1. In class, we represented a Markov Chain with a *directed graph*, where the nodes represented the states and the directed edges (i.e. the arrows that we drew between the states) represented the steps. In this exercise, we will formalize the connection between the graph-theoretic "visualization" of a Markov Chain and the definition of a Markov Chain that we saw in class.

Suppose that a directed graph G has n vertices, denoted $v_1, v_2, \cdots, v_n$ and some number of edges (note: in this model, there is not necessarily an edge between every pair of vertices). We will let $\deg^+(i)$ represent the number of directed edges coming out of the i^{th} vertex, v_i (this is called the *out-degree* of v_i). In this model, we'll let $p_{ij} = \frac{1}{\deg^+(i)}$ if there is an edge from v_i to v_j. If there is no edge from v_i to v_j, we'll define p_{ij} to be 0.

(a)[2 points] Form a matrix P from the probabilities p_{ij} defined above. Prove that P is a *stochastic* matrix (i.e. each row of the matrix represents a probability distribution function).

(b)[1 point] Explain intuitively why the directed graph G that we've defined above should represent a Markov chain with n states and transition matrix P. (Hint: Your explanation should include the other properties of a Markov Chain that were in the formal definition.)

2.[2 points] Find a sequence of uniformly bounded discrete independent random variables $\{X_n\}$ such that the variance of their sums does not tend to infinity as $n \to \infty$ and such that their sum is not asymptotically normally distributed.

Bonus![+2 points] Let P be the following 2×2 matrix:

$$\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

Prove that the powers of P (i.e. $P, P^2, P^3, ...$) converge.