Composition of group-subgroup subfactors

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- Planar algebras and subfactors
- Automorphisms of planar algebras
- Planar fixed-point subfactors
- Examples and applications

Definitions:

- II_1 factor: von Neumann algebra with trivial center, finite trace
- II_1 subfactor: inclusion of II_1 factors $M_0 \subset M_1$
- Jones index: dimension of $L^2(M_1)$ as left M_0 module
- Basic construction: $M_2 = \{M_1, E_{M_0}\}'' \subset B(L^2(M_1))$
- Iterate (with finite index): $M_0 \subset M_1 \subset M_2 \subset M_3 \subset \ldots$

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Standard invariant:

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- Standard invariant $\rightarrow$ planar algebra (Jones)
- Planar algebra $\rightarrow$ standard invariant (Popa)

Today: Construct planar algebras abstractly, obtain new subfactors.

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Planar operad:

- Planar tangles
- Labelled internal disks
- Checkerboard shading
- Distinguished boundary region
- Composition via gluing

Subfactor planar algebra:

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Subfactor planar algebra $\leftrightarrow$ extremal subfactor.
Problem: Constructing SPAs is hard (BDG, Peters, BMPS, B).

Bipartite graph planar algebra P_Γ

- Γ is any locally finite bipartite graph
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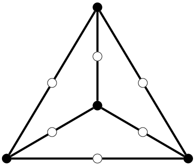
- action of planar operad determined by Γ, μ: Jones '99



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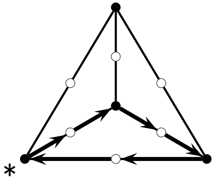

$= l;\ b_l \in V_3^+$

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Theorem (B '10):
Let Γ be a locally finite graph with positive weight vector μ, A a planar subalgebra of P_Γ with $\dim A \cap V_0^{\pm} = 1$.
Then A is a subfactor planar algebra: the other necessary properties are inherited from P_Γ.
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So we can find new subfactor planar algebras as small planar subalgebras of BGPAs. This provides new subfactors.

Automorphisms of planar algebras

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- α is an invertible graded linear map on $V_n^{\pm}$
- $\alpha \in \mathrm{Aut}P$ if it commutes with the entire planar operad
- $G \subset \mathrm{Aut}P \to P^G$ is closed under the planar operad

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To show $\alpha \in \mathrm{Aut}P$, it suffices to show that it commutes with a convenient generating set of the planar operad.
In general computing $\mathrm{Aut}P$ is hard, but the group may be completely described when P is a BGPA (B '10).
$\mathrm{Aut}P$ is generated by

- Graph automorphism operators (from $\mathrm{Aut}\Gamma$)
- Multiplication operators (from unitaries in V_1^+)

Bisch-Haagerup subfactors

- Let G be a group generated by two finite subgroups H and K, with $H \cap K = \{e\}$
- Let H and K have outer actions on some II_1 factor M
- Fixed point subfactor: $M^H \subset M$
- Crossed product subfactor: $M \subset M \rtimes K$

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- Composite subfactor (BH) $M^H \subset M \rtimes K$
- Principal graph determined by $G \subset \mathrm{Out} M$
- Planar algebra determined by groups plus scalar 3-cocycle obstruction: $\omega \in H^3(G, S^1)$
- IRF model for planar algebra (BDG)
- Determined by subfactor pair: $M \subset M \rtimes H$, $M \subset M \rtimes K$

Generalized Bisch-Haagerup subfactors

- Let G be a group generated by two finite subgroups H and K, with $H \cap K = A$.
- Nondegenerate group quadrilateral: $\begin{array}{ccc} H & \subset & G \\ \cup & & \cup \\ A & \subset & K \end{array}$

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- Let M be a II_1 factor admitting outer actions of H and K which agree on A, such that G is the group generated by H and K in $\mathrm{Out} M$
- $M \rtimes A \subset M \rtimes H$ and $M \rtimes A \subset M \rtimes K$ are subfactors.

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- $M \rtimes A \subset M \rtimes H$ and $M \rtimes A \subset M \rtimes K$ are subfactors.
- Let N be the result of the downward basic construction on $M \rtimes A \subset M \rtimes H$. (i.e. $M \rtimes H$ is isomorphic to the basic construction on $N \subset M \rtimes A$).
- Then $N \subset M \rtimes K$ is the group type subfactor obtained from the above group quadrilateral.
- This is a generalization of the Bisch-Haagerup construction $M^H \subset M \rtimes K$.

Irreducible planar fixed point subfactors

The class of planar fixed point subfactors is very large, but a few assumptions make it more tractible.

- Let Γ be a bipartite graph with no multiple edges.
- Let $G \subset \mathrm{Aut} P_\Gamma$ act transitively on V_1^+.

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- Let Γ be a bipartite graph with no multiple edges.
- Let $G \subset \mathrm{Aut} P_\Gamma$ act transitively on V_1^+.
- Then the planar fixed point subfactor obtained from P_Γ^G is a generalized Bisch-Haagerup subfactor (group quadrilateral).
- Moreover, the planar algebra of any such subfactor may be obtained in this way (B '10).

All of these irreducible subfactors have (composite) integer index, and have an intermediate subfactor.

Cocycle perturbations

- Let ρ be a map from some group G to $\mathrm{Aut} P_\Gamma$, such that P_Γ^G produces a subfactor.
- Let E be the center of the group of multiplication operators (a normal subgroup of $\mathrm{Aut} P_\Gamma$)
- Let α be a map from G to E such that $\alpha_{gh} = \alpha_g \rho_g(\alpha_h)$.
- Equivalently, α is a 1-coycle for G with coefficients in E.
- This is a perturbation of the G-action, producing a (potentially) different SPA with the same principal graph.

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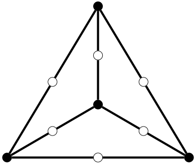
- Generalized Bisch-Haagerup planar algebras are determined by (A, H, K, G) plus $\omega \in H^3(G, S^1)$, with the restriction that ω has trivial image under the inflation map $H^3(G, S^1) \to^3 (H *_A K, S^1)$.
- From some elementary group cohomology, all such cocycles can be obtained from elements of $H^1(G, E)$ (c.f. Jones)
- This sometimes allows enumeration of BH subfactors with specified G, H, K (c.f. IK '93)

Enumeration of Bisch-Haagerup subfactors



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Taking $G = A_4$ produces a Bisch-Haagerup subfactor ($H = Z_2$, $K = Z_3$, $G = A_4$). Here $H^1(G, E) = Z_2$.

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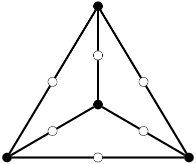

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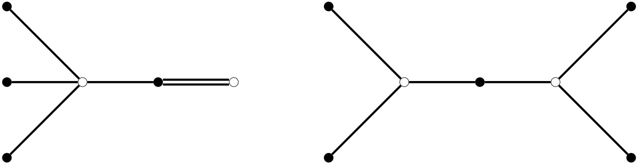

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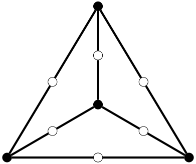

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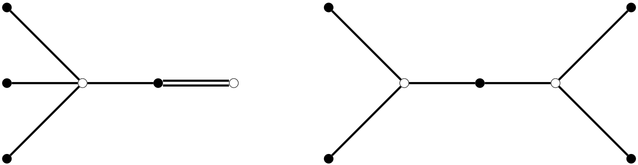

- The cohomology computation implies that there are two (hyperfinite) subfactors with this principal graph.
- They are distinguished by whether or not the representation of G in $\mathrm{Out} R$ lifts to $\mathrm{Aut} R$.

Wildness at $3 + 2\sqrt{2}$

Reducible planar fixed point subfactors are more complicated

- They are determined by the group and the graph, not just the group
- They need not be integer index
- Smallest index: $3 + 2\sqrt{2}$
- Graph: tree branching 3 times at each even vertex, 2 times at each odd vertex

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- Any vertex-transitive color-preserving group of graph automorphisms produces a subfactor
- Infinitely many of these produce mutually nonisomorphic subfactors