

Combinatorics Through Guided Discovery¹

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Preface

This book is an introduction to combinatorial mathematics, also known as combinatorics. The book focuses especially but not exclusively on the part of combinatorics that mathematicians refer to as “counting.” The book consists almost entirely of problems. Some of the problems are designed to lead you to think about a concept, others are designed to help you figure out a concept and state a theorem about it, while still others ask you to prove the theorem. Other problems give you a chance to use a theorem you have proved. From time to time there is a discussion that pulls together some of the things you have learned or introduces a new idea for you to work with. Many of the problems are designed to build up your intuition for how combinatorial mathematics works. There are problems that some people will solve quickly, and there are problems that will take days of thought for everyone. Probably the best way to use this book is to work on a problem until you feel you are not making progress and then go on to the next one. Think about the problem you couldn’t get as you do other things. The next chance you get, discuss the problem you are stymied on with other members of the class. Often you will all feel you’ve hit dead ends, but when you begin comparing notes and listening *carefully* to each other, you will see more than one approach to the problem and be able to make some progress. In fact, after comparing notes you may realize that there is more than one way to interpret the problem. In this case your first step should be to think together about what the problem is actually asking you to do. You may have learned in school that for every problem you are given, there is a method that has already been taught to you, and you are supposed to figure out which method applies and apply it. That is not the case here. Based on some simplified examples, you will discover the method for yourself. Later on, you may recognize a pattern that suggests you should try to use this method again.

The point of learning from this book is that you are learning how to

discover ideas and methods for yourself, not that you are learning to apply methods that someone else has told you about. The problems in this book are designed to lead you to discover for yourself and prove for yourself the main ideas of combinatorial mathematics. There is considerable evidence that this leads to deeper learning and more understanding.

You will see that some of the problems are marked with bullets. Those are the problems that I feel are essential to having an understanding of what comes later, whether or not it is marked by a bullet. The problems with bullets are the problems in which the main ideas of the book are developed. Many of the most interesting problems, in fact entire sections, are not marked in this way, because they use an important idea rather than developing one. Other problems are not marked with bullets because they are designed to provide motivation for the important concepts, motivation with which some students may already be familiar. If you are taking a course, your instructor will choose other problems (especially among the motivational ones) for you to work on based on the prerequisites for and goals of the course. If you are reading the book on your own, I recommend that you try all the problems in a section you want to cover. Make sure you can do the problems with bullets, but by all means don't restrict yourself to them. Sometimes a bulleted problem makes more sense if you have done some of the easier motivational problems that come before it. Problems that are motivational in nature or are introductory to the topic at hand are marked with a small circle. If, after you've tried it, you want to skip over a problem without a bullet or circle, you should not miss out on much by not doing that problem. Also, if you don't find the problems in a section with no bullets interesting, you can skip them, understanding that you may be skipping an entire branch of combinatorial mathematics! And no matter what, read the textual material that comes before, between, and immediately after problems you are working on!

You will also see that some problems are marked with arrows. These point to problems that I think are particularly interesting. Some of them are also difficult, but not all are. A few problems that summarize ideas that have come before but aren't really essential are marked with a plus, and problems that are essential if you want to cover the section they are in or, perhaps, the next section. If a problem is relevant to a much later section in an essential way, I've marked it with a dot and a parenthetical note that explains where it will be essential. Finally, problems that seem unusually hard to me are marked with an asterisk. Some I've marked as hard only because I think

they are difficult in light of what has come before, not because they are intrinsically difficult. In particular, some of the problems marked as hard will not seem so hard if you come back to them after you have finished more of the problems.

One of the downsides of how we learn math in high school is that many of us come to believe that if we can't solve a problem in ten or twenty minutes, then we can't solve it at all. There will be problems in this book that take hours of hard thought. Many of these problems were first conceived and solved by professional mathematicians, and *they* spent days or weeks on them. How can you be expected to solve them at all then? You have a context in which to work, and even though some of the problems are so open ended that you go into them without any idea of the answer, the context and the leading examples that preceded them give you a structure to work with. That doesn't mean you'll get them right away, but you will find a real sense of satisfaction when you see what you can figure out with concentrated thought. Besides, you can get hints!

Some of the questions will appear to be trick questions, especially when you get the answer. They are not intended as trick questions at all. Instead they are designed so that they don't tell you the answer in advance. For example the answer to a question that begins "How many..." might be "none." Or there might be just one example (or even no examples) for a problem that asks you to find all examples of something. So when you read a question, unless it directly tells you what the answer is and asks you to show it is true, don't expect the wording of the problem to suggest the answer. The book isn't designed this way to be cruel. Rather, there is evidence that the more open-ended a question is, the more deeply you learn from working on it. If you do go on to do mathematics later in life, the problems that come to you from the real world or from exploring a mathematical topic are going to be open-ended problems because nobody will have done them before. Thus working on open-ended problems now should help to prepare you to do mathematics later on.

You should try to write up answers to all the problems that you work on. If you claim something is true, you explain why it is true; that is you should prove it. In some cases an idea is introduced before you have the tools to prove it, or the proof of something will add nothing to your understanding. In such problems there is a remark telling you not to bother with a proof. When you write up a problem, remember that the instructor has to be able to "get" your ideas and understand exactly what you are saying. Your instructor is

going to choose some of your solutions to read carefully and give you detailed feedback on. When you get this feedback, you should think it over carefully and then write the solution again! You may be asked not to have someone else read your solutions to some of these problems until your instructor has. This is so that the instructor can offer help which is aimed at your needs. On other problems it is a good idea to seek feedback from other students. One of the best ways of learning to write clearly is to have someone who is as easily confused as you are point out to you where it is hard to figure out what you mean.

As you work on a problem, think about why you are doing what you are doing. Is it helping you? If your current approach doesn't feel right, try to see why. Is this a problem you can decompose into simpler problems? Can you see a way to make up a simple example, even a silly one, of what the problem is asking you to do? If a problem is asking you to do something for every value of an integer n , then what happens with simple values of n like 0, 1, and 2? Don't worry about making mistakes; it is often finding mistakes that leads mathematicians to their best insights. Above all, don't worry if you can't do a problem. Some problems are given as soon as there is one technique you've learned that might help do that problem. Later on there may be other techniques that you can bring back to that problem to try again. The notes have been designed this way on purpose. If you happen to get a hard problem with the bare minimum of tools, you will have accomplished much. As you go along, you will see your ideas appearing again later in other problems. On the other hand, if you don't get the problem the first time through, it will be nagging at you as you work on other things, and when you see the idea for an old problem in new work, you will know you are learning.

There are quite a few concepts that are developed in this book. Since most of the intellectual content is in the problems, it is natural that definitions of concepts will often be within problems. When you come across an unfamiliar term in a problem, it is likely it was defined earlier. Look it up in the index, and with luck (hopefully no luck will really be needed!) you will be able to find the definition.

Above all, this book is dedicated to the principle that doing mathematics is fun. As long as you know that some of the problems are going to require more than one attempt before you hit on the main idea, you can relax and enjoy your successes, knowing that as you work more and more problems and share more and more ideas, problems that seemed intractable at first become

a source of satisfaction later on.

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Chapter 1

What is Combinatorics?

Combinatorial mathematics arises from studying how we can *combine* objects into arrangements. For example, we might be combining sports teams into a tournament, samples of tires into plans to mount them on cars for testing, students into classes to compare approaches to teaching a subject, or members of a tennis club into pairs to play tennis. There are many questions one can ask about such arrangements of objects. Here we will focus on questions about *how many ways* we may combine the objects into arrangements of the desired type. Sometimes, though, we ask if an arrangement is possible (if we have ten baseball teams, and each team has to play each other team once, can we schedule all the games if we only have the fields available at enough times for forty games?). Sometimes we will ask if all the arrangements we might be able to make have a certain desirable property (Do all ways of testing 5 brands of tires on 5 different cars [with certain additional properties] compare each brand with each other brand on at least one common car?). Problems of these sorts come up throughout physics, biology, computer science, statistics, and many other subjects. However, to demonstrate all these relationships, we would have to take detours into all these subjects. While we will give some important applications, we will usually phrase our discussions around everyday experience and mathematical experience so that the student does not have to learn a new context before learning mathematics in context!

1.1 About These Notes

These notes are based on the philosophy that you learn the most about a subject when you are figuring it out directly for yourself, and learn the least when you are trying to figure out what someone else is saying about it. On the other hand, there is a subject called combinatorial mathematics, and that is what we are going to be studying, so we will have to tell you some basic facts. What we are going to try to do is to give you a chance to discover many of the interesting examples that usually appear as textbook examples and discover the principles that appear as textbook theorems. Your main activity will be solving problems designed to lead you to discover the basic principles of combinatorial mathematics. Some of the problems lead you through a new idea, some give you a chance to describe what you have learned in a sequence of problems, and some are quite challenging. When you find a problem challenging, don't give up on it, but don't let it stop you from going on with other problems. Frequently you will find an idea in a later problem that you can take back to the one you skipped over or only partly finished in order to finish it off. With that in mind, let's get started.

1.2 Basic Counting Principles

- 1. Five schools are going to send their baseball teams to a tournament, in which each team must play each other team exactly once. How many games are required?
- 2. Now some number n of schools are going to send their baseball teams to a tournament, and each team must play each other team exactly once. Let us think of the teams as numbered 1 through n .
 - (a) How many games does team 1 have to play in?
 - (b) How many games, other than the one with team 1, does team two have to play in?
 - (c) How many games, other than those with the first $i - 1$ teams, does team i have to play in?
 - (d) In terms of your answers to the previous parts of this problem, what is the total number of games that must be played?

- 3. One of the schools sending its team to the tournament has to send its players from some distance, and so it is making sandwiches for team members to eat along the way. There are three choices for the kind of bread and five choices for the kind of filling. How many different kinds of sandwiches are available?
- +4. An *ordered pair* (a, b) consists of two things we call a and b . We say a is the first member of the pair and b is the second member of the pair. If M is an m element set and N is an n -element set, how many ordered pairs are there whose first member is in M and whose second member is in N ? Does this problem have anything to do with any of the previous problems?
- 5. Since a sandwich by itself is pretty boring, students from the school in Problem 3 are offered a choice of a drink (from among five different kinds), a sandwich, and a fruit (from among four different kinds). In how many ways may a student make a choice of the three items now?
- 6. The coach of the team in Problem 3 knows of an ice cream parlor along the way where she plans to stop to buy each team member a triple decker cone. There are 12 different flavors of ice cream, and triple decker cones are made in homemade waffle cones. Having chocolate ice cream as the bottom scoop is different from having chocolate ice cream as the top scoop. How many possible ice cream cones are going to be available to the team members? How many cones with three different kinds of ice cream will be available?
- 7. The idea of a function is ubiquitous in mathematics. A function f from a set S to a set T is a relationship between the two sets that associates exactly one member $f(x)$ of T with each element x in S . We will come back to the ideas of functions and relationships in more detail and from different points of view from time to time. However, the quick review above should probably let you answer these questions. If you have difficulty with them, it would be a good idea to go now to Appendix ?? and work through Section ?? which covers this definition in more detail.
 - (a) Using f, g, \dots , to stand for the various functions, write down all the different functions you can from the set $\{1, 2\}$ to the set $\{a, b\}$.