\{s_{n_k}\}$ such that there exist $t_{n_k} \in G_{y_{n_k}}$ with $s_{n_k} \cdot t_{n_k} \to s$. However, the continuity of Res implies that

$$(s_n \cdot \text{Res}\, y_n, \text{Res}\, y_n) = (\text{Res}(s_n \cdot y_n), \text{Res}\, y_n) \to (\text{Res}\, w, \text{Res}\, y)$$

in $\hat{A} \times \hat{A}$, so this follows from the properness of $\hat{A}$.

By Proposition 4.3(2) and Theorem 3.1, the spectrum of $A \rtimes_\alpha G$ is Hausdorff. To see it is a $\hat{\sigma}$-proper $\hat{G}$-space, suppose we have sequences $\{z_n\}$, $\{\gamma_n\}$ such that $(\gamma_n \cdot z_n, z_n) \to (w, z)$. As Ind is open, we can by passing to a subsequence suppose there are $\{y_n\} \subseteq (A \rtimes_\alpha \mathscr{P})\hat{}$, $\{s_n\} \subseteq G$, and $y, y' \in (A \rtimes_\alpha \mathscr{P})\hat{}$ such that

$$\text{Ind}(y_n) = z_n, \qquad y_n \to y \qquad \text{and} \qquad s_n \cdot \gamma_n \cdot y_n \to y';$$

because Ind is continuous and $\hat{G}$-equivariant, we must also have $\text{Ind}(y) = z$ and $\text{Ind}(y') = w$. Now we have

$$(\text{Res}(s_n \cdot \gamma_n \cdot y_n), \text{Res}(y_n)) = (s_n \cdot \text{Res}(y_n), \text{Res}(y_n)) \to (\text{Res}(y'), \text{Res}(y)),$$

and because $\hat{A}$ is σ-proper we can by passing to another subsequence suppose that $\{s_n\}$ converges to some s in G. Then we have

$$(\gamma_n \cdot (s_n \cdot y_n), s_n \cdot y_n) = (s_n \cdot (\gamma_n \cdot y_n), s_n \cdot y_n) \to (y', s \cdot y);$$

because $(A \rtimes_\alpha \mathscr{P})\hat{}$ is $\hat{\sigma}$-proper, we can pass to yet another subsequence to ensure that γ_n converges in $\hat{G}$. Thus $(A \rtimes_\alpha G)\hat{}$ is indeed a $\hat{\sigma}$-proper $\hat{G}$-space.

By Proposition 4.3, it will now suffice to show that both Ind and q have local cross sections. Since this is a local problem, we may as well suppose the map p has a global section c, and that there is a section d for Res defined on a neighborhood U of $c(\hat{A}/G)$. Straight away this gives a section $f = \text{Ind} \circ d \circ c$ for q, and it remains to find one for Ind. As in the proof of Proposition 4.3(3), using the section f we define a $\hat{G}$-isomorphism ϕ of $(\hat{A}/G \times \hat{G})/\sim$ onto $(A \rtimes_\alpha G)\hat{}$ by

$$\phi([t, \gamma]) = \gamma \cdot f(t), \qquad \text{for} \quad t \in \hat{A}/G, \quad \gamma \in \hat{G};$$

the inverse is given by

$$\phi^{-1}(x) = [q(x), \gamma], \qquad \text{where } \gamma \text{ satisfies } \gamma \cdot f(q(x)) = x.$$

We take $g\colon (A \rtimes_\alpha G)\hat{} \to (A \rtimes_\alpha \mathscr{P})\hat{}$ to be the composition

$$(A \rtimes_\alpha G)\hat{} \xrightarrow{\phi^{-1}} (\hat{A}/G \times \hat{G})/\sim \xrightarrow{c \times id} (U \times \hat{G})/\sim \xrightarrow{\mu} X,$$

where

$$c \times id([t, \gamma]) = [c(t), \gamma] \qquad \text{and} \qquad \mu([y, \gamma]) = \gamma \cdot d(y).$$

Then g is continuous because all its constituents are, and

$$\begin{aligned} \text{Ind} \circ g(x) &= \text{Ind}(\mu(c \times id([q(x), \gamma]))), \qquad \text{where} \quad \gamma \cdot f(q(x)) = x \\ &= \text{Ind}(\gamma \cdot d(c(q(x)))) \\ &= \gamma \cdot \text{Ind} \circ d \circ c(q(x)) \\ &= \gamma \cdot f(q(x)) \\ &= x, \end{aligned}$$

so g is a section for Ind. This completes the proof of Theorem 6.3. ∎

Remark 6.4. If both $\sigma\colon \hat{A} \to \Sigma_G$ and $\hat{\sigma}\colon \hat{A} \to \Sigma_{\hat{G}}$ are locally liftable, then the diamond in Theorem 6.3 will consist of locally trivial *bundles*. We observe, however, that these seem to be independent properties of the map σ, in the sense that one of σ or $\hat{\sigma}$ can be locally liftable while the other is not. For example, consider the map $\sigma\colon [0, \infty) \to \Sigma_{\mathbb{R}}$ defined by

$$\sigma(r) = \begin{cases} r\mathbb{Z}, & r \neq 0, \\ \mathbb{R}, & r = 0, \end{cases}$$

for which $\hat{\sigma}\colon [0, \infty) \to \Sigma_{\hat{\mathbb{R}}} = \Sigma_{\mathbb{R}}$ is given by

$$\hat{\sigma}(r) = \begin{cases} (1/r)\mathbb{Z}, & r \neq 0, \\ 0, & r = 0. \end{cases}$$

We saw in Example 4.6 that σ is not locally liftable, and we shall now see that $\hat{\sigma}$ is. Suppose $c\colon [0, \infty) \to [0, \infty) \times \mathbb{R}/\sim$ is a continuous section, where the equivalence relation is that determined by $\hat{\sigma}$. There is no problem away from 0—σ and $\hat{\sigma}$ are locally constant so Proposition 4.10 applies. The quotient map q is one-to-one on the neighborhood $U = [0, \frac{1}{2}) \times [0, \frac{1}{2})$ of $(0, 0)$, and hence induces a homeomorphism of U onto a neighborhood W of $[(0, 0)]$ in the quotient. But then $h^{-1} \circ c$ is a continuous map of $c^{-1}(W)$ into $[0, \frac{1}{2}) \times \mathbb{R}$, and the projection of $h^{-1} \circ c$ on the second factor gives the required local lift for c.

We shall now discuss some examples, mostly concerning actions of $\mathbb{R}^n$. Our first observation, however, is that any locally $\hat{\sigma}$-trivial space can arise as the bottom left-hand arrow in our diamond.

PROPOSITION 6.5. *Let Y be a separable locally σ-trivial G-space for a second countable locally compact group G. Then $A = C^*(G, Y)$ has continuous trace, the dual action α of $\hat{G}$ on A is locally unitary on the stabilisers, and $q\colon (A \rtimes_\alpha \hat{G})^\wedge \to \hat{A}/\hat{G} = Y/G$ is G-isomorphic to $Y \to Y/G$.*

Proof. The algebra A has continuous trace by [33, Theorem 5.1] and α is locally unitary by Proposition 5.8. The Takai duality theorem gives a covariant isomorphism

$$(C^*(G, Y) \rtimes_\alpha \hat{G}, G, \hat{\alpha}) \cong (C_0(Y) \otimes \mathcal{K}(L^2(G)), G, \tau \otimes \operatorname{Ad} \rho);$$

since $\tau \otimes \operatorname{Ad} \rho$ clearly induces the original action of G on $Y = (C_0(Y) \otimes \mathcal{K})^\wedge$, this isomorphism induces the required G-isomorphism of $(A \rtimes_\alpha \hat{G})^\wedge$ onto Y. ∎

EXAMPLE 6.6. Suppose α is an action of $\mathbb{R}^n$ on a separable continuous-trace algebra A such that the stabiliser map σ is continuous with values in $\mathrm{Gr}(n, k)$ for some k, the action of $\mathbb{R}^n$ on $\hat{A}$ is σ-proper, and the Mackey obstructions all vanish (equivalently, the action is pointwise unitary on the stabilisers). Then σ is locally constant by Lemma 4.11, and α is therefore locally unitary on the stabilisers by Proposition 5.5. For such a stabiliser map σ, every σ-proper action is locally σ-trivial (Proposition 4.10) and Theorem 6.3 applies. We deduce that $(A \rtimes_\alpha \mathbb{R}^n)^\wedge$ is a locally $\hat{\sigma}$-trivial $\mathbb{R}^n$-space over $\hat{A}/\mathbb{R}^n$. Since $\hat{\sigma}$ is continuous with values in $\mathrm{Gr}(n, n-k)$, such spaces are globally trivial by Proposition 4.13. Thus, $(A \rtimes_\alpha \mathbb{R}^n)^\wedge$ is $\mathbb{R}^n$-isomorphic to the space $(\hat{A}/\mathbb{R}^n \times \mathbb{R}^n)/\sim$ defined by the map $\hat{\sigma}\colon \hat{A}/\mathbb{R}^n \to \mathrm{Gr}(n, n-k) \subseteq \Sigma_{\mathbb{R}^n}$.

EXAMPLE 6.7. *Actions of* $\mathbb{R}$. Suppose X is a locally σ-trivial $\mathbb{R}$-space, and $A = p^*B$ is the pull-back of a continuous trace algebra B with spectrum $X/\mathbb{R}$ along the orbit map $p: X \to X/\mathbb{R}$. Then, as in Remark 5.7, we can compare an action α of $\mathbb{R}$ on A inducing the given action on X with the translation action $\tau = p^*id$. Here the appropriate Moore cohomology goup $H^2(\mathbb{R}, C(X, \mathbb{T}))$ is trivial [22, Theorem 4.1] and $\mathbb{R}$ is connected. Provided $\check{H}^2(X, \mathbb{Z})$ is countable, therefore, α will be exterior equivalent to τ, and

$$A \rtimes_\alpha \mathbb{R} \cong (C_0(X) \otimes_{C(X/\mathbb{R})} B) \rtimes_{\tau \otimes id} \mathbb{R} \cong C^*(\mathbb{R}, X) \otimes_{C(X/\mathbb{R})} B \cong q^*B,$$

where q is the projection of $C^*(\mathbb{R}, X)\hat{} = (X/\mathbb{R} \times \mathbb{R})/\sim$ onto $X/\mathbb{R}$ (see Proposition 4.8).

Now Proposition 6.5 shows that any locally σ-trivial $\mathbb{R}$-space Y can be realized as $(A \rtimes_\alpha \mathbb{R})\hat{}$ for the dual action of $\mathbb{R}$ on $A = C^*(\mathbb{R}, Y)$; if A were the pull-back of some algebra B, then by the argument in the preceding paragraph we would have $(A \rtimes_\alpha \mathbb{R})\hat{}$ trivial. Thus if Y is non-trivial, A cannot be a pull-back. Stabilizing A does not affect the spectrum of $A \rtimes_\alpha \mathbb{R}$, so by [23, Proposition 1.4] this implies that $\delta(A)$ is not in the range of the induced map p^* from $\check{H}^3(\hat{A}/\mathbb{R}, \mathbb{Z})$ to $\check{H}^3(\hat{A}, \mathbb{Z})$. In particular, we have:

PROPOSITION 6.8. *Let Y be a second countable locally σ-trivial $\mathbb{R}$-space, and suppose that $\check{H}^2((Y/\mathbb{R} \times \mathbb{R})/\sim, \mathbb{Z})$ is countable (where the equivalence relation is the one determined by $\hat{\sigma}$). Then $C^*(\mathbb{R}, Y)$ is a continuous trace C^*-algebra with spectrum $(Y/\mathbb{R} \times \mathbb{R})/\sim$, and the Dixmier–Douady class $\delta(C^*(\mathbb{R}, Y))$ vanishes if and only if Y is globally σ-trivial.*

Proof. That $C^*(\mathbb{R}, Y)$ has continuous trace and the given spectrum is proved in [33, Theorem 5.1; 32] (see also Proposition 4.8), and we have just seen that if Y is non-trivial, then $\delta(C^*(\mathbb{R}, Y)) \neq 0$. Conversely, if Y is isomorphic to $(Y/\mathbb{R} \times \mathbb{R})/\sim$ (for $\sim$ defined by σ), then $Y \to Y/\mathbb{R}$ has a continuous cross section and [17, Lemma 3.2 and Theorem 2.3] imply that $\delta(C^*(\mathbb{R}, Y)) = 0$. ∎

Remark 6.9. This result implies, of course, that there can be no non-trivial, locally σ-trivial $\mathbb{R}$-spaces with orbit space T unless the group $\check{H}^3((T \times \mathbb{R})/\sim_{\hat{\sigma}}, \mathbb{Z})$ is non-zero. In fact, we have proved a little more: there are no such $\mathbb{R}$-spaces unless the induced map

$$p^*: \check{H}^3(T, \mathbb{Z}) \to \check{H}^3((T \times \mathbb{R})/\sim_{\hat{\sigma}}, \mathbb{Z})$$

fails to be surjective. To get some feeling for what is happening here, we shall consider the case where $\sigma(t) = \mathbb{Z}$ for all t, and where we can compute all of the invariants involved.

EXAMPLE 6.10. Suppose Y is a principal $\mathbb{T}$-bundle over T—or equivalently, that Y is a locally σ-trivial $\mathbb{R}$-bundle over T for the constant map $\sigma(t)=\mathbb{Z}$. Then by [18, Corollary 2.5]

$$C^*(\mathbb{R}, Y) \cong \operatorname{Ind}_{\mathbb{Z}^\perp}^{\hat{\mathbb{R}}}(C^*(\mathbb{T}, Y), \hat{\alpha}).$$

The group $\hat{\alpha}|_{\mathbb{Z}^\perp} = \hat{\alpha}|_{\mathbb{Z}}$ is locally unitary, and by duality the obstruction $\zeta(\hat{\alpha}|_{\mathbb{Z}})$ of [20] is the class of the $\mathbb{T}$-bundle Y in $\check{H}^2(T, \mathbb{Z})$. By [22, Corollary 3.5], the Dixmier–Douady class $\delta(C^*(\mathbb{R}, Y))$ is the external product $z \times \zeta(\hat{\alpha}|_{\mathbb{Z}})$ of $\zeta(\hat{\alpha}|_{\mathbb{Z}})$ with a generator z for $\check{H}^2(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}$. In this case, $T \times \mathbb{R}/\sim_\sigma \cong T \times \mathbb{T}$, and by the Künneth theorem, the external product induces an injection

$$\bigoplus_{p+q=3} \check{H}^p(\mathbb{T}, \mathbb{Z}) \otimes \check{H}^q(T, \mathbb{Z}) \to \check{H}^3(\mathbb{T} \times T, \mathbb{Z});$$

thus the Dixmier Douady class $\delta(C^*(\mathbb{R}, Y))$ and the range of p^* do indeed lie in the disjoint parts $\check{H}^1 \otimes \check{H}^2$ and $\check{H}^0 \otimes \check{H}^3$ of $\check{H}^3(\mathbb{T} \times T)$.

Remark 6.11. We have now shown that the description of the topology on $(A \rtimes_\alpha G)^\wedge$ given in [22, Sect. 2] extends to the case of continuously varying stabilisers. In [22], however, it was also proved that when A has continuous trace and α is locally unitary on the constant stabiliser H, the crossed product $A \rtimes_\alpha G$ has continuous trace too. There are therefore four topological invariants associated with the diamond: the classes of the principal bundles p and q, and the Dixmier- Douady classes of the algebras A and $A \rtimes_\alpha G$. There are various relationships between these classes, but various examples have been given to show that all four can be non-zero simultaneously [22, Sect. 3(b); 21].

We have not yet proved in our setting that $A \rtimes_\alpha G$ has continuous trace when A does, although such a result would certainly be interesting. However, in the various examples of $\mathbb{R}$-actions studied above, $A \rtimes_\alpha \mathbb{R}$ does have continuous trace and we can easily compute all the invariants. When $A = p^*B$ is a pull-back, A and $A \rtimes_\alpha \mathbb{R} \cong q^*B$ have Dixmier–Douady class $p^*(\delta(B))$ and $q^*(\delta(B))$ (see [23, Proposition 1.4]); the bundle p is the one we started with and $(A \rtimes_\alpha \mathbb{R})^\wedge$ is isomorphic to $C^*(\mathbb{R}, X)^\wedge = (X/\mathbb{R} \times \mathbb{R})/\sim$. When α is the dual action of $\hat{\mathbb{R}}$ on $A = C^*(\mathbb{R}, Y)$, $A \rtimes_\alpha \hat{\mathbb{R}} \cong C_0(Y) \otimes \mathcal{K}$ by duality, and $\delta(A) = 0$. We have already seen that $\delta(C^*(\mathbb{R}, Y)) \neq 0$ if Y is non-trivial, but $\hat{A} = C^*(\mathbb{R}, Y)^\wedge$ is trivial by Example 4.6.

7. APPENDIX

Here we give the example mentioned in Section 4 of a σ-proper G-space in which not all compact sets are G-wandering (when one uses

Definition 2.4 in [33]). The space X is the subspace of $\mathbb{R}^2$ which is the range of the function $x: \{1, 2, ..., \infty\} \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$x(n, t) = \begin{cases} (1, 0), & \text{if } n = 1, \\ (n - (n - 1/n)\cos(t/(n - 1/n)), (n - 1/n)\sin(t/(n - 1/n))), & \text{if } 1 < n < \infty, \text{ and} \\ (0, t), & \text{if } n = \infty. \end{cases}$$

If we let $O_n = \{x(n, t)\}_{t \in \mathbb{R}}$, then we see that, for $n = 2, 3, ...$, each O_n is a circle of radius $n - 1/n$ centered at $(n, 0)$. Furthermore, all the O_n are isolated with the exception of O_∞. Since $x(n_k, t_k) \to x(\infty, t)$ if and only if $n_k \to \infty$ and $t_k \to t$, we get a continuous $\mathbb{R}$-action by defining $r \cdot x(n, t) = x(n, t + r)$. The $\{O_n\}$ then coincide with the $\mathbb{R}$-orbits and, if S_n is the common stabiliser on O_n, then

$$S_n = \begin{cases} \mathbb{R}, & \text{if } n = 1, \\ 2\pi(n - 1/n)\mathbb{Z}, & \text{if } 1 < n < \infty, \text{ and} \\ \{0\}, & \text{if } n = \infty. \end{cases}$$

Thus, the stabiliser map is continuous and it is not hard to verify that X is a σ-proper G-space. On the other hand, if K is the compact set $\{x(n, 0)\}_{n=1}^{n=\infty}$, then

$$S(K) = \{(x, s) \in X \times G/\sim : x \in K \text{ and } rK \cap K \neq \varnothing\}$$

is not relatively compact (in particular, $\{(0, 2\pi(n - 1/n))\}_{n=1}^{\infty} \subseteq S(K)$).

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