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Transformation Group C^*-Algebras with Continuous Trace

DANA P. WILLIAMS

Department of Mathematics, Texas A & M University, College Station, Texas 77843

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We obtain several results characterizing when transformation group C^*-algebras have continuous trace. These results can be stated most succinctly when (G, Ω) is second countable, and the stability groups are contained in a fixed abelian subgroup. In this case, $C^*(G, \Omega)$ has continuous trace if and only if the stability groups vary continuously on Ω and compact subsets of Ω are wandering in an appropriate sense. In general, we must assume that the stability groups vary continuously, and if (G, Ω) is not second countable, that the natural maps of G/S_x onto $G \cdot x$ are homeomorphisms for each x. Then $C^*(G, \Omega)$ has continuous trace if and only if compact subsets of Ω are wandering and an additional C^*-algebra, constructed from the stability groups and Ω, has continuous trace.

1. INTRODUCTION

In [9], Green characterized which C^*-algebras associated to freely acting transformation group C^*-algebras have continuous trace. The purpose of this paper is to characterize those algebras with continuous trace which arise from possibly non-freely acting transformation groups. In [9], Green showed that if G acts freely and every compact subset, K, of Ω is wandering in the sense that $\{s \in G: sK \cap K \neq \varnothing\}$ is relatively compact, then $C^*(G, \Omega)$ has continuous trace. In fact, $C^*(G, \Omega)$ is isomorphic to the C^*-algebra defined by a continuous field of Hilbert spaces [9, Theorem 14]. If (G, Ω) is second countable, then he shows that the wandering hypothesis is also necessary [9, Theorem 17].

With the exception of the statement about continuous fields, the results in this paper contain those mentioned above. The first step in the argument is to find a suitable generalization of wandering for compact subsets of Ω. Notice that with the above definition, any point which has a non-compact stabilty group is a non-wandering compact set. In order to find a workable definition of wandering, it is convenient to assume that the stability groups vary

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continuously on Ω (Definition 2.1). Moreover, the majority of the techniques in this paper depend on this assumption, and the continuity of the stability groups turns out to be a neccessary condition in order that $C^*(G, \Omega)$ has continuous trace in a reasonably large number of cases (Proposition 4.1). It may be a necessary condition in general, but I cannot prove this.

Of course the (group C^*-algebras of the) stability groups must each have continuous trace. However, more must be true: Another C^*-algebra, $C^*(\mathscr{S})$, constructed from the stability groups and Ω in a manner similar to Fell's "sub-group C^*-algebra" [6], must have continuous trace. The advantage in using $C^*(\mathscr{S})$ is that its construction does not depend on the G-action. For example, $C^*(\mathscr{S})$ is always abelian when the stability groups are abelian and, hence, has continuous trace.

The results in this paper are most simply stated when the stability groups are contained in a fixed abelian subgroup and (G, Ω) is second countable. In this case, $C^*(G, \Omega)$ has continuous trace if and only if the stability groups vary continuously on Ω and every compact subset of Ω is wandering (Definition 2.4).

The general result is as follows: Suppose that the stability groups vary continuously, and if (G, Ω) is not second countable, that the natural maps of G/S_x onto $G \cdot x$ are homeomorphisms for each x. Then $C^*(G, \Omega)$ has continuous trace if and only if $C^*(\mathscr{S})$ has continuous trace and every compact subset of Ω is wandering.

The paper is organized as follows. In Section 2 we make the necessary preliminary definitions including the generalized wandering condition. We also construct $C^*(\mathscr{S})$, prove a variety of lemmas needed in the rest of the paper, and state the most general sufficient conditions we obtain for $C^*(G, \Omega)$ to have continuous trace (Theorem 2.7).

Section 3 is devoted to the proof of Theorem 2.7.

In Section 4, we obtain a variety of partial converse to Theorem 2.7. By combining these results with Theorem 2.7, the general characterization of continuous trace transformation group C^*-algebras mentioned above is obtained (Theorem 4.8).

In Section 5, we give a number of examples and summarize the results when G is "almost" abelian (cf. Theorem 5.1) and the action is essentialy free (Theorem 5.2). We also suggest a number of unanswered questions.

In Section 6, we prove some results concerning Morita equivalence which were stated without proof in Section 2.

The arguments used in this paper depend somewhat on the results in [20, 21]. In particular, for the precise definition of transformation group C^*-algebras, and additional references, the reader is encouraged to see Section 2 of [20]. Since it will often be convenient to consider $C_c(G \times \Omega)$ as a dense subalgebra of $C^*(G, \Omega)$, we will denote the latter by $C_c(G, \Omega)$ to distinguish it from the subalgebra of $C_0(G \times \Omega)$. Also some familarity with Rieffel's theory of induced representations of C^*-algebras [15] and Morita equivalence of C^*-algebras [16; 17; 15, Sect. 6] would be helpful.

Isomorphisms, homomorphisms, and representations of C^*-algebras will always be assumed to be *-preserving. Representations are also assumed to be non-degenerate. Ideal will always mean two-sided ideal, but $m(A)$ and $\kappa(A)$ (defined in Section 3) may not be closed ideals.

Finally, our proof of Theorem 2.7 is considerably different from Green's proof of [9, Theorem 14]. However, Proposition 4.2 uses several ideas from [9, Theorem 17]. The proof of Theorem 2.14 in Section 6 appeared in the author's doctoral dissertation at the University of California at Berkeley written under the supervision of Marc Rieffel.

2. Preliminaries

Let (G, Ω) be a locally compact transformation group. That is, G is a locally compact group and Ω is a locally compact Hausdorff space together with a jointly continuous map of $G \times \Omega$ to Ω, which we denote by $(s, x) \mapsto s \cdot x$, such that $t \cdot (s \cdot x) = ts \cdot x$ for all s, $t \in G$ and $x \in \Omega$. If $x \in \Omega$, then S_x will denote the stability group at x. Let Σ denote the space of subgroups of G endowed with the Fell topology (cf. [5]).

Definition 2.1. The stability groups are said to vary continuously if the map $x \mapsto S_x$ from Ω to Σ is continuous.

For the remainder of this section, the stability groups will be assumed to vary continuously. It should be pointed out that this is a rather severe restriction. However, the constructions and techniques of proof in this paper are heavily dependent on this assumption. Moreover, in Section 4 it will be shown that, for abelian groups (or, if the stability groups are contained in a fixed abelian group), the stability groups must vary continuously in order for $C^*(G, \Omega)$ even to have Hausdorff spectrum. Thus, this assumption would seem impossible to ignore.

To find the appropriate generalization of the wandering hypothesis for possibly non-free actions, it will be necessary to consider a quotient topological space.

Definition 2.2. Let $\Omega \times G/\sim$ denote the quotient topological space obtained from $\Omega \times G$ by identifying (x, s) and (y, r) when $x = y$ and $r \cdot x = s \cdot x$. Also, let $\delta: \Omega \times G \to \Omega \times G/\sim$ be the natural map.

Lemma 2.3. *If the stability groups vary continuously, then the natural map* $\delta: \Omega \times G \to \Omega \times G/\sim$ *is open and* $\Omega \times G/\sim$ *is Hausdorff.*

Proof. Let U and V be open in Ω and G, respectively. It will suffice to show that $\mathcal{O} = \delta^{-1}(\delta(U \times V))$ is open in $\Omega \times G$. Note that a typical element of $\mathcal{O}$ has the form (x, st) with $x \in U$, $s \in V$, and $t \in S_x$. Let $\{U_\alpha\}$ be a family of neighborhoods shrinking to x in Ω and let V' be a symmetric neighborhood of e in G such that $s(V')^2 \subseteq V$. It is enough to show that $U_\alpha \times sV't$ is eventually in $\mathcal{O}$.

If the above were false, then, passing to subnets if necessary, there are $x_\alpha \in U_\alpha$ and $s_\alpha \in V'$ such that

$$(x_\alpha, ss_\alpha t) \notin \mathcal{O}.$$

However, $x_\alpha \to x$, so by assumption, $S_{x_\alpha} \to S_x$. Thus, there are $t_\alpha \in S_{x_\alpha}$ with $t_\alpha \to t$. Since

$$(x_\alpha, ss_\alpha t) = (x_\alpha, ss_\alpha tt_\alpha^{-1} t_\alpha)$$

and tt_α^{-1} is eventually in V', $(x_\alpha, s(s_\alpha(tt_\alpha^{-1}))\, t_\alpha)$ is eventually in $(x_\alpha, Vt_\alpha) \subseteq \mathcal{O}$. This is nonsense; therefore, σ is an open map.

Now let (x_α, s_α) denote a net in $\Omega \times G/\sim$ which converges to (x, s) and (y, r). In order to show $\Omega \times G/\sim$ is Hausdorff, it will suffice to show $(x, s) = (y, r)$ in $\Omega \times G/\sim$. Since σ is open, we may assume that (x_α, s_α) converges to (x, s) in $\Omega \times G$, that $x = y$, and that there are $t_\alpha \in S_x$ such that $(x_\alpha, s_\alpha t_\alpha)$ converges to (x, r) in $\Omega \times G$. In particular, $t_\alpha \to s^{-1}r$. Again since $S_{x_\alpha} \to S_x$, it follows that $s^{-1}r \in S_x$, and $s \cdot x = r \cdot x$. Q.E.D.

Notice that if S_x is not compact, then no set containing x can be wandering in the usual sense. The most that can be hoped for is the following.

DEFINITION 2.4. A subset U of Ω is G-wandering if $\{(x, r) \in \Omega \times G/\sim :$ $x \in U$ and $rU \cap U \neq \varnothing\}$ is relatively compact in $\Omega \times G/\sim$.

Then in analogy with Green's work (cf. [9, Theorems 14 and 17], the appropriate condition on (G, Ω) is that compact subsets of Ω be G-wandering.

It will also be necessary to consider a C^*-algebra the construction of which is modeled after Sauvageot's "C^*-algèbre des Stabilisateurs" [19] and Fell's subgroup C^*-algebra [6]. Let

$$\mathscr{S} = \{(x, t) \in \Omega \times G : t \in S_x\}.$$

If $x \to S_x$ is continuous, then one sees easily that $\mathscr{S}$ is a closed, hence locally compact, subset of $\Omega \times G$. In fact, $\mathscr{S}$ is a locally compact groupoid as defined by Renault in [14], and the C^*-algebra described below is simply the C^*-algebra of the groupoid $\mathscr{S}$.

First, we fix a non-genative function, f_0, in $C_c(G)$ which does not vanish at the identity. For the remainder of his paper, we set α_H to be the left Haar measure on $H \in \Sigma$ with the property that

$$\int_H f_0(t)\, d\alpha_H(t) = 1.$$

Such a choice f measures is called a continuous choice of Haar measures, and has the property that

$$H \mapsto \int_H f(t)\, d\alpha_H(t)$$

is continuous on Σ for every $f \in C_c(G)$ [7, p. 908]. For convenience, let α_x denote α_{S_x} and let Δ_x be the modular function on S_x. Notice that $\{\alpha_x\}_{x \in \Omega}$ are a left Haar system on the groupoid $\mathscr{S}$ as defined in [14]. The next lemma summarizes some of the properties that the α_x enjoy.

LEMMA 2.5. *Suppose that the stability groups vary continuously on Ω.*

(i) *Suppose that $\{f_\beta\}$ is a net of functions in $C_c(G)$ converging to f in the inductive limit topology and that $x_\beta \to x$ in Ω. Then*

$$\int_{S_{x_\beta}} f_\beta(t)\, d\alpha_{x_\beta}(t) \to \int_{S_x} f(t)\, d\alpha_x(t).$$

(ii) *$\Delta_x(t)$ is continuous on $\mathscr{S}$.*

(iii) *If W is a locally compact Hausdorff space and $F \in C_c(W \times \mathscr{S})$, then*

$$\int_{S_x} F(w, x, t)\, d\alpha_x(t)$$

defines an element of $C_c(W \times \Omega)$.

(iv) *If W is as above and $G \in C_c(W \times G)$, then*

$$\int_H G(w, s)\, d\alpha_H(s)$$

defines an element of $C_c(W \times \Sigma)$.

Proof. This is essentially [20, Lemma 2.12]. Parts (i) and (ii) follow directly, while (iii) is only slightly more complicated.

On the other hand $w \mapsto G(w, \cdot)$ clearly defines a continuous function from W to $C_c(G)$ with respect to the inductive limit topology. Part (iv) now follows directly from [20, Lemma 2.12(i)]. Q.E.D.

$C_c(\mathscr{S})$ may be given a $*$-algebraic structure in the following manner. If $f, g \in C_c(\mathscr{S})$, then let

$$f * g(x, t) = \int_{S_x} f(x, r)\, g(x, r^{-1}t)\, d\alpha_x(r),$$

and let

$$f^*(x, t) = \Delta_x(t^{-1})\overline{f(x, t^{-1})}.$$

The fact that $f * g$ and f^* are also in $C_c(\mathscr{S})$ follows from Lemma 2.5. Moreover,

$$\|f\|_1 = \int_{S_x} |f(x, t)|\, d\alpha_x(t)$$

defines a norm on $C(\mathscr{S})$. We can also put a norm on $C_c(\mathscr{S})$ making it a pre-C^*-algebra, namely,

$$\|f\| = \sup_L \|L(f)\|,$$

where L runs over $\|\cdot\|_1$-norm decreasing representations of $C_c(\mathscr{S})$.

DEFINITION 2.6. Let $C^*(\mathscr{S})$ denote the C^*-algebra which is the completion of $C_c(\mathscr{S})$ with respect to the norm $\|\cdot\|$.

It is now possible to state one of the main results of this paper.

THEOREM 2.7. *Suppose that the stability groups vary continuously and that $C^*(\mathscr{S})$ has continuous trace. Then if every compact subset of Ω is G-wandering, $C^*(G, \Omega)$ has continuous trace.*

The proof of this theorem will be taken up in Section 3. The remainder of this section will concentrate on the structure of $C^*(\mathscr{S})$.

Suppose that $x \in \Omega$ and that $\pi \in \hat{S}_x$. Then we may define an irreducible representation, M_π^x, of $C^*(\mathscr{S})$ on the space of π as follows. For $f \in C_c(\mathscr{S})$

$$M_\pi^x(f) = \int_{S_x} f(x, t)\, \pi(t)\, d\alpha_x(t).$$

LEMMA 2.8. *Every irreducible representation of $C^*(\mathscr{S})$ is equivalent to a M_π^x for some $x \in \Omega$ and $\pi \in \hat{S}_x$.*

Proof. It is not difficult to see that $C^*(\mathscr{S})$ is isomorphic to a quotient of $C_0(\Omega) \otimes C_s^*(G)$, where $C_s^*(G)$ is the subgroup C^*-algebra constructed in [6, Sect. 2]. To be precise, the primitive ideal space of $C_0(\Omega) \otimes C_s^*(G)$ is parameterized by the set $\{(x, H, \ker\sigma): x \in \Omega,\ H \in \Sigma,\ \sigma \in \hat{H}\}$ [6, Lemma 2.8]. Since the map from Prim $C_s^*(G)$ to Σ defined by $(H, \operatorname{Ker}\sigma) \mapsto H$ is continuous by [6, Lemma 2.5], $\{(x, S_x, \ker\sigma): x \in \Omega, \sigma \in \hat{S}_x\}$ is closed in $\operatorname{Prim}(C_0(\Omega) \otimes C_s^*(G))$; the lemma now follows easily. Q.E.D.

LEMMA 2.9. *The map from* Prim $C^*(\mathcal{S})$ *to* Ω *defined by* $M_\pi^x \mapsto x$ *is continuous.*

Proof. This follows immediately from the isomorphism of $C^*(\mathcal{S})$ with the quotient of $C_0(\Omega) \otimes C_s^*(G)$ described in the proof of the previous lemma. Q.E.D.

Although we cannot prove it, we think of $C^*(\mathcal{S})$ as a field of C^*-algebras over Ω with the fibre over x being $C^*(S_x)$. Recall that $C^*(H)$ is (strongly) Morita equivalent to $C^*(G, G/H)$ [16, Definition 1.1]. Thus, one should expect $C^*(\mathcal{S})$ to be Morita equivalent to a field of C^*-algebras over Ω with fibre $C^*(G, G/S_x)$ over x. Before we proceed with the construction of this algebra, we describe the $C^*(H)$- and $C^*(G, G/H)$-valued inner products implementing the Morita equivalence between $C^*(H)$ and $C^*(G, G/H)$. As in [15], we usually will work with pre-C^*-algebras.

DEFINITION 2.10. Let h, $k \in C_c(G)$ and $H \in \Sigma$.

(a) Let $\gamma_H(t) = (\Delta_G(t)/\Delta_H(t))^{1/2}$, $t \in H$.

(b) $\langle f, g\rangle_H(t) = \gamma_H(t) \int_G f^*(s)\, g(s^{-1}t)\, d\alpha_G(s)$, $t \in H$.

(c) $\langle f, g\rangle_{(G,G/H)}(s, r) = \int_G f(rt)\, g^*(t^{-1}r^{-1}s)\, d\alpha_H(t)$, r, $s \in G$.

Let $x \in \Omega$, $H \subseteq S_x$, and suppose π is a representation of H on V_π. Let N_π^H and $\operatorname{Ind}_{(x,H)}^G(\pi)$ be the representations of $C^*(G, G/H)$ and $C^*(G, \Omega)$ induced from π on H [20, Definition 3.4]. Recall that both N_π^H and $\operatorname{Ind}_{(x,H)}^G(\pi)$ act on the completion of $C_c(G) \otimes V_\pi$ with respect to the pre-inner product given on elementary tensors by

$$\langle f \otimes \xi, g \otimes \eta\rangle_\pi^x = \langle \pi(\langle g, f\rangle_H)\, \xi, \eta\rangle_{V_\pi},$$

where $\langle \cdot, \cdot\rangle_H$ is defined above. If $f \in C_c(G, \Omega)$, then

$$\operatorname{Ind}_{(x,H)}^G(\pi)(h \otimes \xi) = \tilde{h} \otimes \xi,$$

where $\tilde{h}(s) = \int_G f(v, s \cdot x)\, h(v^{-1}s)\, d\alpha_G(v)$. And if $\phi \in C_c(G, G/H)$, then

$$N_\pi^H(\phi)(h \otimes \xi) = \tilde{h} \otimes \xi,$$

where $\tilde{h}(s) = \int_G \phi(v, s)\, h(v^{-1}s)\, d\alpha_G(s)$.

DEFINITION 2.11. Let $\mathscr{E} = \Omega \times G \times G/\sim$ be the quotient topological space obtained from $\Omega \times G \times G$ by identifying (x, r, s) and (y, u, v) if and only if $x = y$, $u = r$, and $s \cdot x = v \cdot x$.

Notice that $\mathscr{E}$ is simply the product of G with the quotient space defined in Definition 2.2. Therefore, the next lemma is clear.

LEMMA 2.12. *The natural map, σ, from $\Omega \times G \times G$ to $\mathscr{E}$ is open. Moreover, $\mathscr{E}$ is a locally compact Hausdorff space.*

In view of the openness of σ, elements of $C_c(\mathscr{E})$ may always be identified with continuous functions, f, on $\Omega \times G \times G$ such that $f(x, r, \cdot)$ defines an element of $C_c(G/S_x)$ and there is a compact subset of $\Omega \times G \times G$ of the form $C \times K \times K_0$ such that $\operatorname{supp} f \subseteq \sigma^{-1}(\sigma(C \times K \times K_0))$.

Now $C_c(\mathscr{E})$ may be given a $*$-algebraic structure. For $f, g \in C_c(\mathscr{E})$ define

$$f * g(x, s, r) = \int_G f(x, v, r)\, g(x, v^{-1}s, v^{-1}s, v^{-1}r)\, d\alpha_G(v)$$

and

$$f^*(x, s, r) = \Delta_G(s^{-1})\overline{f(x, s^{-1}, s^{-1}r)}.$$

Using the above remarks it is not hard to see that $f * g$ and f^* are again elements of $C_c(\mathscr{E})$.

Of course if $f \in C_c(\mathscr{E})$, then f_x defined by $f_x(r, s) = f(x, r, s)$ defines an element of $C_c(G, G/S_x)$.

DEFINITION 2.13. Let U_π^x denote the representation of $C_c(\mathscr{E})$ defined by $U_\pi^x(f) = N_\pi^{S_x}(f_x)$. Also, let L_π^x denote $\operatorname{Ind}_{(x, S_x)}^G(\pi)$.

Of course, since $C^*(S_x)$ and $C^*(G, G/S_x)$ are Morita equivalent, it follows that if π is irreducible, then $N_\pi^{S_x}$ is irreducible as well. It follows that U_π^x is irreducible. Moreover, if we give $C_c(\mathscr{E})$ the norm defined by

$$\|f\| = \sup_{U_\pi^x} \| U_\pi^x(f)\|,$$

where the U_π^x run over all $x \in \Omega$ and $\pi \in S_x$, then $C_c(\mathscr{E})$ becomes a pre-C^*-algebra. Thus, the completion is a C^*-algebra which will be denoted by $C^*(\mathscr{E})$.

There is a natural action of $C_c(G, \Omega)$ on $C_c(\mathscr{E})$. Namely, if $f \in C_c(G, \Omega)$ and $e \in C_c(\mathscr{E})$, then define

$$f \cdot e(x, s, r) = \int_G f(v, r \cdot x)\, e(x, v^{-1}s, v^{-1}r)\, d\alpha_G(v).$$

A straightforward computation shows that

$$U_\pi^x(f \cdot e) = L_\pi^x(f)\, U_\pi^x(e). \tag{1}$$

Thus,

$$\|f \cdot e\|_{\mathscr{E}} \leqslant \|f\|_{C^*(G,\Omega)}\, \|e\|_{\mathscr{E}}.$$

Therefore the action defined above extends to one of all of $C^*(G, \Omega)$ on $C^*(\mathscr{E})$ and, in fact, gives a homomorphism of $C^*(G, \Omega)$ into $M(C^*(\mathscr{E}))$ such that Eq. (1) holds for every $f \in C^*(G, \Omega)$ and $e \in C^*(\mathscr{E})$. Recall that if A is a C^*-algebra, then $M(A)$ is the algebra of double centralizers [2].

The next two theorems explain the need for introducing the algebra $C^*(\mathscr{E})$.

THEOREM 2.14. *$C^*(\mathscr{S})$ and $C^*(\mathscr{E})$ are Morita equivalent. In particular, the representation of $C^*(\mathscr{E})$ induced from M_π^x via the above Morita equivalence is unitarily equivalent to U_π^x.*

Since the proof, although not difficult, is rather long, technical, and somewhat peripheral to the main results of the paper, the proof will be postponed until Section 6.

The next theorem is part of some unpublished work of Green's on C^*-algebras with continuous trace [11]. His proof is also given in Section 6.

THEOREM 2.15. *If A and B are Morita equivalent C^*-algebras, then A has continuous trace if and only if B has continuous trace. In particular, $C^*(\mathscr{E})$ has continuous trace whenever $C^*(\mathscr{S})$ has.*

COROLLARY 2.16. *The map Γ from $C^*(\mathscr{E})^\wedge$ to Ω defined by $U_\pi^x \mapsto x$ is continuous.*

Proof. By Theorem 2.14 and Corollary 6.27 of [15], the map $M_\pi^x \to U_\pi^x$ is a homeomorphism of $C^*(\mathscr{S})^\wedge$ onto $C^*(\mathscr{E})^\wedge$. The corollary now follows immediately from Lemma 2.9. Q.E.D.

The next proposition highlights some direct consequences of the wandering hypothesis.

PROPOSITION 2.17. *Suppose that the subgroups vary continuously and that every compact set of Ω is G-wandering, then each of the following hold.*

(i) *Ω/G is Hausdorff.*

(ii) *The natural map of G/S_x onto $G \cdot x$ is a homeomorphism for each x.*

(iii) *$C^*(G, \Omega)$ is EH-regular. In particular, every irreducible representation of $C^*(G, \Omega)$ is (equivalent to) a L_π^x for some $x \in \Omega$ and $\pi \in \hat{S}_x$.*

Proof. The proofs of parts (i) and (ii) can be taken from the beginning of [9, Theorem 14] with only minor modifications.

Since part (i) implies that each orbit is closed and, by [10, Corollary 19], that $C^*(G, \Omega)$ is quasi-regular (cf. [10, p. 221]). It follows, just as in the proof of Proposition 3.2 in [20], that every irreducible representation of $C^*(G, \Omega)$ factors through $C^*(G, G \cdot x)$. However, $C^*(G, G \cdot x)$ is isomorphic to $C^*(G, G/S_x)$ since the homeomorphism from part (ii) is G-equivariant. Since every irreducible representation of the latter algebra is equivalent to a N_π^x and the isomorphism clearly carries L_π^x (on $C^*(G, G \cdot x)$) to N^x, part (iii) has been proved. Q.E.D.

Recall that a Bruhat approximate cross section for G with respect to G/H is a continuous, bounded, non-negative function, b, on G with the following properties: First the intersection of the support of b with the saturant of any compact set, C, in G (i.e., CH) is compact. And second

$$\int_H b(st)\, d\alpha_H(t) = 1 \qquad \text{for all } s.$$

Such functions are shown to exist in [1, Proposition 8], for example. It will be frequently necessary to make use of the existence of a cut-down generalized version of the above.

PROPOSITION 2.18. *Let C and K be compact subsets of G and Ω, respectively.*

(i) *There is a non-negative function $b \in C_c(\Sigma \times G)$ such that*

$$\int_H b(H, st)\, d\alpha_H(t) = 1 \qquad \textit{for all } s \in CH.$$

(ii) *If the stability groups vary continuously, then there is a non-negative function $b \in C_c(\Omega \times G)$ such that*

$$\int_{S_x} b(x, st)\, d\alpha_x(t) = 1 \qquad \textit{for all } s \in CS_x \textit{ and } x \in K.$$

Proof. It clearly suffices to prove only part (i). Fix $H \in \Sigma$, and let $b^H \in C_c(G)$ be a cut-down Bruhat approximate cross section for G with respect to G/H. That is, $b^H \geqslant 0$ and

$$\int_H b^H(st)\, d\alpha_H(t) = 1 \qquad \text{for all } s \in CH. \tag{2}$$

Since $f(s,t) = b^H(st)$ may be viewed as an element of $C_c(C \times G)$, Lemma 2.5(iv) implies Eq. (2) is continuous in H and s. Moreover, since C is compact, there is a neighborhood, U_H, of H such that

$$\int_K b^H(s,t)\, d\alpha_K(t) > \tfrac{1}{2}$$

for every $K \in U_H$ and $s \in C$. Of course the left invariance of the α_K implies the above actually holds for all $s \in CK$.

Since Σ is compact, suppose $U_{H_1},\ldots, U_{H_n}$ cover Σ. If $f_1,\ldots,f_n$ are a partition of unity on Σ such that $\operatorname{supp} f_i \subseteq U_{H_i}$, then

$$b(H,s) = \sum_{i=1}^{n} f_i(H) \frac{b^{H_i}(s)}{\int_H b^{H_i}(st)\, d\alpha_H(t)}$$

will do. Q.E.D.

It will also be necessary to recall some basic observations from [18, 19] concerning the appropriate choice of quasi-invariant measures on the quotient spaces G/S_x.

DEFINITION 2.19. (a) Define $\omega = G \times \Omega \to R^+$ by

$$\int_{S_x} f(tst^{-1})\, d\alpha_x(s) = \omega(t,x) \int_{S_{t\cdot x}} f(s)\, d\alpha_{t\cdot x}(s).$$

(b) Define $\rho: G \times \Omega \to R^+$ by $\rho(s,x) = \Delta_G(s^{-1})\, \omega(s,x)$.

LEMMA 2.20. *Suppose the stability groups vary continuously. Then the following statements hold.*

(i) *Both ρ and ω are jointly continuous on $G \times \Omega$.*

(ii) *For all $s, r \in G$ and $x \in \Omega$; $\rho(rs,x) = \rho(s,x)\, \rho(r, s\cdot x)$.*

(iii) *For all $x \in \Omega$, $s \in G$, and $t \in S_x$; $\rho(st,x) = \Delta_{S_x}(t)\, \Delta_G(t^{-1})\, \rho(s,x)$.*

(iv) *There is a unique quasi-invariant measure, μ_x, on G/S_x such that, for all $f \in C_c(G)$,*

$$\int_G f(s)\, \rho(s,x)\, d\alpha_G(s) = \int_{G/S_x} \int_{S_x} f(st)\, d\alpha_x(t)\, d\mu_x(s).$$

Proof. Let f_0 be the function in $C_c(G)$ defining the continuous choice of measures α_x. Lemma 2.5 implies

$$\int_{S_x} f_0(tst^{-1})\, d\alpha_x(s) \tag{3}$$

is continuous in x and t. However, Eq. (3) is equal to $\omega(t, x)$ since $\int_{S_y} f_0(s)\, d\alpha_y(s) = 1$ for all y. This proves part (i).

The proofs of parts (ii) and (iii) are simple computations, and part (iv) follows from [1, Chap. VII, par. 2, no. 5, Théorème 2]. Q.E.D.

The essential properties of the μ_x needed in the following are outlined in the next two lemmas.

LEMMA 2.21. *If $f \in C_c(G/S_x)$ and $r \in G$, then $s \mapsto f(sr)$ is an element of $C_c(G/S_{r \cdot x})$ and*

$$\int_{G/S_x} f(s)\, d\mu_x(s) = \int_{G/S_{r \cdot x}} f(sr)\, d\mu_{r \cdot x}(s).$$

Proof. Let b be a Bruhat approximate cross section for G/S_x. Then the left-hand side of the above equation equals

$$\begin{aligned} &\int_G f(s)\, b(s)\, \rho(s, x)\, d\alpha_G(s) \\ &= \Delta_G(r) \int_G f(sr)\, b(sr)\, \rho(sr, x)\, d\alpha_G(s) \\ &= \omega(r, x) \int_G f(sr)\, b(sr)\, \rho(s, r \cdot x)\, d\alpha_G(s) \\ &= \int_{G/S_{r \cdot x}} f(sr)\, \omega(r, x) \int_{S_{r \cdot x}} b(str)\, d\alpha_{r \cdot x}(t)\, d\mu_{r \cdot x}(s). \end{aligned}$$

which equals the right-hand side. Q.E.D.

LEMMA 2.22. *Suppose that the stability groups vary continuously and that compact subsets of Ω are G-wandering.*

(i) *If $\phi \in C_c(\Omega)$, then*

$$\tilde{\phi}(x) = \int_{G/S_x} \phi(s \cdot x)\, du_x(s)$$

is an element of $C_c(G/\Omega)$.

(ii) *If $C \subseteq \Omega$ is compact and $\phi_x \colon G/S_x \to \Omega$ is the natural map, then $\mu_x(\phi_x^{-1}(C))$ is bounded on Ω. In fact, if $\phi_C \in C_c(\Omega)$ is any non-negative function which is identically one on C, then*

$$\mu_x(\phi_x^{-1}(C)) \leqslant \|\tilde{\phi}_C\|_\infty.$$

Proof. $\tilde{\phi}$ is constant on orbits by the previous lemma and clearly has compact support. For $x_0 \in \Omega$, let U be a compact neighborhood of x_0. Let $C = \operatorname{supp} \phi$ and $K = C \cup U$. Since

$$\{(x, s) \in \Omega \times G/\sim : x \in K, sK \cap K \neq \varnothing\}$$

has compact closure in $\Omega \times G/\sim$, it follows from Lemma 2.3 and Proposition 2.18 that there is a $b \in C_c(\Omega \times G)$ such that

$$\phi(r \cdot x) = \phi(r \cdot x) \int_{S_x} b(x, rt)\, d\alpha_x(t)$$

provided $x \in U$. In particular, for every $x \in U$,

$$\tilde{\phi}(x) = \int_G \phi(r \cdot x)\, b(x, r)\, \rho(r, x)\, d\alpha_G(r),$$

which is clearly continuous in x. This establishes (i).

Let ϕ_C be as in (ii). Then

$$\tilde{\phi}_C(x) = \int_{G/S_x} \phi(r \cdot x)\, d\mu_x(r) \geqslant \mu_x(\phi_x^{-1}(C)). \qquad \text{Q.E.D.}$$

3. Proof of Theorem 2.7

If A is a C^*-algebra, let $m(A)$ denote the two-sided ideal of continuous trace elements in A. That is, the set of $a \in A$ such that $\pi \mapsto \operatorname{tr}(\pi(a))$ is continuous from $\hat{A}$ to R^+ [3, 4.5.2]. Let $\kappa(A)$ denote the dense, hereditary, two-sided ideal in A which is minimal among all dense two-sided ideals (i.e., the "Pedersen ideal," cf. [12, 5.6.1]. If B is an ideal in A, then B^+ will always denote the intersection of B with the positive cone of A.

Since $C^*(\mathscr{S})$ and $C^*(\mathscr{E})$ are defined only when the stability groups vary continuously, we shall assume this throughout this section.

The idea of the proof will be to use the fact that $m(C^*(\mathscr{E}))^+$ spans a dense set to produce sufficiently many continuous trace elements in $C^*(G, \Omega)$. Towards this end, we define a linear map, P, from $C_c(\mathscr{E})$ to $C_c(G, \Omega)$ by

$$P(f)(s, x) = \int_{G/S_x} f(r \cdot x, s, r^{-1})\, d\mu_x(r). \tag{4}$$

It is not difficult to check that $r \mapsto f(r \cdot x, s, r^{-1})$ defines an element of $C_c(G/S_x)$ and, hence, that $P(f)(s, x)$ is well defined.

LEMMA 3.1. *If $f \in C_c(\mathscr{E})$, then $P(f) \in C_c(G, \Omega)$ and $P(f^*) = P(f)^*$.*

Proof. Suppose that $\operatorname{supp} f \subseteq \sigma(C \times K \times K_0)$. Then, $\operatorname{supp}(P(f)) \subseteq K \times K_0 C$. Let $b \in C_c(\Omega \times G)$ be a generalized cut-down Bruhat approximate crosssection such that

$$\int_{S_x} b(x, rt)\, d\alpha_x(t) = 1,$$

if $x \in K_0 C$ and $r \in K_0^{-1}$.

It follows that, for every $x \in \Omega$ and $s, r \in G$,

$$f(r \cdot x, s, r^{-1}) = f(r \cdot x, s, r^{-1}) \int_{S_x} b(x, rt)\, d\alpha_x(t).$$

Thus, one can show that $P(f)$ is continuous in much the same way as in Lemma 2.22.

The assertion about adjoints follows quickly from Lemma 2.21. Q.E.D.

For the moment, fix $x \in \Omega$ and $\omega \in \hat{S}_x$. Also, let L^x_ω be the corresponding element of $C^*(G, \Omega)^\wedge$ (L^x_ω is irreducible by [20, Proposition 4.2]) and denote the space of L^x_ω by simply V. If $r \in G$, let ${}^r\omega$ be the representation of $S_{r \cdot x}$ defined by ${}^r\omega(t) = \omega(r^{-1}tr)$. It will be convenient to realize ${}^rU^x_\omega = U^{r \cdot x}_{{}^r\omega}$ as a representation on V. Thus, if $F \otimes \xi$ is an elementary tensor in $C_c(G) \otimes V_\omega$, then define T from the space of ${}^rU^x_\omega$ to V by $T(f \otimes \xi) = (\omega(r, x)^{-1/2} \rho(r^{-1}) F) \otimes \xi$, where $\rho(r) F(v) = \Delta(s)^{1/2} F(vr)$.

Now,

$$\begin{aligned}
&\langle T(F \otimes \xi), T(G \otimes \eta)\rangle_V \\
&\quad = \omega(r, x)^{-1} \int_{S_x} \langle \rho(r^{-1}) G, \rho(r^{-1}) F\rangle_{S_x}(t) \langle \omega(t)\xi, \eta\rangle\, d\alpha_x(t) \\
&\quad = \omega(r, x)^{-1} \int_{S_x} \gamma_{S_x}(t) \int_G G^*(s) F(s^{-1}rtr^{-1})\, d\alpha_G(r) \langle \omega(t)\xi, \eta\rangle\, d\alpha_x(t).
\end{aligned}$$

And, since $\Delta_{S_x}(r^{-1}tr) = \Delta_{S_{r \cdot x}}(t)$, the above equals

$$\langle F \otimes \xi, G \otimes \eta\rangle^{r \cdot x}_{{}^r\omega}.$$

Thus T extends to a unitary map onto V.

Let ${}^rR^x_\omega$ denote $T\,{}^rU^x_\omega T^*$. It is not difficult to compute that, if $e \in C_c(\mathscr{E})$,

$${}^rR^x_\omega(e)(F \otimes \eta) = \tilde{F} \otimes \eta, \tag{5}$$

where

$$\tilde{F}(s) = \int_G e(r \cdot x, x, sr^{-1}) F(v^{-1}s)\, d\alpha_G(v). \tag{6}$$

Or, more simply, ${}^rR^x_\omega(e) = N^{S_x}_\omega(e_r)$, where $e_r(v, s) = e(r \cdot x, sr^{-1})$. It follows from Eq. (6) that ${}^rR^x_\omega$ depends only on the class of r in G/S_x and $\omega \in \hat{S}_x$. Moreover it is not hard to see that if $r_\alpha \to r$ in G, then $e_{r_\alpha} \to e_r$ in the inductive limit topology on $C_c(G, G/S_x)$. Thus, $r \mapsto {}^rR^x_\omega(e)$ is norm continuous. In particular,

$$\int_{G/S_x} {}^rR^x_\omega(e)\, du_x(r)$$

is a well-defined operator in $B(V)$.

LEMMA 3.2. *For each* $f \in C_c(\mathscr{E})$, $L^x_\omega(P(f)) = \int_{G/S_x} {}^rR^x_\omega(f)\, du_x(r)$.

Proof. By definition $L^x_\omega(P(f))(F \times \xi) = \tilde{F} \otimes \xi$, where

$$\tilde{F}(s) = \int_G P(f)(v, s \cdot x)\, F(v^{-1}s)\, d\alpha_G(v)$$

$$= \int_G \int_{G/S_{s \cdot x}} f(rs \cdot x, v, r^{-1})\, F(v^{-1}s)\, du_{s \cdot x}(r)\, d\alpha_G(v),$$

which by Lemma 2.21 is

$$\tilde{F}(s) = \int_{G/S_x} \left(\int_G f(r \cdot x, v, sr^{-1})\, F(v^{-1}s)\, d\alpha_G(v) \right) du_x(r).$$

By comparing with Eq. (6) above, one gets

$$L^x_\omega(P(f))(F \otimes \xi) = \left(\int_{G/S_x} {}^rR^x_\omega\, du_x(r) \right) (F \otimes \xi).$$

Q.E.D.

Now, suppose $C^*(\mathscr{S})$ has continuous trace. Then certainly each stability group has continuous trace, and in particular each stability group is C.C.R. It follows from Proposition 2.17 and [20, Proposition 3.2] that if every compact subset of Ω is G-wandering, then $C^*(G, \Omega)$ is C.C.R. Therefore, $L^x_\omega(P(f))$ is a compact operator.

Suppose $\{v_\alpha\}_{\alpha \in \Lambda}$ is an orthonormal basis for V. Since $L^x_\omega(P(f))$ is compact, there are only countably many v_α such that

$$\langle L^x_\omega(P(f))\, v_\alpha, v_\alpha \rangle_V \neq 0.$$

Moreover, if $f \in C_c(\mathscr{E}) \cap C^*(\mathscr{E})^+$, then

$$\int_{G/S_x} \langle {}^rR^x(f)\, v_\alpha, v_\alpha \rangle_V\, du_x(r) = 0$$

if and only if $\langle {}^rR^x_\omega(f)\, v_\alpha, v_\alpha\rangle_{V_\omega} = 0$ for all $r \in G$. Thus, for fixed $f \in C_c(\mathscr{E}) \cap C^*(\mathscr{E})^+$, there are $\{\alpha_i\}_{i=1}^{\infty} \subseteq A$ such that

$$\operatorname{tr}(L^x_\omega(P(f))) = \sum_{i=1}^{\infty} \langle L(P(f))\, v_{\alpha_i}, v_{\alpha_i}\rangle_{V_\omega}$$

and

$$\operatorname{tr}({}^rR^x_\omega(f)) = \sum_{i=1}^{\infty} \langle {}^rR^x_\omega(f)\, v_{\alpha_i}, v_{\alpha_i}\rangle_{V_\omega}.$$

Therefore, the next lemma follows by the monotone convergence theorem. Of course, $\operatorname{tr}({}^sU^x_\omega) = \operatorname{tr}({}^sR^x_\omega)$.

LEMMA 3.3. *If $f \in C_c(\mathscr{E}) \cap C^*(\mathscr{E})^+$, then*

$$\operatorname{tr}(L^x_\omega(P(f)) = \int_{G/S_x} \operatorname{tr}({}^sU^x_\omega(f))\, d\mu_x(s).$$

Unfortunately, it does not seem possible to extend P to all of $C^*(\mathscr{E})$. In fact, if $\operatorname{supp} f \subseteq \sigma(C \times K \times K_0)$, then it is clear from Eqs. (5) and (6) that $R^{s\cdot x}_{s_\omega}(f) = 0$ if $s \cdot x \notin C$. In particular,

$$\|L^x(P(f))\| \leqslant \int_{G/S_x} \|{}^sR^x_\omega(f)\|\, d\mu_x(s) \leqslant \|f\|_{\mathscr{E}}\, \mu_x(\phi_x^{-1}(C)).$$

Thus, bu Lemma 2.22 if $\phi_C \in C(\Omega)$ with ϕ_C identically one on C and $\phi_C \geqslant 0$, then

$$\|L^x_\omega(P(f))\| \leqslant \|f\|_{\mathscr{E}}\, \|\tilde{\phi}_C\|_\infty. \tag{7}$$

Since the compact set C varies with f, there seems to be no reason to suspect P is bounded. However, if $g \in C_c(\mathscr{E})$, then let P_g denote the map $f \mapsto P(g * f * g^*)$.

PROPOSITION 3.4. *Suppose that the stability groups vary continuously on Ω and that every compact subset of Ω is G-wandering. Then P_g is bounded and extends to a positive linear map from $C^*(\mathscr{E})$ to $C^*(G, \Omega)$.*

Proof. Let $g \in C_c(\mathscr{E})$ be fixed with $\operatorname{supp}(g) \subseteq \sigma(C \times K \times K_0)$. Let $f \in C_c(\mathscr{E})$ have support in $\sigma(C_1 \times K_1 \times K_2)$. Then, $g * f * g^*$ has support in $\sigma(C \times KK_1K^{-1} \times K_0)$. Since Eq. (7) above holds for any L^x_ω, it follows that

$$\|P_g(f)\|_{C^*(G,\Omega)} \leqslant \|g\|^2_{\mathscr{E}}\, \|\tilde{\phi}_C\|_\infty\, \|f\|_{\mathscr{E}},$$

where ϕ_C is any non-negative function in $C_c(\Omega)$ which is identically one on C (Lemma 2.22). Thus, P_g is bounded and can be extended in the usual way.

On the other hand, if $f \in C_c(\mathscr{E}) \cap C^*(\mathscr{E})^+$, then it follows immediately from Lemma 3.2 that

$$\langle L_\omega^x(P_g(f)), v\rangle_V \geqslant 0$$

for every $L_\omega^x \in C^*(G, \Omega)^\wedge$ and $v \in V$; that is, P_g is a positive map. Q.E.D.

Remark. It is possible to show that P is a generalized conditional expectation as defined in [15], but this fact is not needed in what follows.

LEMMA 3.5. *For every $g \in C_c(\mathscr{E})$ and $f \in C^*(\mathscr{E})$,*

(i) $L_\omega^x(P_g(f)) = \int_{G/S_x} {}^sR_\omega^x(g * f * g^*)\, d\mu_x(s)$.

And, if $f \in C^*(\mathscr{E})^+$,

(ii) $\operatorname{tr}(L_\omega^x(P_g(f))) = \int_{G/S_x} \operatorname{tr}({}^sU_\omega^x(g * f * g^*) d\mu_x(s)$.

Proof. Let f_n be a sequence of functions in $C_c(\mathscr{E})$ converging to f in $C^*(\mathscr{E})$. Then $L_\omega^x(P_g(f_n)) \to L_\omega^x(P_g(f))$. But, by Lemma 3.2

$$L_\omega^x(P_g(f_n)) = \int_{G/S_x} {}^sR_\omega^x(g * f_n * g^*)\, d\mu_x(s).$$

Since $\|{}^sR_\omega^x(g * f_n * g^*)\|$ is bounded by a multiple of the characteristic function of $\phi_x^{-1}(C)$ for all n, the right-hand side of the above equation converges to

$$\int_{G/S_x} {}^sR_\omega^x(g * f * g^*)\, d\mu_x(s),$$

by the Lebesque dominated convergence theorem.

Part (ii) now follows from part (i) as in Lemma 3.3, since $L_\omega^x(P_g(f))$ is compact. Q.E.D.

Now suppose $f \in m(C^*(\mathscr{E}))^+$. Then

$$U_\omega^x \mapsto \operatorname{tr}(U_\omega^x(g * f * g^*))$$

defines a non-negative continuous function on $C^*(\mathscr{E})^\wedge$. Consider

$$\Phi(U_\omega^x) = \int_{G/S_x} \operatorname{tr}({}^sU_\omega^x(g * f * g^*)\, d\mu_x(s) \tag{8}$$

as a function on $C^*(\mathscr{E})^\wedge$.

LEMMA 3.6. *The map from $G \times C^*(\mathscr{E})^\wedge$ to $C^*(\mathscr{E})^\wedge$ defined by $(s, U_\omega^x) \mapsto {}^sU_\omega^x$ is continuous.*

Proof. By Theorem 2.14 and [15, Corollary 6.27], it suffices to show $(s, M_\omega^x) \mapsto M_{s_\omega}^{s \cdot x}$ is jointly continuous on $G \times C^*(\mathscr{S})$. However, this follows from [6], Lemma 2.9 and the fact that $C^*(\mathscr{S})$ is isomorphic to a quotient of $C_0(\Omega) \otimes C_s^*(G)$. Q.E.D.

PROPOSITION 3.7. *Suppose $g \in C_c(\mathscr{E})$ and $f \in C^*(\mathscr{E})^+$. If, in addition, every compact subset of Ω is G-wandering, then Φ, as defined in Eq.* (8), *is continuous from $C^*(\mathscr{E})^\wedge$ to R^+.*

Proof. Fix $U_{\omega_0}^{x_0}$ in $C^*(\mathscr{E})^\wedge$ and let W be a compact neighborhood of $U_{\omega_0}^{x_0}$. Let Γ denote the map of $C^*(\mathscr{E})^\wedge$ onto Ω defined by $\Gamma(U_\omega^x) = x$. Γ is continuous by Corollary 2.16. In particular, $\Gamma(W)$ is a compact set containing x_0. Suppose g has support in $\sigma(C \times K \times K_0)$. Then $U_\pi^y(g * f * g^*)$ is zero if $y \notin C$.

If $K = \Gamma(W) \cup C$, then

$$\{(x, s) \in \Omega \times G/\sim : x \in K, sK \cap K \neq \varnothing\}$$

has compact closure in $\Omega \times G/\sim$. Using this fact and Proposition 2.18, one can find $b \in C_c(\Omega \times G)$ such that, for all $U_\pi^y \in W$ and $s \in G$,

$$\operatorname{tr}({}^sU_\pi^y(g * f * g^*)) = \operatorname{tr}({}^sU_\pi^y(g * f * g^*)) \int_{S_y} b(y, st)\, d\alpha_y(t).$$

It follows that if $U_\pi^y \in W$, then

$$\Phi(U_\pi^y) = \int_G \operatorname{tr}({}^sU_\pi^y(g * f * g^*)\, b(y, s)\, \rho(s, y)\, d\mu_G(s).$$

In view of the previous lemma and the compact support of b, the right-hand side is continuous in U_π^y and y. On the other hand, if $U_{\pi_\alpha}^{y_\alpha}$ converges to $U_{\pi_0}^{x_0}$, then the continuity of Γ implies that $y_\alpha \to x_0$. Thus, Φ is continuous at $U_{\pi_0}^{x_0}$. Q.E.D.

Notice that by Lemma 2.21, $\Phi({}^sU_\pi^y) = \Phi(U_\pi^y)$.

PROPOSITION 3.8. *Suppose that the stability groups vary continuously and that compact subsets of Ω are G-wandering. If $g \in C_c(\mathscr{E})$ and $f \in m(C^*(\mathscr{E}))^+$, then $P_g(f) \in m(C^*(G, \Omega))^+$.*

Proof. Since L_ω^x is a typical element of $C^*(G, \Omega)$ (Proposition 2.17), it suffices to show that $L_\omega^x \mapsto \operatorname{tr}(L_\omega^x(P_g(f)))$ is continuous at L_ω^x.

Suppose, to the contrary, that there were a net $\{L^{x_\alpha}_{\omega_\alpha}\}_{\alpha\in\Lambda}$, converging to L^x_ω such that

$$|\operatorname{tr}(L^{x_\alpha}_{\omega_\alpha}(P_g(f))) - \operatorname{tr}(L^x_\omega(P_g(f)))| \geqslant \varepsilon > 0$$

for every $\alpha \in \Lambda$. Or, equivalently (Lemma 3.5),

$$|\Phi(U^{x_\alpha}_{\omega_\alpha}) - \Phi(U^x_\omega)| \geqslant \varepsilon > 0 \tag{9}$$

for every $\alpha \in \Lambda$.

Suppose in addition that U^x_ω is not in the closure of

$$\{{}^sU^{x_\alpha}_{\omega_\alpha} : \alpha \in \Lambda \text{ and } s \in G\}$$

in $C^*(\mathscr{E})^\wedge$. In particular, there would be an $e \in C^*(\mathscr{E})^+$ such that

$${}^sU^{x_\alpha}_{\omega_\alpha}(e) = 0$$

for every $\alpha \in \Lambda$ and $s \in G$, while

$$U^x_\omega(e) \neq 0.$$

By Lemma 3.5(i) (and the fact ${}^sU^x_\omega$ is equivalent to $R^{s\cdot x}_{s_\omega}$),

$$L^{x_\alpha}_{\omega_\alpha}(P_{g_0}(e)) = 0$$

for every $\alpha \in \Lambda$ and $g_0 \in C_c(\mathscr{E})$. Moreover, since $P_{g_0}(e)$ is in the common kernel of the $L^{x_\alpha}_{\omega_\alpha}$,

$$L^x_\omega(P_{g_0}(e)) = 0$$

as well. On the other hand, there is a $g_0 \in C_0(\mathscr{E})$ and a $v \in V$ such that

$$\langle U^x_\omega(g_0 * e * g_0^*)\, v, v\rangle_V > 0.$$

And by Lemma 3.5, $\langle L^x_\omega(P_{g_0}(e))\, v, v\rangle$ is the integral of a continuous, nonnegative, and non-zero function on G/S_x; hence, $L^x_\omega(P(e)) \neq 0$.

In view of the above contradiction, it may be assumed that there is a net, $\{{}^{s_\beta}B^x_{\omega_\beta}\}_{\beta\in\Lambda'}$, converging to U^x_ω with $\{U^{x_\beta}_{\omega_\beta}\}_{\beta\in\Lambda'} \subseteq \{U^{x_\alpha}_{\omega_\alpha}\}_{\alpha\in\Lambda}$. By the previous proposition anf the remark following, the $\Phi({}^{s_\beta}U^{x_\beta}_{\omega_\beta}) = \Phi(U^{x_\beta}_{\omega_\beta})$ converge to $\Phi(U^x_\omega)$. This contradicts Eq. (9) and completes the proof of the proposition.

Q.E.D.

To complete the proof of Theorem 2.7, it will suffice to show that for an arbitrary $L^x_\omega \in C^*(G, \Omega)^\wedge$ there is a $f \in m(C^*(G, \Omega))$ such that $L^x_\omega(f) \neq 0$ [3, 4.5.2]. Since $C^*(\mathscr{S})$ and $C^*(\mathscr{E})$ are Morita equivalent, $m(C^*(\mathscr{E}))$ is dense in $C^*(\mathscr{E})$. Thus, there is an $e \in m(C^*(\mathscr{E}))^+$ such that $U^x_\omega(e) \neq 0$ and a

$g \in C_c(\mathscr{E})$ such that $U_\omega^x(g * e * g^*) \neq 0$. It follows from Lemma 3.5, just as in the proof of the previous proposition, that $L_\omega^x(P_g(e)) \neq 0$. Of course, $P_g(e) \in m(C^*(G, \Omega))$ by the last proposition.

In the next section, a variety of partial converses will be established.

4. Converses to Theorem 2.7

As mentioned earlier, the assumption that the stability groups vary continuously is a strong one. However, the next proposition will show that the assumption is necessary in a large number of cases. (Recall that C^*-algebras with continuous trace have Hausdorff spectrum.)

PROPOSITION 4.1. *Suppose that the stability groups are contained in a fixed abelian subgroup H. In addition, suppose that the natural maps of G/S_x onto $G \cdot x$ are homeomorphisms for each x, and if G is not abelian, that $C^*(H, \Omega)$ is quasi-regular. Then, if $C^*(G, \Omega)$ has Hausdorff spectrum, the stability groups vary continuously on Ω.*

Remark. If (G, Ω) is second countable, then $C^*(H, \Omega)$ is automatically quasi-regular [9, Corollary 19], and the assumption on the natural maps from G/S_x onto $G \cdot x$ is unnecessary [8].

Proof. Note that the maps from H/S_x to $H \cdot x$ are also homeomorphisms. It follows from [10, Corollary 19 and Proposition 20] that both $C^*(G, \Omega)$ and $C^*(H, \Omega)$ are *EH*-regular. If $H = G$, then the proposition follows from [21]. In general, the arguments in [21] show that if $x \to S_x$ has a point of discontinuity, then there is a net in $\Omega \times \hat{H}$ such that $(x_\alpha, \sigma_\alpha)$ converges to (x, σ) and σ_α is identically one on S_{x_α}, while σ is not identically one on S_x. It follows from [20, Lemma 4.9] and [10, Proposition 8] that $L_1^{x_\alpha}$ converges to both $L_{\sigma_{S_x}}^x$ and L_1^x, where 1 denotes the trivial representation and σ_{S_x} denotes the restrictions of σ to S_x. However, $L_{\sigma_{S_x}}^x$ and L_1^x are not equivalent since $1 \neq \sigma_{S_x}$ implies that their unitary parts are not equivalent. Q.E.D.

For the remainder of this section, it will be assumed that the stability groups vary continuously. The proof of the necessity of the wandering hypothesis is inspired by Green's proof in [9]. The basic idea is the same: To produce an element in the Pedersen ideal which is not continuous trace. The minimality of the Pedersen ideal among dense ideals then implies that $C^*(G, \Omega)$ does not have continuous trace.

PROPOSITION 4.2. *Suppose that the stability groups vary continuously. If (G, Ω) is not second countable, then suppose that the natural maps*

$\phi_x: G/S_x \to G \cdot x$ are homeomorphisms for each x. Then if $C^*(G, \Omega)$ has continuous trace, every compact subset of Ω is G-wandering.

Before proceeding with the proof, it will be necessary to prove a number of lemmas.

LEMMA 4.3. *Suppose that the stability groups vary continuously. If $C^*(G, \Omega)$ is C.C.R. with Hausdorff spectrum, then Ω/G is Hausdorff.*

Proof. Follows immediately from [20, Theorem 4.11 and Proposition 4.16]. Q.E.D.

Suppose that K is a non-G-wandering compact subset in Ω. That is, if

$$S_K = \{(x, s) \in \Omega \times G/\sim : x \in K \; sK \cap K \neq \varnothing\},$$

then S_K is not relatively compact in $\Omega \times G/\sim$. In particular, given $C \subseteq G$ compact, there is $x_C \in K$ and a $r_C \notin CS_{x_C}$ such that $r_C x_C \in K$. Using compactness, we may assume $x_C \to z \in K$ and $r_C x_C \to y \in K$, where the nets are directed by increasing compact sets. Moreover, since Ω/G is Hausdorff, there is an $s_0 \in G$ such that $z = s_0 y$. In particular, if $s_C = s_0 r_C$, then $s_C x_C \to z$ and

$$s_C \notin s_0 C S_{x_C}. \tag{10}$$

Fix $z \in \Omega$ and let $f \in C_0(\Omega)$ be a non-negative function which is identically one on an open neighborhood, W, of z. Let N be a compact set containing the support of f. For each $x \in \Omega$ let $f_x(s) = f(s \cdot x)$ and let F be a compact set in G such that FS_z contains the support of f_z. F and N may be taken so that $F = F^{-1}$, $e \in F$, and $W \subseteq N$.

It follows from [20, Lemma 4.14] that L_1^x is equivalent to a representation, L^x, on $L^2(G/S_x, \mu_x)$ such that, if $h \in L^2(G/S_x)$ and $G \in C_C(G, \Omega)$, then

$$L^x(G)\, h(s) = \int_G G(v, s \cdot x)\, \rho(v^{-1}, s \cdot x)^{1/2}\, h(v^{-1}s)\, d\alpha_G(v).$$

Of course, $\rho(v^{-1}, s \cdot x)^{1/2} = \rho(v^{-1}s, x)^{1/2}\, \rho(s, x)^{-1/2}$ by Lemma 2.20.

Suppose $b \in C_C(\Omega \times G)$ has the property that

$$\int_{S_x} b(x, rt)\, d\alpha_x(t) = 1$$

whenever $x \in N$ and $r \in F^2$. Also let

$$G(r, y) = f(y) f(r^{-1} \cdot y)\, b(y, r^{-1})\, \Delta_G(r^{-1})\, \rho(r^{-1}, y)^{1/2}, \tag{11}$$

where f is as above. Then,

$$L^x(G)(h)(s) = \int f(s \cdot x) f(rs \cdot x)\, b(s \cdot x, r)\, \rho(r, s \cdot x)\, h(r \cdot s)\, d\alpha_G(r),$$

which, using Lemma 2.20 and sending $r \mapsto rs^{-1}$,

$$L^x(G)(h)(s)$$
$$= \omega(s, x)^{-1} \int_G f(s \cdot x) f(r \cdot x)\, b(s \cdot x, rs^{-1})\, h(r)\, \rho(r, x)\, d\alpha_G(r),$$

which is equal to

$$\int_{G/S_x} f(s \cdot x) f(r \cdot x) \int_{S_{s \cdot x}} b(s \cdot x, rs^{-1}t)\, d\alpha_{s \cdot x}(t)\, h(r)\, d\mu_x(r). \tag{12}$$

And in particular, if $x = z$, then

$$L^z(G)(h)(s) = \int_{G/S_z} f(s \cdot z) f(r \cdot z)\, h(r)\, d\mu_z(r), \tag{13}$$

which is a rank one positive operator.

Let V_0 be a symmetric neighborhood of the identity in G with compact closure. V. Also, let $K = V^2F^2V^3F$. Notice that $\Phi\colon G \times \Omega \to \Omega$, defined by $(s, x) \mapsto (s \cdot x)$, may also be viewed as a function on $\Omega \times G/\sim$. Thus, $\delta(z \times K) \backslash \delta(z \times V_0F) = \delta(z \times (K \backslash V_0FS_z))$ is compact and is contained in $\Phi^{-1}(\Omega \backslash N)$, an open set. Therefore, for each $(z, s) \in \delta(z \times (K \backslash V_0FS_z))$, there is an open neighborhood, $U_s \times V_s \subseteq \Omega \times G$, such that $\delta(U_s \times V_s) \subseteq \Phi^{-1}(\Omega \backslash N)$. Suppose that $\delta(U_{s_1} \times V_{s_1}),\dots, \delta(U_{s_n} \times V_{s_n})$ cover $\delta(z \times (K \backslash V_0FS_z))$. Let $U = \bigcap_{i=1}^n U_{s_i}$ and $V_1 = \bigcup_{i=1}^n V_{s_i}$. Then,

$$\delta(U \times V_1) \supseteq \delta(z \times (K \backslash V_0FS_z))$$

and

$$\delta(U \times V_1) \subseteq \Phi^{-1}(\Omega \backslash N).$$

Moreover, $\delta^{-1}(\delta(U \times (V_1 \cup V_0F)))$ is a saturated open set in $\Omega \times G$ which contains $z \times K$ Thus, there is a neighborhood of x, $U_0 \subseteq U$, such that

$$\delta(U_0 \times K) \subseteq \delta(U \times (V_1 \cup V_0F)).$$

Notice that $\delta(U_0 \times V_0F) \subseteq \delta(U_0 \times V_1 \cup V_0F) \backslash \delta(U_0 \times V_0F) \subseteq \delta(U_0 \times V_1) \subseteq \delta(U \times V_1) \subseteq \Phi^{-1}(\Omega \backslash N)$. Therefore if $x \in U_0$, then

$$f_x^1(s) = \begin{cases} f(s \cdot x) & \text{if } s \in KS_x, \\ 0 & \text{if } s \notin V_0FS_x \end{cases}$$

is a well-defined, continuous function on $\Omega \times G$.

Now assume the function $b \in C_c(\Omega \times G)$ in Eq. (11) has been chosen so that

$$\int_{S_y} b(y, rt)\, d\alpha_y(t) = 1$$

for all $y \in N$ and $r \in VF^2V$. And that the support has been cut down, if necessary, so that

$$b(y, r) = 0$$

if $(y, r) \notin \delta^{-1}(\delta(M \times V^2F^2V^2))$, where M is compact and $M \supseteq N$.

LEMMA 4.4. *With the notation and definitions established above and if* $x \in U_0$, *then*

$$f(s \cdot x) f(r \cdot x) \int_{S_{s \cdot x}} b(s \cdot x, rs^{-1}t)\, d\alpha_{s \cdot x}(t) f_x^1(r) \tag{14}$$

and

$$f_x^1(s) f(r \cdot x) f_x^1(r) \tag{15}$$

are equal for all $s, r \in G$.

Proof. Notice that both equations depend only on the classes of s and r in G/S_x. If $r \notin V_0FS_x$, then $f_x^1(r) = 0$, so both equations are zero. Thus, it can be assumed that $r \in V_0F$.

If $s \in V_0F$, then $f_x^1(s) = f(s \cdot x)$ and $rs^{-1} \in V_0F^2V_0$. Thus,

$$\int_{S_{s \cdot x}} b(s \cdot x, rs^{-1}t)\, d\alpha_{s \cdot x}(t) = 1.$$

Of course, the above holds for all $s \in V_0FS_x$. And in that case, Eqs. (14) and (15) agree.

If $s \in K \backslash V_0FS_x$, then $f_x^1(s) = f(s \cdot x) = 0$. It follows that Eqs. (14) and (15) are both zero.

Finally if $s \notin K$, then $f_x^1(s)$ is zero as is Eq. (15). On the other hand, whenever $rs^{-1} \in V^2F^2V^2$, it follows that $s \in V^2F^2V^2r$. But if $r \in V_0F$, that implies $s \in K$. Thus, by our choice of b,

$$\int_{S_{s \cdot x}} b(s \cdot x, rs^{-1}t)\, d\alpha_{s \cdot x}(t) = 0$$

if $s \notin K$ and Eq. (14) is also zero. Q.E.D.

It follows from the above lemma and Eq. (12) that f_x^1 is an eigenvector for $L^x(G)$ for each $x \in U_0$. Moreover, the corresponding eigenvalue is

$$\lambda_1^x = \int_{G/S_x} f(r \cdot x) f_x^1(r)\, d\mu_x(r).$$

On the other hand, assuming $U_0 \subseteq W \subseteq N$,

$$f_x^1(r) \int_{S_x} b(x, rt)\, d\alpha_x(t) = f_x^1(r)$$

for all $x \in U_0$ and $r \in G$ (since $e \in F$ implies $V_0 F \subseteq VF^2V$). Thus,

$$\lambda_1^x = \int_G f(r \cdot x) f_x^1(r)\, b(x, r)\, \rho(r, x)\, d\alpha_G(r).$$

In particular, $\lambda_1^{x_C}$ converges to $\lambda_1^z = \operatorname{tr}(L^2(G))$.

LEMMA 4.5. *Let W be as above. There is an open, symmetric neighborhood, Q, of e in G and a neighborhood of z, $U_1 \subseteq U_0$, such that $r \cdot x \in U_1$ implies $Qrx \subseteq W$.*

Proof. If the lemma were false, there would exist s_α, $t_\alpha \in G$ and $x_\alpha \in \Omega$ such that $t_\alpha \to e$, $r_\alpha \cdot x_\alpha \to z$, and $t_\alpha r_\alpha \cdot x_\alpha$ is not in W. But, $t_\alpha \cdot (r_\alpha \cdot x_\alpha) \to z$. Q.E.D.

Let Q and U_1 be as above with $Q = Q^{-1}$ and $Q^2 \subseteq V$. Let K_0 be a compact set containing $s_0^{-1}QVF$ such that x_C and $s_C x_C$ are both in U_1 if $C \supseteq K_0$. Notice that $s_C \notin QVF$ and that Qs_C and VF are disjoint.

Let h_C be the characteristic function of $Qs_C S_{x_C}$ in $L^2(G/S_{x_C})$. Notice that if $s \in Qs_C$, then

$$\int_{S_{s \cdot x_C}} b(s \cdot x_C, rs^{-1}t)\, d\alpha_{s \cdot x_C}(t)\, h_C(r) = h_C(r),$$

because $Q^2 \subseteq V \subseteq VF^2V$ and $s \cdot x_C \in Qs_C x_C \subseteq W \subseteq N$. But since f is identically one on W, for all $s \in Qs_C$ it follows that

$$L^{x_C}(G)(h_C)(s) = \langle h_C, h_C \rangle = \|h_C\|_C^2.$$

Of course $\|\cdot\|_C$ denotes the norm in $L^2(G/S_{x_C})$. In particular, viewed as an operator on the orthogonal complement of $f_{x_C}^1$, $L^{x_C}(G)$ is a positive compact operator of norm at least $\|h_C\|_C$ and, therefore, has an eigenvalue λ_2^C, such that

$$\lambda_2^C \geqslant \|h_C\|^2 = \mu_x(Qs_C S_{x_C}). \tag{16}$$

Let Q_0 be a neighborhood of e in G with $Q_0^2 \subseteq Q$. Then it is possible to find $b^0 \in C_c(\Omega \times G)$ with the properties that if $x \in U_1$, then

$$\int_{S_x} b^0(x, rt)\, d\alpha_x(t) = 1$$

for all $r \in Q_0 S_x$, and

$$\int_{S_x} b^0(x, rt)\, d\alpha_x(t) = 0$$

if $r \notin QS_x$. Define $b^C(x, r)$ to be $\omega(s_C^{-1}, x)\, b^0(s_C \cdot x, rs_C^{-1})$. Also let

$$\bar{b}^C(x, r) = \int_{S_x} b^C(x, rt)\, d\alpha_x(t).$$

Since

$$\bar{b}^C(x, r) = \int_{S_{s_C \cdot x}} b^0(s_C \cdot x, rs_C^{-1}t)\, d\alpha_{s_C \cdot x}(t),$$

it follows that $\bar{b}^C(x, \cdot)$ is a non-negative function in $C_c(G/S_x)$ which is one on $Q_0 s_C S_x$ and zero off $Qs_C S_x$. In particular,

$$\mu_{x_C}(Qs_C S_{x_C}) \geqslant \int_{G/S_x} \bar{b}^C(x_C, r)\, d\mu_{x_C}(r). \tag{17}$$

LEMMA 4.6. *There is a compact set, K_1, containing K_0 and an $a > 0$ such that $C \supseteq K_1$ implies $\lambda_1^z \geqslant 3a$, $\lambda_2^C \geqslant 2a$, and $\lambda_1^{x_C} > \lambda_1^z - a$.*

Proof. The right-hand side of Eq. (17) is, by Lemma 2.21, equal to

$$\int_{G/S_{s_C \cdot x_C}} \bar{b}^C(x_C, rs_C)\, d\mu_{s_C \cdot x_C}(r),$$

which, by definition, is equal to

$$\int_{G/S_{s_C \cdot x_C}} \omega(s_C^{-1}, x) \int_{S_x} b^0(s_C \cdot x_C, rs_C t s_C^{-1})\, d\alpha_x(t)\, d\mu_{s_C \cdot x_C}(t)$$

$$= \int_G b^0(s_C \cdot x_C, r)\, \rho(r, s_C \cdot x_C)\, d\alpha_G(r).$$

However, the last equation approaches

$$\int_G b^0(z,r)\,\rho(r,z)\,d\alpha_G(r) > 0$$

as C increases. The assertions of the lemma now follow easily in the light of Eq. (16). Q.E.D.

Finally, let r be the function on $[0, \infty)$ such that

$$r(t) = \begin{cases} 0, & 0 \leqslant t \leqslant a, \\ 2(t-a), & a \leqslant t \leqslant 2a, \\ t, & 2a \leqslant t. \end{cases}$$

Now, $r(G)$ is in the Pedersen ideal of $C^*(G, \Omega)$ by [13, p. 134]. Moreover, by [20, Lemma 4.9] L^{x_C} converges to L^z in $C^*(G, \Omega)^\wedge$ since $S_{x_C} \to S_z$ by assumption. Therefore, the next lemma completes the proof of Proposition 4.2.

LEMMA 4.7. *With the above definitions and notation,* $\operatorname{tr}(L^{x_C}(r(G)))$ *does not converge to* $\operatorname{tr}(L^z(r(G)))$.

Proof. Notice that $L^x(r(G)) = r(L^x(G))$. In particular, $L^z(r(G)) = L^z(G)$ because $L^z(G)$ is a rank one operator with eigenvalue $\lambda_1^z \geqslant 2a$. But if $C \supseteq K_1$, then $r(\lambda_1^{x_C}) = \lambda_1^{x_C}$ and $r(\lambda_2^C) = \lambda_2^C$. In particular,

$$\begin{aligned} \operatorname{tr}(L^{x_C}(r(G))) &= \operatorname{tr}(r(L^{x_C}(G))) \geqslant \lambda_1^{x_C} + \lambda_2^C \geqslant \lambda_1^z + a \\ &= \operatorname{tr}(L^z(r(G))) + a. \end{aligned}$$

Q.E.D.

Combining the results of the previous sections with a final observation one obtains the following theorem.

THEOREM 4.8. *Suppose that the stability groups vary continuously and, if* (G, Ω) *is not second countable, that the natural maps of* G/S_x *onto* $G \cdot x$ *are homeomorphisms for each* x. *Then* $C^*(G, \Omega)$ *has continuous trace if and only if* $C^*(\mathscr{S})$ *has continuous trace and every compact subset of* Ω *is* C-*wandering.*

Proof. In view of Theorem 2.7 and Proposition 4.2, it will suffice to show that, with the given hypotheses, $C^*(G, \Omega)$ having continuous trace implies $C^*(\mathscr{S})$ has continuous trace. By Theorems 2.14 and 2.15, it will be enough to show that $C^*(\mathscr{E})$ has continuous trace. In the following, if $f \in C^*(G, \Omega)$, then let $\bar{f}$ denote the image of f in $M(C^*(\mathscr{E}))$.

LEMMA 4.9. *Suppose that* $f \in m(C^*(G,\Omega))^+$ *and that* $e \in C^*(\mathscr{E})^+$ *with* $\bar{f} \geqslant e$ *in* $M(C^*(\mathscr{E}))$. *Then* $e \in m(C^*(\mathscr{E}))^+$.

Proof. Recall that $C^*(\mathscr{E})^\wedge$ can be viewed as an open subset of $M(C^*(\mathscr{E}))^\wedge$. Then, if $y = \bar{f} - e \in M(C^*(\mathscr{E}))^+$, the (possibly infinite-valued) functions

$$\pi \longmapsto \operatorname{tr}(\pi(y))$$

and

$$\pi \longmapsto \operatorname{tr}(\pi(e))$$

are non-negative, lower semi-continuous on $M(C^*(\mathscr{E}))^+$ [3, 3.5.9]. In particular, when restricted to $C^*(\mathscr{E})^\wedge$, they are still lower semicontinuous and have sum

$$U^x_\omega \longmapsto \operatorname{tr}(\tilde{U}^x_\omega(\bar{f})) = \operatorname{tr}(L^x_\omega(f)),$$

where $\tilde{U}^x_\omega$ denotes the canonical extension of U^x_ω to $M(C^*(\mathscr{E}))$. Applying [10, Proposition 9(i)] to the homomorphism of $C^*(G,\Omega)$ into $M(C^*(\mathscr{E}))$, it is not difficult to check that the map of $C^*(\mathscr{E})^\wedge$ onto $C^*(G,\Omega)^\wedge$ defined by $U^x_\omega \mapsto L^x_\omega$ is continuous. Thus, the sum described above is finite and continuous on $C^*(\mathscr{E})^\wedge$. It follows that both summands are continuous; thus, $e \in m(C^*(\mathscr{E}))^+$. Q.E.D.

To finish the proof of Theorem 4.8, let U^x_ω be an arbitrary element of $C^*(\mathscr{E})^\wedge$. Since $C^*(G,\Omega)^\wedge$ has continuous trace, there is a $f \in m(C^*(G,\Omega))$ such that $L^x_\omega(f) \neq 0$. Let $e \in C^*(\mathscr{E})$ be such that $U^x_\omega(f \cdot e) \neq 0$. It follows that $d = (f \cdot e) * (f \cdot e)^* \leqslant \| e \|^2_{\mathscr{E}} \, f * f^* \in m(C^*(G,\Omega))^+$. Thus, $d \in m(C^*(\mathscr{E}))^+$ by the last lemma, and

$$\| U^x_\omega(d) \| = \| U^x_\omega(f \cdot e) \|^2 \neq 0.$$

This suffices (cf. [3, 4.5.2]). Q.E.D.

5. EXAMPLES AND QUESTIONS

One important class of examples is the case when all the stability groups are abelian. Then $C^*(\mathscr{S})$ is commutative, and clearly has continuous trace. In particular, the hypothesis on $C^*(\mathscr{S})$ may be omitted from Theorems 2.7 and 4.8 in this case. If in addition the stability groups are contained in a fixed abelian subgroup H, then the next theorem and remark summarizes the conclusions of Theorems 2.7 and 4.8 and Proposition 4.1.

THEOREM 5.1. *Suppose that all the stability groups are contained in a fixed abelian subgroup and that (G, Ω) is second countable. It follows that $C^*(G, \Omega)$ has continuous trace if and only if the stability groups all vary continuously on Ω and every compact subset of Ω is G-wandering.*

Remark. The hypothesis that (G, Ω) is second countable may be replaced by assuming that the natural maps of G/S_x onto $G \cdot x$ are homeomorphisms for each x and, if G itself is not abelian, that $C^*(H, \Omega)$ is quasi-regular.

Another class of examples is the case when the action is essentially free. That is, all of the stability groups are the same, say $S_x = H$ for all x. Then the stability groups obviously vary continuously and $C^*(\mathscr{S})$ has continuous trace if and only if $C^*(H)$ does. Notice that H is normal in G.

THEOREM 5.2. *Suppose (G, Ω) is an essentially free transformation group with stability group H. If (G, Ω) is not second countable, then assume each orbit is homeomorphic to G/H. Then the following are equivalent:*

(i) *$C^*(G, \Omega)$ has continuous trace.*

(ii) *Every compact subset of Ω G-wandering and $C^*(H)$ has continuous trace.*

(iii) *$C^*(G/H, \Omega)$ and $C^*(H)$ have continuous trace.*

Proof. This is a direct result of the above remarks and Theorem 2.7 as well as Proposition 4.2. Of course, (ii) ⇒ (iii) is just [9, Theorem 14] and, in the second countable case, (iii) ⇒ (ii) is [9, Theorem 17]. Q.E.D.

EXAMPLE 5.3. Let $G = \mathbf{T}$, the 1-dimensional torus, and $\Omega = \mathbf{C}$. Let G act by multiplication. That is, $e^{i\theta}(re^{i\psi}) = re^{i(\theta+\psi)}$. Then the orbits are concentric circles about the origin together with the origin. Thus, the orbit space is (homeomorphic to) $[0, \infty)$, but the stability groups do not vary continuously since the action is free everywhere except at the origin. It follows that $C^*(\mathbf{T}, \mathbf{C})$ does not have Hausdorff spectrum and, in particular, does not have continuous trace. Note that by [9, Theorem 14], freely acting compact groups always result in algebras with continuous trace. In general, if G is a compact abelian group, then $C^*(G, \Omega)$ has continuous trace if and only if the stability groups vary continuously.

I know of no examples of transformation group C^*-algebras having continuous trace and stability groups which do not vary continuously. On the other hand, I can find no proof of the necessity of this condition in the general case. It would be very interesting to know whether or not this condition is necessary for any class of non-abelian groups.

EXAMPLE 5.4. Let $G = R$ and $\Omega = \mathbf{C}$. Let R act on $\mathbf{C}$ by fixing the origin and, if $|\xi| \neq 0$, $r \cdot \xi = \exp(2\pi i(r/|\xi|))\,\xi$. The orbits again are concentric circles and the orbit space is homeomorphic to $[0, \infty)$. Notice that points rotate about the origin much more quicly as one approaches the origin. Moreover,

$$S_\xi = \begin{cases} |\xi|\,\mathbf{Z} & \text{if } \xi \neq 0, \\ \mathbf{R} & \text{if } \xi = 0. \end{cases}$$

Thus, the S_ξ vary continuously. It is not hard to see that compact sets are $\mathbf{R}$-wandering. In fact, let $K \subseteq \mathbf{C}$ be compact. If $0 \notin K$, then $(\xi, r) \mapsto (\xi, r/|\xi|)$ defines a homeomorphism of $\delta(K \times \mathbf{R})$ with $K \times \mathbf{T}$. In particular, K is $\mathbf{R}$-wandering. If $0 \in K$, then let $\{(\xi_\alpha, r_\alpha)\}$ be a net in $\{(\xi, r): \xi \in K, rK \cap K \neq \varnothing\}$. It may be assumed that $\xi_\alpha \to \xi$ in K. If $\xi = 0$, then (ξ_α, s_α) converges to $(0, 0)$ in $\mathbf{C} \times \mathbf{R}/\!\sim$. Otherwise, we may pass to a subnet which is bounded away from the origin. This subnet must have a convergent subnet by the argument above.

It follows that $C^*(\mathbf{R}, \mathbf{C})$ has continuous trace.

Green gives an example in [9, cf. pp. 95–96] where the action is free and the orbit space, hence the spectrum, is Hausdorff, but $C^*(G, \Omega)$ does not have continuous trace.

There are several questions suggested by the results in this paper which I cannot answer.

Q1: If $C^*(G, \Omega)$ has continuous trace, then do the stability groups have to vary continuously?

An answer to Q1 would be of interest even in the case when all the stability groups are abelian.

Since the condition that $C^*(\mathcal{S})$ has continuous trace is rather difficult to check, it would be convenient to have a theorem giving simpler conditions which would guarantee this. In particular, one might ask the following.

Q2: If the stability groups vary continuously on Ω and $C^*(S_x)$ has continuous trace for each $x \in \Omega$, then does $C^*(\mathcal{S})$ have continuous trace?

A simpler version of Q2, but also interesting, is the following:

Q3: If G is compact and the stability groups vary continuously, then does $C^*(\mathcal{S})$ have continuous trace?

6. THE IMPRIMITIVITY BIMODULE

If one keeps the $C^*(H) - C^*(G, G/H)$ case in mind (cf. Definition 2.10), then the following definitions seem quite natural. The imprimitivity bimodule will be $C_c(\Omega \times G)$, which will be denoted by X. Of course, through this section it will be assumed that the stability groups vary continuously.

If $f \in C_c(\mathcal{S})$ and $F \in X$, then the right $C_c(\mathcal{S})$-action is given by

$$F \cdot f(x, s) = \int_{S_x} F(x, st^{-1})\, \Delta_G(t^{-1})\, \gamma_{S_x}(t) f(x, t)\, d\alpha_x(t). \tag{18}$$

If $e \in C_c(\mathcal{E})$ and $F \in X$, then the left $C_c(\mathcal{E})$-action is given by

$$e \cdot F(x, s) = \int_G e(x, r, s)\, F(x, r^{-1}s)\, d\alpha_G(r). \tag{19}$$

Finally if F and $G \in X$, then the $C_c(\mathcal{S})$- and $C_c(\mathcal{E})$-valued inner products on X are given by

$$\langle F, G\rangle_{\mathcal{S}}\,(x, t) = \gamma_x(t) \int_G F^*(x, s)\, G(x, s^{-1}t)\, d\alpha_G(s) \tag{20}$$

and

$$\langle F, G\rangle_{\mathcal{E}}\,(x, r, s) = \int_{S_x} F(x, st)\, G^*(x, t^{-1}s^{-1}r)\, d\alpha_x(t), \tag{21}$$

where $F^*(x, s) = \Delta(s^{-1})\, F(x, s^{-1})^-$.

To see that the above actions and inner products actually take values in the appropriate spaces of functions, one may appeal to Lemma 2.5.

The following formulae follow from routine calculations. Let $f \in C_c(\mathcal{S})$, F, G, $H \in X$, and $e \in C_c(\mathcal{E})$.

$$\langle F, G \cdot f\rangle_{\mathcal{S}} = \langle F, G\rangle_{\mathcal{S}} * f, \tag{22}$$

$$\langle e \cdot F, G\rangle_{\mathcal{E}} = e * \langle F, G\rangle_{\mathcal{E}}, \tag{23}$$

$$\langle F, G\rangle_{\mathcal{E}} \cdot H = F \cdot \langle G, H\rangle_{\mathcal{S}}, \tag{24}$$

$$\langle F, G\rangle_{\mathcal{S}}^* = \langle G, F\rangle_{\mathcal{S}}, \tag{25}$$

$$\langle F, G\rangle_{\mathcal{E}}^* = \langle G, F\rangle_{\mathcal{E}}. \tag{26}$$

LEMMA 6.1. *For each $F \in X$, $\langle F, F\rangle_{\mathcal{S}}$ and $\langle F, F\rangle_{\mathcal{E}}$ are positive elements of $C^*(\mathcal{S})$ and $C^*(\mathcal{E})$, respectively.*

Proof. Let F_x denote the element of $C_c(G)$ defined by $F_x(s) = F(x, s)$. Then notice that

$$\langle F, F\rangle_{\mathcal{S}}\,(x, t) = \langle F_x, F_x\rangle_{S_x}\,(t)$$

and

$$\langle F, F\rangle_{\mathscr{E}}(x, r, s) = \langle F_x, F_x\rangle_{(G, G/S_x)}(r, s).$$

By Lemma 2.8, it will suffice to show that $M_\pi^x(\langle F, F\rangle_{\mathscr{S}}) \geqslant 0$ for each $x \in \Omega$ and $\pi \in \hat{S}_x$. However,

$$M_\pi^x(\langle F, F\rangle_{\mathscr{S}}) = \pi(\langle F_x, F_x\rangle),$$

which is positive by [15, Theorem 4.4].

To show that $\langle F, F\rangle_{\mathscr{E}}$ is positive, it will be enough to show that $U_\pi^x(\langle F, F\rangle_{\mathscr{E}})$ is positive for any $x \in \Omega$ and $\pi \in \hat{S}_x$. This follows from [15, Proposition 7.9] in much the same way as above. Q.E.D.

In order to show that X is both a right $C_c(\mathscr{S})$-rigged space and a left $C_c(\mathscr{E})$-rigged space we must show that $\langle X, X\rangle_{\mathscr{S}}$ and $\langle X, X\rangle_{\mathscr{E}}$ span dense subsets of $C^*(\mathscr{S})$ and $C^*(\mathscr{E})$, respectively. To do this, several lemmas are needed.

LEMMA 6.2. *$C_c(\mathscr{S})$ contains an approximate identity for the inductive limit topology, and hence for either the $\|\cdot\|_1$-norm or the C^*-norm topology, of the form $\langle F_\alpha, F_\alpha\rangle_{\mathscr{S}}$ with $F_\alpha \in X$.*

Proof. Let K be compact in G, and $f \in C_c(G)$ identically one on K with $0 \leqslant f \leqslant 1$. Since $x \mapsto \int_G f\, d\alpha_x$ is continuous on Ω, it follows that $\alpha_x(K)$ is bounded on Ω. If $F \in C_c(\mathscr{S})$ and if the support of F is contained in $A \times K$ with K compact in G, then

$$\|F\| \leqslant \|F\|_1 \leqslant M_\infty \cdot M_K,$$

where M_∞ is the supremum of F on $\mathscr{S}$ and M_K is the supremum of $\alpha_x(K)$. The fact that the inductive limit topology is stronger than the norm topologies now follows easily.

Let U be a compact neighborhood of the identity in G, and V a symmetric neighborhood of the identity such that $V^2 \subseteq U$. Let $f \in C_c(G)$ be non-negative with $\operatorname{supp} f \subseteq V$ and $f \neq 0$. Also, for each $C \subseteq \Omega$ compact, let ψ_C be an element of $C_c(\Omega)$ such that $0 \leqslant \psi_C \leqslant 1$ and $\psi_C \equiv 1$ on C.

Now define

$$\theta_{(C,U)}(x, t) = \psi_C(x)\langle f, f\rangle_{S_x}(t)\left(\int_{S_x} \langle f, f\rangle_{S_x}(t)\, d\alpha_x(t)\right)^{-1}.$$

Recall that $\langle \cdot, \cdot\rangle_{S_x}$ is defined in Definition 2.10. In particular, $\theta_{(C,U)} \in C_c(\mathscr{S})$ by Lemma 2.5(ii), and has the proper form. Namely,

$$\theta_{(C,U)}(x, t) = \langle F_{(C,U)}, F_{(C,U)}\rangle_{\mathscr{S}}(x, t),$$

where

$$F_{(C,U)}(x, s) = \psi_C(x)^{1/2} \left(\int_{S_x} \langle f, f \rangle_{S_x}(t)\, d\alpha_x(t) \right)^{-1/2} f(s).$$

Notice that $\theta_{(C,U)}$ has the property that

$$\int_{S_x} \theta_{(C,U)}(x, t)\, d\alpha_x(t) = 1 \qquad \text{for every } x \in C.$$

Finally, it must be shown that, for each $f \in C_c(\mathscr{S})$, $\theta_{(C,U)} * f \to f$ in the inductive limit topology as C increases to Ω and U shrinks to the identity. In fact, suppose that f has support in $C \times U$. Then $\theta_{(C',U')} * f$ has support in $C \times U'U$. Thus, it will suffice to show uniform convergence. But, using standard compactness arguments, there is a compact neighborhood of e in G, V_ϵ, such that

$$|f(x, r) - f(x, s)| < \varepsilon$$

if $sr^{-1} \in V_\epsilon$. Then if $C' \supseteq C$ and $U' \subseteq V_\epsilon$,

$$|\,\theta_{(C',U')} * f(x, t) - f(x, t)| \leqslant \varepsilon.$$

Q.E.D.

LEMMA 6.3. *Given $f \in C_c(\mathscr{E})$ and $\varepsilon > 0$, there is a neighborhood of the identity V_ϵ such that $|f(x, r, s) - f(x, u, v)| < \varepsilon$ if $ur^{-1} \in V_\epsilon$ and $vs^{-1} \in V_\epsilon$.*

Proof. Let $\sigma(A \times K \times C)$ contain the support of f. Let V be a fixed symmetric compact neighborhood of e. If the lemma were false, then for each $V_\alpha \subseteq V$ we could find $x_\alpha \in \Omega$ and r_α, s_α, u_α, $v_\alpha \in G$ such that $u_\alpha r_\alpha^{-1}$, $v_\alpha s_\alpha^{-1} \in V_\alpha$, and

$$|f(x_\alpha, r_\alpha, s_\alpha) - f(x_\alpha, u_\alpha, v_\alpha)| \geqslant \varepsilon \qquad \text{for every } \alpha.$$

However, we must have $(x_\alpha, r_\alpha, s_\alpha)$ and $(x_\alpha, u_\alpha, v_\alpha)$ in $\sigma(A \times VK \times VC)$, a compact subset of $\Omega \times G \times G/\!\sim$. From this observation, together with Lemma 2.12, we may assume that $(x_\alpha, r_\alpha, s_\alpha)$ converges to (x, r, s) in $\Omega \times G \times G$ and $(x_\alpha, u_\alpha, v_\alpha)$ converges to (x, u, v). Since $u_\alpha r_\alpha^{-1}$ and $v_\alpha s_\alpha^{-1}$ converge to the identity, $u = r$ and $s = v$. The continuity of f now provides a contradiction. Q.E.D.

LEMMA 6.4. *$C_c(\mathscr{E})$ has an approximate identity for the inductive limit topology, and hence for the C^*-norm topology, of the form $\sum_{i=1}^{n} \langle F_\alpha^i, G_\alpha^i \rangle$ for F_α^i and $G_\alpha^i \in X$.*

Proof. An argument similar to that of Lemma 6.1 shows that the inductive limit topology is stronger than the norm tppology. More precisely, recall from [4, Lemma 3.21] that if L is a representation of $C^*(G, G/H)$ and $\phi \in C_c(G, G/H)$, then $\|L(\phi)\| \leqslant \|\phi\|_1$, where $\|\phi\|_1 = \int_G \|\phi(s, \cdot)\|_\infty \, d\alpha_G(s)$. Thus, if $f \in C_c(\mathscr{E})$ with support contained in $\sigma(A \times K \times C)$, we see that $\|f\| \leqslant M_\infty M_K$ where M_∞ is the supremum of f and M_K is an upper bound for $\alpha_x(K)$.

The reader will notice the similarity of the construction below with that in [15, Proposition 7.11].

Let $A \subseteq \Omega$ and $C \subseteq G$ be compact subsets. Let N be a compact neighborhood of e in G with M a neighborhood of e such that $M^2 \subseteq N$. Pick a partition of unity in G subordinate to the right translates of M (i.e., each element of the partition is supported in a translate of M). Let b be the function constructed in Lemma 2.18 for $A \times C \subseteq \Omega \times G$.

Then, multiplying the functions in the partition pointwise by b, we obtain only a finite number of non-zero functions, $F_1, ..., F_n \in X$, with the properties

$$\operatorname{supp} F_i \subseteq A \times M \cdot z_i \qquad \text{some} \quad z_i \in G$$

and

$$\sum_{i=1}^{n} \int_{S_x} F_i(x, st) \, d\alpha_x(t) = 1 \qquad \text{for } s \in CH, \ x \in A.$$

Now take $g_i \in C_c(G)$ with $\operatorname{supp} g_i^* \subseteq z_i \cdot M$ and $\int_G g_i^* \, d\mu_G = 1$. Also, let $\psi_A \in C_c(\Omega)$ be identically 1 on A. We define

$$G_i(x, s) = \psi_A(x) \, g_i(s),$$

$$\theta_{(A,N,C)} = \sum_{i=1}^{n} \langle F_i, G_i \rangle_E.$$

Notice that the support of $\langle F_i, G_i \rangle_E$ is contained in $\sigma(A \times N \times C)$ for each i, so that the support of $\theta_{(A,N,C)}$ is as well. Also,

$$\theta_{(A,N,C)}(x, r, s) = 0 \qquad \text{if} \quad r \notin N$$

and

$$\int_G \theta_{(A,N,C)}(x, r, s) \, d\alpha_G(r) = 1 \qquad \text{if} \quad s \in CH \text{ and } x \in A.$$

These are precisely the conditions in [15, Proposition 7.10] which imply that $\theta_{(A,N,C)}(x, \cdot, \cdot)$ is an approximate identity for $C_c(G, G/S_x)$. Thus, as in [15], the proof that $\theta_{(A,N,C)}$ forms the required approximate identity, when directed by increasing A and C and decreasing N, will be similar to Lemma 3.27 of [4].

Let $f \in C_c(\mathscr{E})$ with support of $\theta_{(A,N,C)} * f \supseteq \sigma(A \times KN \times C)$. It follows that we need only show that $\theta_{(A,N,C)} * f$ converges uniformly to f on $\sigma(A \times KN \times C)$. Let $\varepsilon > 0$ and let V_ϵ be a compact neighborhood as defined in Lemma 6.2. Then if $N \subseteq V_\epsilon$,

$$\begin{aligned} &|\theta_{(A,N,C)} * f(x, r, s) - f(x, r, s)| \\ &\quad = \left| \int_G \theta_{(A,N,C)}(x, t, s)[f(x, t^{-1}r, t^{-1}s) - f(x, r, s)]\, d\alpha_G(t) \right| \\ &\quad \leqslant \varepsilon \int_G \theta_{(A,N,C)}(x, t, s)\, d\alpha_G(t) = \varepsilon. \end{aligned}$$

Q.E.D.

PROPOSITION 6.5. *X is an $C_c(\mathscr{E}) - C_c(\mathscr{S})$-imprimitivity bimodule. That is, X is a right $C_c(\mathscr{S})$-rigged space and a left $C_c(\mathscr{E})$-rigged space with the following additional properties.*

(a) *For $F, G, H \in X$, $\langle F, G\rangle_{\mathscr{E}} \cdot H = F \cdot \langle G, H\rangle_{\mathscr{S}}$.*

(b) *For $f \in C_c(\mathscr{E})$, $F \in X$, $\langle f \cdot F, f \cdot F\rangle_{\mathscr{S}} \leqslant \|f\|^2_{C^*(\mathscr{E})} \langle F, F\rangle_{\mathscr{S}}$.*

(c) *For $g \in C_c(\mathscr{S})$, $F \in X$, $\langle F \cdot g, F \cdot g\rangle \leqslant \|g\|^2_{C^*(\mathscr{S})} \langle F, F\rangle_{\mathscr{E}}$.*

Proof. It follows from Eqs. (22), (23), (25), and (26) that $\langle X, X\rangle_{\mathscr{E}}$ and $\langle X, X\rangle_{\mathscr{S}}$ are ideals in X. From Lemmas 6.2 and 6.4 it follows that these ideals are dense. In particular, the statements about rigged spaces have been demonstrated.

Part (a) is Eq. (24). To prove (b), it suffices, by Lemma 2.8, to show that

$$P(\langle f \cdot F, f \cdot F\rangle_{\mathscr{S}}) \subseteq \|f\|^2\, P(\langle F, F\rangle_{\mathscr{S}}), \tag{27}$$

where P is a state of $C^*(\mathscr{S})$ of the form $P(F) = \rho(F_x)$ for ρ a state of $C^*(S_x)$ and $x \in \Omega$. Let f_x denote the element of $C_c(G, G/S^x)$ defined by $f_x(r, s) = f(x, r, s)$. Then the left-hand side of (27) is

$$\rho(\langle f_x \cdot F_x, f_x \cdot F_x\rangle_{S_x}). \tag{28}$$

The action of $C_c(G, G/S_x)$ is as defined in 7.4 of [15]. Then by Proposition 2.6 of [16]

$$(28) \leqslant \|f_x\|_{C^*(G, G/S_x)}\, \rho(\langle F_x, F_x\rangle_{S_x})$$

(cf. Definition 6.10 of [15]). The desired result now follows.

To show (c), note that it suffices to show that $P(\langle F \cdot g, F \cdot g\rangle_{\mathscr{E}}) \leqslant \|g\|^2_{C^*(\mathscr{S})} P(\langle F, F\rangle_{\mathscr{S}})$, where P is a state of $C^*(\mathscr{E})$ of the form $P(f) = \rho(f_x)$, for $x \in \Omega$, and ρ is a state of $C^*(G, G/S_x)$. The proof now proceeds exactly as in part (b) and depends on the fact $C_c(G)$ is a $C_c(S_x) - C_c(G, G/S_x)$ imprimitivity bimodule. Q.E.D.

If R is a representation of $C^*(\mathscr{S})$ in V_R, then let $\text{Ind}(R)$ denote the representation of $C^*(\mathscr{E})$ induced from R via X. Recall from [15] that $\text{Ind}(R)$ acts on the completion of $X \otimes_{C^*(\mathscr{S})} V_R$ with respect to the inner product

$$\langle F \otimes \xi, G \otimes \eta \rangle = \langle R(\langle G, F \rangle_{\mathscr{S}})\, \xi, \eta \rangle_{V_R}.$$

Moreover,

$$\text{Ind}(R)(e)(F \otimes \xi) = (e \cdot F) \otimes \xi,$$

where $e \cdot F$ is defined by Eq. (19).

The next lemma together with the previous proposition completes the proof of Theorem 2.14.

LEMMA 6.6. *Let $x \in \Omega$ and π a unitary representation of S_x. Then $\text{Ind}(M^x_\pi)$ is unitarily equivalent to U^x_π.*

Proof. Let $V_\pi = V_{M^x_\pi}$, the space of π. Then define U from $X \otimes V_\pi$ to $C_c(G) \otimes V_\pi$ by sending $F \otimes \xi$ to $F_x \otimes \xi$. Since

$$\begin{aligned}\langle F \otimes \xi, G \otimes \eta \rangle &= \langle M^x_\pi(\langle G, F \rangle_{\mathscr{S}})\, \xi, \eta \rangle_{V_\pi} \\ &= \langle \pi(\langle G_x, F_x \rangle_{S_x})\, \xi, \eta \rangle \\ &= \langle U(F \otimes \xi), U(G \otimes \eta) \rangle_V,\end{aligned}$$

U extends to a unitary map of the space of $\text{Ind}(M^x_\pi)$ onto the space of U^x. It is a simple matter to check that U intertwines the appropriate actions.Q.E.D.

Recall that if X is a $B - A$ imprimitivity bimodule, then X admits a seminorm [17, Sect. 3]. Namely,

$$\|x\|_X^2 = \|\langle x, x \rangle_A\|_A = \|\langle x, x \rangle_B\|_B.$$

Thus, we may assume that X is complete.

Proof of Theorem 2.15. Let X be a complete $B - A$ imprimitivity bimodule and suppose that A has continuous trace. Let $\pi \in \hat{A}$ and let $\text{Ind}(\pi)$ denote the irreducible representation of B induced from π via X. Also, let H be the space of π and V the space of $\text{Ind}(\pi)$.

For each $x \in X$, T_x denotes the bounded operator from H to V defined by $T_x(\xi) = x \otimes \xi$. Then

$$T_x^* T_x = \pi(\langle x, x \rangle_A) \tag{29}$$

and

$$T_x T_x^* = \mathrm{Ind}(\pi)(\langle x, x\rangle_B). \tag{30}$$

In particular, $\mathrm{tr}(\pi(\langle x, x\rangle_A)) = \mathrm{tr}(\mathrm{Ind}(\pi)(\langle x, x\rangle_B))$. Since $\pi \mapsto \mathrm{Ind}(\pi)$ defines a homeomorphism of $\hat{A}$ onto $\hat{B}$ [15, Corollary 6.29], it follows that $\langle x, x\rangle_B \in m(B)^+$ if $\langle x, x\rangle_A \in m(A)^+$.

Since $\mathrm{Ind}(\pi)$ is a typical element of $\hat{B}$, to show that B has continuous trace it will suffice to produce a $b \in m(B)$ such that $\mathrm{Ind}(\pi)(b) \neq 0$ [3, 4.5.2]. But, since sums of the form $\sum_{i=1}^n \langle x_i, x_i\rangle_A$ are dense in A [17, Theorem 3.1 and Lemma 3.1], there is an $x \in X$ such that $\pi(\langle x, x\rangle_A) \neq 0$. Moreover $m(A)^+$ contains a self-adjoint approximate identity, $\{e_\alpha\}$, for A. In particular, $\langle x \cdot e_\alpha, x \cdot e_\alpha\rangle_A = e_\alpha \langle x, x\rangle_A e_\alpha$ is in $m(A)^+$ and converges to $\langle x, x\rangle_A$. Thus, we may assume that $\langle x, x\rangle_A \in m(A)^+$. Since $\pi(\langle x, x\rangle_A) \neq 0$, Eq. (29) implies $T_x \neq 0$, and by (30), $\mathrm{Ind}(\pi)(\langle x, x\rangle_B) \neq 0$. Q.E.D.

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