Combinatorics Through Guided Discovery[1]

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Contents

Preface

This book is an introduction to combinatorial mathematics, also known as combinatorics. The book focuses especially but not exclusively on the part of combinatorics that mathematicians refer to as "counting." The book consists almost entirely of problems. Some of the problems are designed to lead you to think about a concept, others are designed to help you figure out a concept and state a theorem about it, while still others ask you to prove the theorem. Other problems give you a chance to use a theorem you have proved. From time to time there is a discussion that pulls together some of the things you have learned or introduces a new idea for you to work with. Many of the problems are designed to build up your intuition for how combinatorial mathematics works. There are problems that some people will solve quickly, and there are problems that will take days of thought for everyone. Probably the best way to use this book is to work on a problem until you feel you are not making progress and then go on to the next one. Think about the problem you couldn't get as you do other things. The next chance you get, discuss the problem you are stymied on with other members of the class. Often you will all feel you've hit dead ends, but when you begin comparing notes and listening *carefully* to each other, you will see more than one approach to the problem and be able to make some progress. In fact, after comparing notes you may realize that there is more than one way to interpret the problem. In this case your first step should be to think together about what the problem is actually asking you to do. You may have learned in school that for every problem you are given, there is a method that has already been taught to you, and you are supposed to figure out which method applies and apply it. That is not the case here. Based on some simplified examples, you will discover the method for yourself. Later on, you may recognize a pattern that suggests you should try to use this method again.

The point of learning from this book is that you are learning how to discover ideas and methods for yourself, not that you are learning to apply

Table 1: The meaning of the symbols to the left of problem numbers.

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$\cdot$	essential for this or the next section

methods that someone else has told you about. The problems in this book are designed to lead you to discover for yourself and prove for yourself the main ideas of combinatorial mathematics. There is considerable evidence that this leads to deeper learning and more understanding.

You will see that some of the problems are marked with bullets. Those are the problems that I feel are essential to having an understanding of what comes later, whether or not it is marked by a bullet. The problems with bullets are the problems in which the main ideas of the book are developed. Your instructor may leave out some of these problems because he or she plans not to cover future problems that rely on them. Many problems, in fact entire sections, are not marked in this way, because they use an important idea rather than developing one. Some other special symbols are described in what follows; a summary appears in Table 1.

Some problems are marked with open circles. This indicates that they are designed to provide motivation for, or an introduction to, the important concepts, motivation with which some students may already be familiar. You will also see that some problems are marked with arrows. These point to problems that I think are particularly interesting. Some of them are also difficult, but not all are. A few problems that summarize ideas that have come before but aren't really essential are marked with a plus, and problems that are essential if you want to cover the section they are in or, perhaps, the next section, are marked with a dot (a small bullet). If a problem is relevant to a much later section in an essential way, I've marked it with a dot and a parenthetical note that explains where it will be essential. Finally, problems that seem unusually hard to me are marked with an asterisk. Some I've marked as hard only because I think they are difficult in light of what has come before, not because they are intrinsically difficult. In particular, some of the problems marked as hard will not seem so hard if you come back to them after you have finished more of the problems.

If you are taking a course, your instructor will choose problems for you to work on based on the prerequisites for and goals of the course. If you are reading the book on your own, I recommend that you try all the problems in a section you want to cover. Try to do the problems with bullets, but by all means don't restrict yourself to them. Often a bulleted problem makes more sense if you have done some of the easier motivational problems that come before it. If, after you've tried it, you want to skip over a problem without a bullet or circle, you should not miss out on much by not doing that problem. Also, if you don't find the problems in a section with no bullets interesting, you can skip them, understanding that you may be skipping an entire branch of combinatorial mathematics! And no matter what, read the textual material that comes before, between, and immediately after problems you are working on!

One of the downsides of how we learn math in high school is that many of us come to believe that if we can't solve a problem in ten or twenty minutes, then we can't solve it at all. There will be problems in this book that take hours of hard thought. Many of these problems were first conceived and solved by professional mathematicians, and *they* spent days or weeks on them. How can you be expected to solve them at all then? You have a context in which to work, and even though some of the problems are so open ended that you go into them without any idea of the answer, the context and the leading examples that precede them give you a structure to work with. That doesn't mean you'll get them right away, but you will find a real sense of satisfaction when you see what you can figure out with concentrated thought. Besides, you can get hints!

Some of the questions will appear to be trick questions, especially when you get the answer. They are not intended as trick questions at all. Instead they are designed so that they don't tell you the answer in advance. For example the answer to a question that begins "How many..." might be "none." Or there might be just one example (or even no examples) for a problem that asks you to find all examples of something. So when you read a question, unless it directly tells you what the answer is and asks you to show it is true, don't expect the wording of the problem to suggest the answer. The book isn't designed this way to be cruel. Rather, there is evidence that the more open-ended a question is, the more deeply you learn from working on it. If you do go on to do mathematics later in life, the problems that come to you from the real world or from exploring a mathematical topic are going to be open-ended problems because nobody will have done them before. Thus working on open-ended problems now should help to prepare you to do mathematics and apply mathematics in other areas later on.

You should try to write up answers to all the problems that you work on. If you claim something is true, you should explain why it is true; that is you should prove it. In some cases an idea is introduced before you have the tools to prove it, or the proof of something will add nothing to your understanding. In such problems there is a remark telling you not to bother with a proof. When you write up a problem, remember that the instructor has to be able to "get" your ideas and understand exactly what you are saying. Your instructor is going to choose some of your solutions to read carefully and give you detailed feedback on. When you get this feedback, you should think it over carefully and then write the solution again! You may be asked not to have someone else read your solutions to some of these problems until your instructor has. This is so that the instructor can offer help which is aimed at your needs. On other problems it is a good idea to seek feedback from other students. One of the best ways of learning to write clearly is to have someone point out to you where it is hard to figure out what you mean. The crucial thing is to make it clear to your reader that you really want to know where you may have left something out, made an unclear statement, or failed to support a statement with a proof. It is often very helpful to choose people who have not yet become an expert with the problems, as long as they realize it will help you most for them to tell you about places in your solutions they do not understand, even if they think it is their problem and not yours!

As you work on a problem, think about why you are doing what you are doing. Is it helping you? If your current approach doesn't feel right, try to see why. Is this a problem you can decompose into simpler problems? Can you see a way to make up a simple example, even a silly one, of what the problem is asking you to do? If a problem is asking you to do something for every value of an integer n, then what happens with simple values of n like 0, 1, and 2? Don't worry about making mistakes; it is often finding mistakes that leads mathematicians to their best insights. Above all, don't worry if you can't do a problem. Some problems are given as soon as there is one technique you've learned that might help do that problem. Later on there may be other techniques that you can bring back to that problem to try again. The notes have been designed this way on purpose. If you happen to get a hard problem with the bare minimum of tools, you will have accomplished much. As you go along, you will see your ideas appearing again later in other problems. On the other hand, if you don't get the problem the first time through, it will be nagging at you as you work on other things, and when you see the idea for an old problem in new work, you will know you are learning.

There are quite a few concepts that are developed in this book. Since most of the intellectual content is in the problems, it is natural that definitions of concepts will often be within problems. When you come across an unfamiliar term in a problem, it is likely it was defined earlier. Look it up in the index, and with luck (hopefully no luck will really be needed!) you will be able to find the definition.

Above all, this book is dedicated to the principle that doing mathematics is fun. As long as you know that some of the problems are going to require more than one attempt before you hit on the main idea, you can relax and enjoy your successes, knowing that as you work more and more problems and share more and more ideas, problems that seemed intractable at first become a source of satisfaction later on.

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Chapter 1

What is Combinatorics?

Combinatorial mathematics arises from studying how we can *combine* objects into arrangements. For example, we might be combining sports teams into a tournament, samples of tires into plans to mount them on cars for testing, students into classes to compare approaches to teaching a subject, or members of a tennis club into pairs to play tennis. There are many questions one can ask about such arrangements of objects. Here we will focus on questions about *how many ways* we may combine the objects into arrangements of the desired type. These are called *counting problems*. Sometimes, though, combinatorial mathematicians ask if an arrangement is possible (if we have ten baseball teams, and each team has to play each other team once, can we schedule all the games if we only have the fields available at enough times for forty games?). Sometimes they ask if all the arrangements we might be able to make have a certain desirable property (Do all ways of testing 5 brands of tires on 5 different cars [with certain additional properties] compare each brand with each other brand on at least one common car?). Counting problems (and problems of the other sorts described) come up throughout physics, biology, computer science, statistics, and many other subjects. However, to demonstrate all these relationships, we would have to take detours into all these subjects. While we will give some important applications, we will usually phrase our discussions around everyday experience and mathematical experience so that the student does not have to learn a new context before learning mathematics in context!

1.1 About These Notes

These notes are based on the philosophy that you learn the most about a subject when you are figuring it out directly for yourself, and learn the least when you are trying to figure out what someone else is saying about it. On the other hand, there is a subject called combinatorial mathematics, and that is what we are going to be studying, so we will have to tell you some basic facts. What we are going to try to do is to give you a chance to discover many of the interesting examples that usually appear as textbook examples and discover the principles that appear as textbook theorems. Your main activity will be solving problems designed to lead you to discover the basic principles of combinatorial mathematics. Some of the problems lead you through a new idea, some give you a chance to describe what you have learned in a sequence of problems, and some are quite challenging. When you find a problem challenging, don't give up on it, but don't let it stop you from going on with other problems. Frequently you will find an idea in a later problem that you can take back to the one you skipped over or only partly finished in order to finish it off. With that in mind, let's get started. In the problems that follow, you will see some problems marked on the left with various symbols. The preface gives a full explanation of these symbols and discusses in greater detail why the book is organized as it is! Table 1.1, which is repeated from the preface, summarizes the meaning of the symbols.

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1.2 Basic Counting Principles

$\circ$1. Five schools are going to send their baseball teams to a tournament, in which each team must play each other team exactly once. How many games are required?

Solution: Think of numbering the five teams. The first team must play all four others, the second team will be in one of these games but must play in three more games, with the third, fourth and fifth team. The third team is in two of the games we've already mentioned, and must still play the fourth and fifth team for two more games, and the fourth team must play the fifth team in addition to playing in three of the games already mentioned. Thus there are $4 + 3 + 2 + 1 = 10$ games. Alternatively, there are five teams, each of which must play in four games, giving us 20 pairings of two teams each. However, each game involves two of these pairings, so there are $20/2 = 10$ games. ▮

•2. Now some number n of schools are going to send their baseball teams to a tournament, and each team must play each other team exactly once. Let us think of the teams as numbered 1 through n.

(a) How many games does team 1 have to play in?

Solution: $n - 1$ ▮

(b) How many games, other than the one with team 1, does team two have to play in?

Solution: $n - 2$ ▮

(c) How many games, other than those with the first $i - 1$ teams, does team i have to play in?

Solution: $n - i$ ▮

(d) In terms of your answers to the previous parts of this problem, what is the total number of games that must be played?

Solution: $1 + 2 + \cdots + n - 1$. Although this need not be part of the answer, a formula that we usually use in both algebra and calculus courses tells us this sum is $n(n - 1)/2$. ▮

•3. One of the schools sending its team to the tournament has to send its players from some distance, and so it is making sandwiches for team members to eat along the way. There are three choices for the kind of bread and five choices for the kind of filling. How many different kinds of sandwiches are available?

Solution: $3 \cdot 5 = 15$, or $5 + 5 + 5 = 15$. ▮

+4. An *ordered pair* (a, b) consists of two things we call a and b. We say a is the first member of the pair and b is the second member of the pair. If M is an m-element set and N is an n-element set, how many ordered pairs are there whose first member is in M and whose second member is in N? Does this problem have anything to do with any of the previous problems?

Solution: $m \cdot n$. This is because if $M = \{x_1, x_2, \ldots, x_m\}$, then we have n ordered pairs starting with x_1, n ordered pairs starting with x_2, and so on, so the total number of ordered pairs is a sum of m terms, all equal to n. In problem 3 we were looking at ordered pairs of bread and filling. Less directly, and so not required for the answer, in Problem 1 we have 20 ordered pairs, and each baseball game involved two of the ordered pairs so we had 10 baseball games. The same argument applies to Problem 2; namely we have n teams each of which is in an ordered pair with $n-1$ other teams, so we have $n(n-1)$ ordered pairs, and each game corresponds to two ordered pairs so we have $n(n-1)/2$ games. This gives us one proof of the formula we mentioned in the solution to that problem. ▮

∘5. Since a sandwich by itself is pretty boring, students from the school in Problem 3 are offered a choice of a drink (from among five different kinds), a sandwich, and a fruit (from among four different kinds). In how many ways may a student make a choice of the three items now?

Solution: $5 \cdot 15 \cdot 4 = 300$. Why do we multiply? Multiplying five by 15 is equivalent to adding 15, the number of sandwiches, once for each drink, giving us 75 combinations of drink and sandwich. For each such pair we have 4 choices of fruit, and we can either think of adding 75 fours or adding four 75s to get three hundred. Thus we multiply because multiplication is repeated addition. ▮

•6. The coach of the team in Problem 3 knows of an ice cream parlor along the way where she plans to stop to buy each team member a triple decker cone. There are 12 different flavors of ice cream, and triple decker cones are made in homemade waffle cones. Having chocolate ice cream as the bottom scoop is different from having chocolate ice cream as the top scoop. How many possible ice cream cones are going to be available to the team members? How many cones with three different kinds of ice cream will be available?

Solution: $12 \cdot 12 \cdot 12 = 1728$ possible cones. If the flavors must be different, we have $12 \cdot 11 \cdot 10 = 1320$ possible cones. In both cases, the reason we are multiplying is as a shortcut for repeated addition. ▮

•7. The idea of a function is ubiquitous in mathematics. A function f from a set S to a set T is a relationship between the two sets that associates exactly one member $f(x)$ of T with each element x in S. We will come back to the ideas of functions and relationships in more detail and from different points of view from time to time. However, the quick review above should probably let you answer these questions. If you have difficulty with them, it would be a good idea to go now to Appendix A and work through Section A.1.1 which covers this definition in more detail. You might also want to study Section A.1.3 to learn to visualize the properties of functions. We will take up the topic of this section later in this chapter as well, but in less detail than is in the appendix.

(a) Using f, g, $\ldots$, to stand for the various functions, write down all the different functions you can from the set $\{1,2\}$ to the set $\{a,b\}$. For example, you might start with the function f given by $f(1)=a$, $f(2)=b$. How many functions are there from the set $\{1,2\}$ to the set $\{a,b\}$?

Solution: $f(1)=a$, $f(2)=b$. Or, $h(1)=a$, $h(2)=a$. Or, $g(1)=b$, $g(2)=a$. Or $j(1)=b$, $j(2)=b$. We have exhausted all the possibilities for functions that associate a with 1 and all possibilities for functions that associate b with 1, so we have exhausted all possibilities. There are four such functions. ▮

(b) How many functions are there from the three element set $\{1,2,3\}$ to the two element set $\{a,b\}$?

Solution: $2\cdot 2\cdot 2=8$ ▮

(c) How many functions are there from the two element set $\{a,b\}$ to the three element set $\{1,2,3\}$?

Solution: $3\cdot 3=9$ ▮

(d) How many functions are there from a three element set to a 12 element set?

Solution: $12\cdot 12\cdot 12=1728$ ▮

(e) A function f is called **one-to-one** or an *injection* if whenever x is different from y, $f(x)$ is different from $f(y)$. How many one-to-one functions are there from a three element set to a 12 element set?

Solution: $12\cdot 11\cdot 10=1320$ ▮

(f) Explain the relationship between this problem and Problem 6.

Solution: When we counted the number of possible ice cream cones we were counting functions from the three places in the cone where ice cream would sit to the 12 flavors. When we counted the number of possible ice cream cones with different flavors, we were counting the number of one-to-one functions from the three places in the cone where ice cream would sit to the 12 flavors. ▮

•8. A group of hungry team members in Problem 6 notices it would be cheaper to buy three pints of ice cream for them to split than to buy a triple decker cone for each of them, and that way they would get more ice cream. They ask their coach if they can buy three pints of ice cream.

(a) In how many ways can they choose three pints of different flavors out of the 12 flavors?

Solution: There are $12 \cdot 11 \cdot 10 = 1320$ ways to make a list of three flavors. But a choice of three flavors accounts for $3 \cdot 2 \cdot 1 = 6$ of those lists. Therefore there are $1320/6 = 220$ ways to choose the pints if the flavors are different. We will discuss the idea behind this solution technique in great detail shortly. ▮

(b) In how many ways may they choose three pints if the flavors don't have to be different?

Solution: If the flavors need not be different, we must add in the number of ways to choose two pints of one flavor and one of a second and also the number of ways to choose three pints of one flavor. The first of these is $12 \cdot 11 = 132$ and the second is 12, so we have $220 + 132 + 12 = 364$ ways to choose three pints. We can do a more elegant solution after we learn about multisets in Problem 125. ▮

•9. Two sets are said to be *disjoint* if they have no elements in common. For example, $\{1, 3, 12\}$ and $\{6, 4, 8, 2\}$ are disjoint, but $\{1, 3, 12\}$ and $\{3, 5, 7\}$ are not. Three or more sets are said to be *mutually disjoint* if no two of them have any elements in common. What can you say about the size of the union of a finite number of finite (mutually) disjoint sets? Does this have anything to do with any of the previous problems?

Solution: The size of a union of disjoint sets is the sum of their sizes. We used this principle in Problems 1 and 2 directly, and indirectly in every other problem when we multiplied the number of ways of doing one thing times the number of ways of doing another. Note that we used this principle informally in the explanation in the solution of Problem 4. ▮

•10. Disjoint subsets are defined in Problem 9. What can you say about the size of the union of m (mutually) disjoint sets, each of size n? Does this have anything to do with any of the previous problems?

Solution: The size of the union is $m \cdot n$. This is because the size of a union of disjoint sets is the sum of their sizes, and a sum of m terms each equal to n is $m \cdot n$. ▮

1.2.1 The sum and product principles

These problems contain among them the kernels of many of the fundamental ideas of combinatorics. For example, with luck, you just stated the sum principle (illustrated in Figure 1.1), and product principle (illustrated in Figure 1.2) in Problems 9 and 10. These are two of the most basic principles of combinatorics. These two counting principles are the basis on which we will develop many other counting principles.

Figure 1.1: The union of these two disjoint sets has size 17.

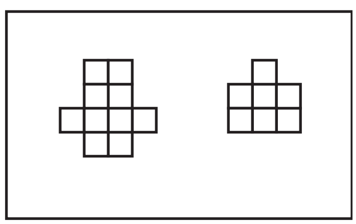
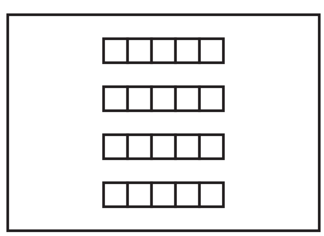


Figure 1.2: The union of four disjoint sets of size five.



You may have noticed some standard mathematical words and phrases such as *set*, *ordered pair*, *function* and so on creeping into the problems. One of our goals in these notes is to show how most counting problems can be recognized as counting all or some of the elements of a set of standard mathematical objects. For example, Problem 4 is meant to suggest that the question we asked in Problem 3 was really a problem of counting all the ordered pairs consisting of a bread choice and a filling choice. We use $A \times B$ to stand for the set of all ordered pairs whose first element is in A and whose second element is in B and we call $A \times B$ the *Cartesian product* of A and B. Thus you can think of Problem 4 as asking you for the size of the Cartesian product of M and N, that is, asking you to count the number of elements of this Cartesian product.

When a set S is a union of disjoint sets $B_1, B_2, \ldots, B_m$ we say that the sets $B_1, B_2, \ldots, B_m$ are a **partition** of the set S. Thus a partition of S is a (special kind of) set of sets. So that we don't find ourselves getting confused between the set S and the sets B_i into which we have divided it, we often call the sets $B_1, B_2, \ldots, B_m$ the *blocks* of the partition. In this language, the **sum principle** says that

> if we have a partition of a finite set S, then the size of S is the sum of the sizes of the blocks of the partition.

The **product principle** says that

> if we have a partition of a finite set S into m blocks, each of size n, then S has size mn.

You'll notice that in our formal statement of the sum and product principle we talked about a partition of a finite set. We could modify our language a bit to cover infinite sizes, but whenever we talk about sizes of sets in what follows, we will be working with finite sets. So as to avoid possible complications in the future, let us agree that when we refer to the size of a set, we are implicitly assuming the set is finite. There is another version of the product principle that applies directly in problems like Problem 5 and Problem 6, where we were not just taking a union of m disjoint sets of size n, but rather m disjoint sets of size n, each of which was a union of m' disjoint sets of size n'. This is an inconvenient way to have to think about a counting problem, so we may rephrase the product principle in terms of a sequence of decisions:

•11. If we make a sequence of m choices for which

- there are k_1 possible first choices, and
- for each way of making the first $i-1$ choices, there are k_i ways to make the ith choice,

then in how many ways may we make our sequence of choices? (You need not prove your answer correct at this time.)

Solution: There are $k_1 \cdot k_2 \cdot \cdots \cdot k_m = \prod_{i=1}^{m} k_i$ ways to make the sequence of choices. ▮

The counting principle you gave in Problem 11 is called the *general product principle*. We will outline a proof of the general product principle from the original product principle in Problem 80. Until then, let us simply accept it as another counting principle. For now, notice how much easier it makes it to explain why we multiplied the things we did in Problem 5 and Problem 6.

➔ 12. A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

(a) In how many ways may we pair up all the members of the club? (Hint: consider the cases of 2, 4, and 6 members.)

Solution: Suppose we list the people in the club in some way (perhaps in alphabetical order), and keep that list for the remainder of the problem. Take the first person from the list and pair that person with any of the $2n-1$ remaining people. Now take the next *unpaired* person from the list and pair that person with any of the remaining $2n-3$ unpaired people. Continuing in this way, once k pairs have been selected, take the next unpaired person from the list and pair that person with any of the remaining $2n-2k-1$ unpaired people. Every pairing can arise in this way, and no pairing can arise twice in this process. Thus the number of outcomes is $\prod_{i=0}^{n-1} 2n-2i-1$. This is the product of the odd numbers between 1 and $2n-1$. It is also the product of every second term of $(2n)!$. Notice that the other n terms of $(2n)!$ are even. In fact, if we double each term of $n!$, we would get the missing terms of $(2n)!$. Thus

$$(2n)! = 2^n n! \prod_{i=0}^{n-1} 2n-2i-1.$$

Therefore the number of tennis pairings is $\frac{(2n)!}{2^n n!}$. While this seems to be an algebraic trick here, in Problem 44 we will have tools to explain it combinatorially. ▮

(b) Suppose that in addition to specifying who plays whom, for each pairing we say who serves first. Now in how many ways may we specify our pairs?

Solution: Now in addition to our pairings, for each pairing we have two choices of who will serve first, so after setting up the pairings, we have 2^n ways to decide which member of each pair serves first. Thus we can do the pairings and choose first servers in $2^n \prod_{i=0}^{n-1} 2n - 2i - 1$ ways. By the analysis of $(2n)!$ in the previous part, it is also $\frac{(2n)!}{n!}$. ▮

+13. Let us now return to Problem 7 and justify—or perhaps finish—our answer to the question about the number of functions from a three-element set to a 12-element set.

(a) How can you justify your answer in Problem 7 to the question "How many functions are there from a three element set (say $[3] = \{1, 2, 3\}$) to a twelve element set (say $[12]$)? "

Solution: For a function f, we can decide on $f(1)$ in twelve ways, then, given the decision we make for $f(1)$, we have 12 ways to decide on $f(2)$, and given the decisions we have made for $f(1)$ and $f(2)$, we have 12 ways to decide on $f(3)$. Therefore by the general product principle, there are $12^3 = 1728$ functions from $[3]$ to $[12]$, or from any three element set to any twelve element set. ▮

(b) Based on the examples you've seen so far, make a conjecture about how many functions there are from the set

$$[m] = \{1, 2, 3, \ldots, m\}$$

to $[n] = \{1, 2, 3, \ldots, n\}$ and prove it.

Solution: n^m. We can think of choosing a function f as making a sequence of m decisions, namely deciding on $f(1), f(2), \ldots, f(m)$. We have n choices for $f(1)$. Given the choices we have made for $f(1)$ through $f(i-1)$, we have n choices for $f(i)$. Thus by the general product principle we have a product of m terms each equal to n, which is n^m, as the number of ways to choose f. ▮

(c) A common notation for the set of all functions from a set M to a set N is N^M. Why is this a good notation?

Solution: Because there are n^m such functions. ▮

+14. Now suppose we are thinking about a set S of functions f from $[m]$ to some set X. (For example, in Problem 6 we were thinking of the set of functions from the three possible places for scoops in an ice-cream cone to 12 flavors of ice cream.) Suppose there are k_1 choices for $f(1)$. (In Problem 6, k_1 was 12, because there were 12 ways to choose the first scoop.) Suppose that for each choice of $f(1)$ there are k_2 choices for $f(2)$. (For example, in Problem 6 k_2 was 12 if the second flavor could be the same as the first, but k_2 was 11 if the flavors had to be different.) In general, suppose that for each choice of $f(1)$, $f(2)$, $\ldots f(i-1)$, there are k_i choices for $f(i)$. (For example, in Problem 6, if the flavors have to be different, then for each choice of $f(1)$ and $f(2)$, there are 10 choices for $f(3)$.)

What we have assumed so far about the functions in S may be summarized as

- There are k_1 choices for $f(1)$.
- For each choice of $f(1)$, $f(2)$, $\ldots$, $f(i-1)$, there are k_i choices for $f(i)$.

How many functions are in the set S? Is there any practical difference between the result of this problem and the general product principle?

Solution: The number of functions in S is $\prod_{i=1}^{m} k_i$. No, there is no practical difference, because given a sequence of k possible decisions, we have a function from the set $[k]$ to the set of decisions, and the number of choices for $f(i)$ is the number of ways we can make the ith decision. Similarly, given a function f from the set $[k]$ to some set X, the number of choices for $f(i)$ is the number of possible results of deciding on the value of $f(i)$. ▮

The point of Problem 14 is that the general product principle can be stated informally, as we did originally, or as a statement about counting sets of standard concrete mathematical objects, namely functions.

➔15. A roller coaster car has n rows of seats, each of which has room for two people. If n men and n women get into the car with a man and a woman in each row, in how many ways may they choose their seats?

Solution: $(n!)^2 2^n$ ▮

+16. How does the general product principle apply to Problem 6?

Solution: By the general product principle, there are $12 \cdot 11 \cdot 10$ triple decker cones. ▮

17. In how many ways can we pass out k distinct pieces of fruit to n children (with no restriction on how many pieces of fruit a child may get)?

Solution: Either by the formula for the number of functions from an m-element set to an n-element set or the general product principle, there are k^n ways. (Each distribution is a function from the set of fruit to the set of children, because each piece of fruit goes to one and only one child.) ▮

•18. How many subsets does a set S with n elements have?

Solution: To choose a subset, we must decide, for each element of S, whether or not it is in the subset. Thus we have to make a sequence of n decisions, and each one of them has two possible outcomes (take the element, don't take the element), regardless of our previous decisions. Therefore by the general product principle, there are 2^n subsets of an n-element set. ▮

∘19. Assuming $k \leq n$, in how many ways can we pass out k distinct pieces of fruit to n children if each child may get at most one? What is the number if $k > n$? Assume for both questions that we pass out all the fruit.

Solution: There are n choices for the child to whom the first piece of fruit goes, then $n-1$ choices for the second, and, in general, $n-i+1$ choices for the ith piece of fruit. By the general product principle, this gives us $\prod_{i=1}^{k} n-i+1$ ways to pass out the fruit. The number of ways to pass out the fruit is zero if $k > n$, because the problem says each child has to get at most one piece of fruit, and that all the fruit must be passed out. This is impossible if $k > n$, so there are zero ways to pass out the fruit. It is a nice coincidence that our formula for the first question gives 0 if $k > n$. ▮

•20. Another name for a list, in a specific order, of k distinct things chosen from a set S is a **k-element permutation of S**. We can also think of a k-element permutation of S as a one-to-one function (or, in other words, injection) from $[k] = \{1, 2, \ldots, k\}$ to S. How many k-element permutations does an n-element set have? (For this problem it is natural to assume $k \leq n$. However, the question makes sense even if $k > n$.) What is the number of k-element permutations of an n-element set if $k > n$?

Solution: By the general product principle, the number is

$$\prod_{i=1}^{k} n - i + 1.$$

In the case that $k > n$, there are no such lists with distinct entries, and that is what the formula gives us, because $n - (n+1) + 1 = 0$. ∎

There are a variety of different notations for the number of k-element permutations of an n-element set. The one we shall use was introduced by Don Knuth; namely $n^{\underline{k}}$, read "n to the k falling" or "n to the k down." In Problem 20 you may have shown that

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1) = \prod_{i=1}^{k}(n-i+1). \tag{1.1}$$

It is standard to call $n^{\underline{k}}$ the **k-th falling factorial power of** n, which explains why we use exponential notation. We call it a *factorial* power since $n^{\underline{n}} = n(n-1)\cdots 1$, which we call *$n$-factorial* and denote by $n!$. If you are unfamiliar with the Pi notation, or *product notation* we introduced for products in Equation 1.1, it works just like the Sigma notation works for summations.

•21. Express $n^{\underline{k}}$ as a quotient of factorials.

Solution: $n^{\underline{k}} = n!/(n-k)!$ ∎

➔22. How should we define $n^{\underline{0}}$?

Solution: Based on the the quotient of factorials formula, we would expect to define $n^{\underline{0}} = 1$. This says there should be one one-to-one function from the empty set into an n-element set. Those who are familiar with the ordered pairs definition of relations and functions may recognize that the empty set of ordered pairs whose first element is in $\emptyset$ and whose second element is in $[n]$ is a function, and it satisfies the rule defining one-to-one functions, because there are not two elements x and y in $\emptyset$ such that $f(x) = f(y)$. ∎

1.2.2 Functions and directed graphs

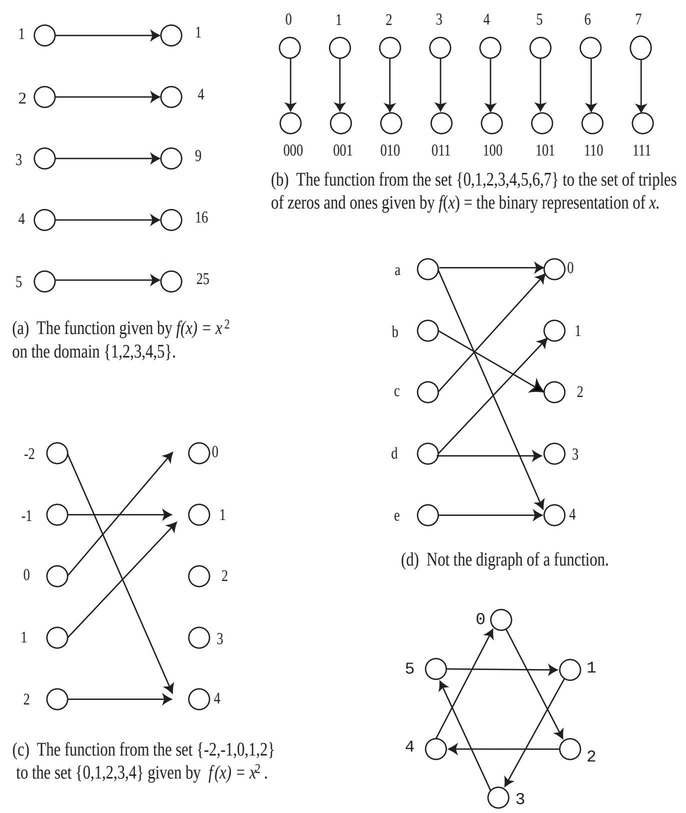
As another example of how standard mathematical language relates to counting problems, Problem 7 explicitly asked you to relate the idea of counting functions to the question of Problem 6. You have probably learned in algebra or calculus how to draw graphs in the cartesian plane of functions from a set of numbers to a set of numbers. You may recall how we can determine whether a graph is a graph of a function by examining whether each vertical straight line crosses the graph at most one time. You might also recall how we can determine whether such a function is one-to-one by examining whether each horizontal straight line crosses the graph at most one time. The functions we deal with will often involve objects which are not numbers, and will often be functions from one finite set to another. Thus graphs in the cartesian plane will often not be available to us for visualizing functions.

However, there is another kind of graph called a *directed graph* or *digraph* that is especially useful when dealing with functions between finite sets. We take up this topic in more detail in Appendix A, particularly Section A.1.2 and Section A.1.3. In Figure 1.3 we show several examples of digraphs of functions. If we have a function f from a set S to a set T, we draw a line of dots or circles, called *vertices* to represent the elements of S and another (usually parallel) line of vertices to represent the elements of T. We then draw an arrow from the vertex for x to the vertex for y if $f(x) = y$. Sometimes, as in part (e) of the figure, if we have a function from a set S to itself, we draw only one set of vertices representing the elements of S, in which case we can have arrows both entering and leaving a given vertex. As you see, the digraph can be more enlightening in this case if we experiment with the function to find a nice placement of the vertices rather than putting them in a row.

Notice that there is a simple test for whether a digraph whose vertices represent the elements of the sets S and T is the digraph of a function from S to T. There must be one and only one arrow leaving each vertex of the digraph representing an element of S. The fact that there is one arrow means that $f(x)$ is defined for each x in S. The fact that there is only one arrow means that each x in S is related to exactly one element of T. (Note that these remarks hold as well if we have a function from S to S and draw only one set of vertices representing the elements of S.) For further discussion of functions and digraphs see Sections A.1.1 and A.1.2 of Appendix A.

∘23. Draw the digraph of the function from the set {Alice, Bob, Dawn,

Figure 1.3: What is a digraph of a function?



(a) The function given by $f(x) = x^2$ on the domain {1,2,3,4,5}.

(b) The function from the set {0,1,2,3,4,5,6,7} to the set of triples of zeros and ones given by $f(x)$ = the binary representation of x.

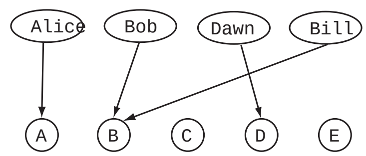
(c) The function from the set {-2,-1,0,1,2} to the set {0,1,2,3,4} given by $f(x) = x^2$.

(d) Not the digraph of a function.

(e) The function from {0, 1, 2, 3, 4, 5} to {0, 1, 2, 3, 4, 5} given by $f(x) = x + 2 \bmod 6$

Bill} to the set {A, B, C, D, E} given by

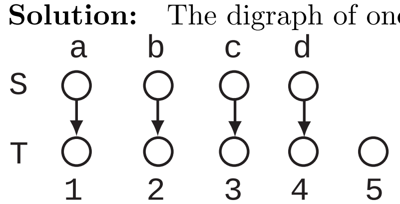
$$f(X) = \text{the first letter of the name } X.$$

Solution: 

∎

•24. A function $f : S \to T$ is called an *onto function* or *surjection* if each element of T is $f(x)$ for some $x \in S$. Choose a set S and a set T so that you can draw the digraph of a function from S to T that is one-to-one but not onto, and draw the digraph of such a function.

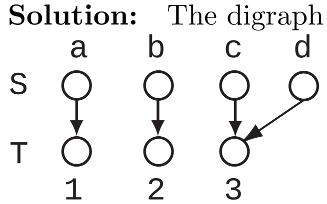
Solution: The digraph of one such function follows.



∎

○25. Choose a set S and a set T so that you can draw the digraph of a function from S to T that is onto but not one-to-one, and draw the digraph of such a function.

Solution: The digraph of one such function follows.



∎

•26. Digraphs of functions help us visualize the ideas of one-to-one functions and onto functions.

(a) What does the digraph of a one-to-one function (injection) from a finite set X to a finite set Y look like? (Look for a test somewhat similar to the one we described for when a digraph is the digraph of a function.)

Solution: A function from X to Y is one-to-one if, in its digraph, at most one arrow goes into each vertex representing a member of Y. (For a digraph to be the digraph of a function from X to Y, one and only one arrow must come out of each vertex representing a member of X.) ▮

(b) What does the digraph of an onto function look like?

Solution: A function is onto if, in its digraph, at least one arrow goes into each vertex representing a member of Y. (For a digraph to be the digraph of a function from X to Y, one and only one arrow must come out of each vertex representing a member of X.) ▮

(c) What does the digraph of a one-to-one and onto function from a finite set S to a set T look like?

Solution: A function from X to Y is one-to one and onto if, in its digraph, exactly one arrow goes into each vertex representing a member of Y. (For a digraph to be the digraph of a function from X to Y, one and only one arrow must come out of each vertex representing a member of X.) ▮

•27. The word *permutation* is actually used in two different ways in mathematics. A **permutation of a set** S is a one-to-one function from S onto S. How many permutations does an n-element set have?

Solution: $n!$, by the general product principle. ▮

Notice that there is a great deal of consistency between the use of the word permutation in Problem 27 and the use in the Problem 20. If we have some way $a_1, a_2, \ldots, a_n$ of listing our set S, then any other list $b_1, b_2, \ldots, b_n$ gives us the permutation of S whose rule is $f(a_i) = b_i$, and any permutation of S, say the one given by $g(a_i) = c_i$ gives us a list $c_1, c_2, \ldots, c_n$ of S. Thus there is really very little difference between the idea of a permutation of S and an n-element permutation of S when n is the size of S.

1.2.3 The bijection principle

Another name for a one-to-one and onto function is **bijection**. The digraphs marked (a), (b), and (e) in Figure 1.3 are digraphs of bijections. The description in Problem 26c of the digraph of a bijection from X to Y illustrates one of the fundamental principles of combinatorial mathematics, the **bijection principle**:

> Two sets have the same size if and only if there is a bijection between them.

It is surprising how this innocent sounding principle guides us into finding insight into some otherwise very complicated proofs.

1.2.4 Counting subsets of a set

28. The *binary* representation of a number m is a list, or string, $a_1a_2\ldots a_k$ of zeros and ones such that $m = a_1 2^{k-1} + a_2 2^{k-2} + \cdots + a_k 2^0$. Describe a bijection between the binary representations of the integers between 0 and $2^n - 1$ and the subsets of an n-element set. What does this tell you about the number of subsets of the n-element set $[n]$?

    Solution: The sequence $a_1a_2\ldots a_k$ corresponds to the set of i such that $a_i = 1$. This is a bijection because each sequence gives a subset of $[n]$, and each subset of $[n]$ is the set of places where exactly one sequence has its ones. Since there are 2^n integers which are between 0 and $2^n - 1$, and they correspond to sequences of length n (notice, we have another bijection, the one between a number and its binary representation), there are 2^n subsets of an n-element set. ▮

Notice that the first question in Problem 8 asked you for the number of ways to choose a three element subset from a 12 element subset. You may have seen a notation like $\binom{n}{k}$, $C(n,k)$, or ${}_nC_k$ which stands for the number of ways to choose a k-element subset from an n-element set. The number $\binom{n}{k}$ is read as "n choose k" and is called a **binomial coefficient** for reasons we will see later on. Another frequently used way to read the binomial coefficient notation is "the number of combinations of n things taken k at a time." We won't use this way of reading the notation. You are going to be asked to construct two bijections that relate to these numbers and figure out what famous formula they prove. We are going to think about subsets of the n-element set $[n] = \{1, 2, 3, \ldots, n\}$. As an example, the set of two-element subsets of $[4]$ is

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

This example tells us that $\binom{4}{2} = 6$.

•29. Let C be the set of k-element subsets of $[n]$ that contain the number n, and let D be the set of k-element subsets of $[n]$ that don't contain n.

    (a) Let C' be the set of $(k-1)$-element subsets of $[n-1]$. Describe a bijection from C to C'. (A verbal description is fine.)

    Solution: Let $f(X) = X - \{n\}$, the set X with n removed. This is a bijection because two different sets containing n must yield different sets when n is removed (one-to-one), and each $(k-1)$-element subset X of $[n-1]$ may be obtained from the k-element subset $X \cup \{n\}$ of $[n]$ by removing n (onto). ▮

(b) Let D' be the set of k-element subsets of $[n-1] = \{1, 2, \ldots n-1\}$. Describe a bijection from D to D'. (A verbal description is fine.)

Solution: Simply let $f(X) = X$. This is one-to-one by definition, and onto because the subsets of $[n-1]$ are identical with the subsets of $[n]$ not containing n. ▮

(c) Based on the two previous parts, express the sizes of C and D in terms of binomial coefficients involving $n-1$ instead of n.

Solution: $|C| = \binom{n-1}{k-1}$; $|D| = \binom{n-1}{k}$ ▮

(d) Apply the sum principle to C and D and obtain a formula that expresses $\binom{n}{k}$ in terms of two binomial coefficients involving $n-1$. You have just derived the Pascal Equation that is the basis for the famous Pascal's Triangle.

Solution: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. ▮

1.2.5 Pascal's Triangle

The Pascal Equation that you derived in Problem 29 gives us the triangle in Figure 1.4. This figure has the number of k-element subsets of an n-element set as the kth number over in the nth row (we call the top row the zeroth row and the beginning entry of a row the zeroth number over). You'll see that your formula doesn't say anything about $\binom{n}{k}$ if $k = 0$ or $k = n$, but otherwise it says that each entry is the sum of the two that are above it and just to the left or right.

Figure 1.4: Pascal's Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

30. Just for practice, what is the next row of Pascal's triangle?

    Solution: 1,8,28,56,70,56,28,8,1 ▮

➔ 31. Without writing out the rows completely, write out enough of Pascal's triangle to get a numerical answer for the first question in Problem 8.

    Solution: Starting with row 9, we get

$$\begin{array}{cccccccccc}
 & & & 1 & & 9 & & 36 & & 84 \\
 & & 1 & & 10 & & 45 & & 120 & \\
 & 1 & & 11 & & 55 & & 165 & & \\
1 & & 12 & & 66 & & 220 & & &
\end{array}$$

    so the answer is 220. We actually didn't need the 1, 12, and 66 in the last row, or the 1 and 11 in the second last row, or the 1 in the third last row. ▮

It is less common to see Pascal's triangle as a right triangle, but it actually makes your formula easier to interpret. In Pascal's Right Triangle, the element in row n and column k (with the convention that the first row is row zero and the first column is column zero) is $\binom{n}{k}$. In this case your formula says each entry in a row is the sum of the one above and the one above and to the left, except for the leftmost and right most entries of a row, for which that doesn't make sense. Since the leftmost entry is $\binom{n}{0}$ and the rightmost entry is $\binom{n}{n}$, these entries are both one (to see why, ask yourself how many 0-element subsets and how many n-element subsets an n-element set has), and your formula then tells how to fill in the rest of the table.

Figure 1.5: Pascal's Right Triangle

	$k=0$	1	2	3	4	5	6	7
$n=0$	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Seeing this right triangle leads us to ask whether there is some natural way to extend the right triangle to a rectangle. If we did have a rectangular table of binomial coefficients, counting the first row as row zero (i.e., $n = 0$) and the first column as column zero (i.e., $k = 0$), the entries we don't yet have are values of $\binom{n}{k}$ for $k > n$. But how many k-element subsets does an n-element set have if $k > n$? The answer, of course, is zero, so all the other entries we would fill in would be zero, giving us the rectangular array in Figure 1.6. It is straightforward to check that Pascal's Equation now works for all the entries in the rectangle that have an entry above them and an entry above and to the left.

Figure 1.6: Pascal's Rectangle

	$k=0$	1	2	3	4	5	6	7
$n=0$	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0
3	1	3	3	1	0	0	0	0
4	1	4	6	4	1	0	0	0
5	1	5	10	10	5	1	0	0
6	1	6	15	20	15	6	1	0
7	1	7	21	35	35	21	7	1

➔ 32. Because our definition told us that $\binom{n}{k}$ is 0 when $k > n$, we got a rectangular table of numbers that satisfies the Pascal Equation.

(a) Is there any other way to define $\binom{n}{k}$ when $k > n$ in order to get a rectangular table that agrees with Pascal's Right Triangle for $k \le n$ and satisfies the Pascal Equation?

Solution: No, because there must be a zero directly above each one not in column zero. That is, there must be a zero in row zero and column 1, row 1 and column 2, and so forth. Then above each zero not in column zero or one, there must be yet another zero and so on. ▮

(b) Suppose we want to extend Pascal's Rectangle to the left and define $\binom{n}{-k}$ for $n \ge 0$ and $k > 0$ so that $-k < 0$. What should we put into row n and column $-k$ of Pascal's Rectangle in order for the Pascal Equation to hold true?

Solution: To the left of all the ones in column zero, we must have zeros for the Pascal Equation to hold. To the left of those zeros, we must again have zeros, and so on. ▮

$*$(c) What should we put into row $-n$ (assume n is positive) and column k or column $-k$ in order for the Pascal Equation to continue to hold? Do we have any freedom of choice?

Solution: Above row zero, we have some freedom. The -1,-1 and the (-1,0)-entry must add to one, so they can be $-x$ and $x+1$ for any number x. To the right of the -1,0 entry they must alternate between $(-x-1)$ and $x+1$ while to the left of the (-1,1)-entry they must alternate between $-x$ and x, ending with the $-x$ in position (-1,1). Now if we know the entry in row -2 and column zero, we can use the Pascal equation (in the form $\binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k}$) to compute all the entries to the left of it, and (in a different form) to compute all the entries to the right of it. Thus we may be arbitrary about the entries in column 0 (or, in fact, one entry in each row) and then the Pascal Equation tells us how to fill in the rest of each row. We shall see later on that there is one very natural choice for how to fill in all the rows above row zero. ∎

33. There is yet another bijection that lets us prove that a set of size n has 2^n subsets. Namely, for each subset S of $[n] = \{1, 2, \ldots, n\}$, define a function (traditionally denoted by χ_S) as follows.[1]

$$\chi_S(i) = \begin{cases} 1 \text{ if } i \in S \\ 0 \text{ if } i \notin S \end{cases}$$

The function χ_S is called the *characteristic function* of S. Notice that the characteristic function is a function from $[n]$ to $\{0, 1\}$.

(a) For practice, consider the function $\chi_{\{1,3\}}$ for the subset $\{1, 3\}$ of the set $\{1, 2, 3, 4\}$. What are

i. $\chi_{\{1,3\}}(1)$?
Solution: 1 ∎

ii. $\chi_{\{1,3\}}(2)$?
Solution: 0 ∎

iii. $\chi_{\{1,3\}}(3)$?
Solution: 1 ∎

iv. $\chi_{\{1,3\}}(4)$?
Solution: 0 ∎

[1]The symbol χ is the Greek letter chi that is pronounced Ki, with the *i* sounding like "eye."

(b) We define a function f from the set of subsets of $[n] = \{1, 2, \ldots, n\}$ to the set of functions from $[n]$ to $\{0, 1\}$ by $f(S) = \chi_S$. Explain why f is a bijection.

Solution: Suppose S and T are subsets of $[n]$. If $i \in S$ but $i \notin T$, then $\chi_S(i) = 1$ but $\chi_T(i) = 0$. Thus if $S \neq T$, then $\chi_S \neq \chi_T$. Therefore f is one-to-one. Given a function g from $[n]$ to $\{0, 1\}$, let $S = \{i | g(i) = 1\}$. Then by definition, $g = \chi_S = f(S)$. Therefore f is onto, so it is a bijection. ▮

(c) Why does the fact that f is a bijection prove that $[n]$ has 2^n subsets?

Solution: We have seen that there are 2^n functions from $[n]$ to the two-element set $\{0, 1\}$, and we have just described a bijection between the set of all such functions and the subsets of $[n]$. ▮

In Problems 18, 28, and 33 you gave three proofs of the following theorem.

Theorem 1 *The number of subsets of an n-element set is 2^n.*

The proofs in Problem 28 and 33 use essentially the same bijection, but they interpret sequences of zeros and ones differently, and so end up being different proofs. We will give yet another proof, using bijections similar to those we used in proving the Pascal Equation, at the beginning of Chapter 2.

1.2.6 The quotient principle

•34. As we noted in Problem 29, the first question in Problem 8 asked us for the number of three-element subsets of a twelve-element set. We were able to use the Pascal Equation to get a numerical answer to that question. Had we had twenty or thirty flavors of ice cream to choose from, using the Pascal Equation to get our answer would have entailed a good bit more work. We have seen how the general product principle gives us an answer to Problem 6. Thus we might think that the number of ways to choose a three element set from 12 elements is the number of ways to choose the first element times the number of ways to choose the second element times the number of ways to choose the third element, which is $12 \cdot 11 \cdot 10 = 1320$. However, our result in Problem 29 shows that this is wrong.

(a) What is it that is different between the number of ways to stack ice cream in a triple decker cone with three different flavors of ice cream and the number of ways to simply choose three different flavors of ice cream?

Solution: What is different is that the order in which we put the scoops into the cone matters, but for simply choosing three flavors, the order of the choices doesn't matter. ▮

(b) In particular, how many different triple decker cones use vanilla, chocolate, and strawberry? (Of course any three distinct flavors could substitute for vanilla, chocolate and strawberry without changing the answer.)

Solution: Six different triple decker cones have the same three flavors. ▮

(c) Using your answer from part 34b, compute the number of ways to choose three different flavors of ice cream (out of twelve flavors) from the number of ways to choose a triple decker cone with three different flavors (out of twelve flavors).

Solution: Since each choice of three flavors corresponds to six cones, we have $1320/6 = 220$ different ways to choose three flavors of ice cream from 12 flavors. ▮

•35. Based on what you observed in Problem 34c, how many k-element subsets does an n-element set have?

Solution: Following the reasoning of Problem 34, the number of k-element permutations of an n-element set is $n^{\underline{k}}$, and each k-element subset of $[n]$ is listed by $k!$ of these permutations, so the number of k-element subsets is $\frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$. ▮

➔36. The formula you proved in Problem 35 is symmetric in k and $n-k$; that is, it gives the same number for $\binom{n}{k}$ as it gives for $\binom{n}{n-k}$. Whenever two quantities are counted by the same formula it is good for our insight to find a bijection that demonstrates the two sets being counted have the same size. In fact this is a guiding principle of research in combinatorial mathematics. Find a bijection that proves that $\binom{n}{k}$ equals $\binom{n}{n-k}$.

Solution: For each k-element subset K of the n-element set N, define $f(K)$ to be the set of all elements of N *not* in K. Then f is the desired bijection. ▮

•37. In how many ways can we pass out k (identical) ping-pong balls to n children if each child may get at most one?

Solution: $\binom{n}{k}$, because we choose the k children to whom we give ping-pong balls. ▮

•38. In how many ways may n people sit around a round table? (Assume that when people are sitting around a round table, all that really matters is who is to each person's right. For example, if we can get one arrangement of people around the table from another by having everyone get up and move to the right one place and sit back down, then we get an equivalent arrangement of people. Notice that you can get a list from a seating arrangement by marking a place at the table, and then listing the people at the table, starting at that place and moving around to the right.) There are at least two different ways of doing this problem. Try to find them both.

Solution: The total number of ways to list how the n people sit around the table is $n!$. However, two lists represent the same seating arrangement if we get one from the other by shifting everyone right the same number of places. (And if, in two lists, the same person is to each person's right, this is the only way the lists can differ.) This divides the set of lists up into blocks of n mutually equivalent lists. The number m of such blocks is the number of seating arrangements. However, by the product principle, $mn = n!$, because we have partitioned up the set of $n!$ lists into m sets of size n. Therefore $m = (n-1)!$

A second solution may be obtained by choosing one of the n people and letting this person sit anywhere. Since all that matters is who is to the right of each person, it doesn't matter where this person sits. Once this person is seated, let everybody else sit down. If the remaining people sit down first in one order clockwise around the table and then in some other order, the person to the right of somebody has changed. Thus there are $(n-1)!$ ways (the number of ways to seat everybody else) to seat the people around the table. ▮

We are now going to analyze the result of Problem 35 in more detail in order to tease out another counting principle that we can use in a wide variety of situations.

In Table 1.2 we list all three-element permutations of the 5-element set $\{a,b,c,d,e\}$. Each row consists of all 3-element permutations of some subset of $\{a,b,c,d,e\}$. Because a given k-element subset can be listed as a k-element permutation in $k!$ ways, there are $3! = 6$ permutations in each row. Because each 3-element permutation appears exactly once in the table, each row is a block of a partition of the set of 3-element permutations of $\{a,b,c,d,e\}$.

Table 1.2: The 3-element permutations of $\{a, b, c, d, e\}$ organized by which 3-element set they permute.

abc	acb	bac	bca	cab	cba
abd	adb	bad	bda	dab	dba
abe	aeb	bae	bea	eab	eba
acd	adc	cad	cda	dac	dca
ace	aec	cae	cea	eac	eca
ade	aed	dae	dea	ead	eda
bcd	bdc	cbd	cdb	dbc	dcb
bce	bec	cbe	ceb	ebc	ecb
bde	bed	dbe	deb	ebd	edb
cde	ced	dce	dec	ecd	edc

Each block has size six. Each block consists of all 3-element permutations of some three element subset of $\{a, b, c, d, e\}$. Since there are ten rows, we see that there are ten 3-element subsets of $\{a, b, c, d, e\}$. An alternate way to see this is to observe that we partitioned the set of all 60 three-element permutations of $\{a, b, c, d, e\}$ into some number q of blocks, each of size six. Thus by the product principle, $q \cdot 6 = 60$, so $q = 10$.

•39. Rather than restricting ourselves to $n = 5$ and $k = 3$, we can partition the set of all k-element permutations of an n-element set S up into blocks. We do so by letting B_K be the set (block) of all k-element permutations of K for each k-element subset K of S. Thus as in our preceding example, each block consists of all permutations of some subset K of our n-element set. For example, the permutations of $\{a, b, c\}$ are listed in the first row of Table 1.2. In fact each row of that table is a block. The questions that follow are about the corresponding partition of the set of k-element permutations of S, where S and k are arbitrary.

(a) How many permutations are there in a block?

Solution: The number of permutations in a block is $k!$. ▮

(b) Since S has n elements, what does Problem 20 tell you about the total number of k-element permutations of S?

Solution: Problem 20 tells us that the total number of k-element permutations is $n^{\underline{k}} = \frac{n!}{(n-k)!}$. ▮

(c) Describe a bijection between the set of blocks of the partition and the set of k-element subsets of S.

Solution: Each k-element set corresponds to the block of all permutations of that set. It is immediate that this is a bijection. ▮

(d) What formula does this give you for the number $\binom{n}{k}$ of k-element subsets of an n-element set?

Solution: Assuming there are s subsets, we have $s \cdot k!$ permutations in total, so by the product principle, $s \cdot k! = \frac{n!}{(n-k)!}$ or $s = \frac{n!}{k!(n-k)!}$. ▮

➔ 40. A basketball team has 12 players. However, only five players play at any given time during a game.

(a) In how may ways may the coach choose the five players?

Solution: $\binom{12}{5}$. ▮

(b) To be more realistic, the five players playing a game normally consist of two guards, two forwards, and one center. If there are five guards, four forwards, and three centers on the team, in how many ways can the coach choose two guards, two forwards, and one center?

Solution: In the more realistic version, $\binom{5}{2}\binom{4}{2}\binom{3}{1} = 180$. ▮

(c) What if one of the centers is equally skilled at playing forward?

Solution: Either the versatile player is playing center or not, and in the second case is available to play forward. This gives us $\binom{5}{2}\binom{4}{2}\binom{1}{1} + \binom{5}{2}\binom{5}{2}\binom{2}{1} = 260$ ways to choose the players. ▮

• 41. In Problem 38, describe a way to partition the n-element permutations of the n people into blocks so that there is a bijection between the set of blocks of the partition and the set of arrangements of the n people around a round table. What method of solution for Problem 38 does this correspond to?

Solution: Put two permutations in the same block if we can get one from the other by moving everyone (circularly) some number r places to the right. This corresponds to the method that gives $n!/n$ as the answer. Many students should be able to answer this question by saying "See the answer to Problem 38." ▮

• 42. In Problems 39d and 41, you have been using the product principle in a new way. One of the ways in which we previously stated the product principle was "If we partition a set into m blocks each of size n, then the set has size $m \cdot n$." In problems 39d and 41 we knew the size p of a set P of permutations of a set, and we knew we had partitioned P into some unknown number of blocks, each of a certain known size r. If we let q stand for the number of blocks, what does the product principle tell us about p, q, and r? What do we get when we solve for q?

Solution: $p = qr$, so that $q = p/r$. ▮

The formula you found in Problem 42 is so useful that we are going to single it out as another principle. The **quotient principle** says:

> If we partition a set P of size p into q blocks, each of size r, then $q = p/r$.

The quotient principle is really just a restatement of the product principle, but thinking about it as a principle in its own right often leads us to find solutions to problems. Notice that it does not always give us a formula for the number of blocks of a partition; it only works when all the blocks have the same size. In Chapter 6, we develop a way to solve problems with different block sizes in cases where there is a good deal of symmetry in the problem. (The roundness of the table was a symmetry in the problem of people at a table; the fact that we can order the sets in any order is the symmetry in the problem of counting k-element subsets.)

In Section A.2 of Appendix A we introduce the idea of an equivalence relation, see what equivalence relations have to do with partitions, and discuss the quotient principle from that point of view. While that appendix is not required for what we are doing here, if you want a more thorough discussion of the quotient principle, this would be a good time to work through that appendix.

➔ •43. In how many ways may we string n distinct beads on a necklace without a clasp? (Perhaps we make the necklace by stringing the beads on a string, and then carefully gluing the two ends of the string together so that the joint can't be seen. Assume someone can pick up the necklace, move it around in space and put it back down, giving an apparently different way of stringing the beads that is equivalent to the first.)

Solution: We can obtain a permutation of the beads by cutting the necklace and stretching it out in a straight line. We can partition the permutations according to which necklace they come from in this process. Two permutations are in the same block if we get one either by circularly permuting the other and/or by reversing the other (this corresponds to flipping the necklace over in space). Thus each necklace corresponds to $2n$ permutations so by the quotient principle we have $n!/2n = (n-1)!/2$ ways to string n distinct beads on a necklace. ▮

➔ 44. We first gave this problem as Problem 12a. Now we have several ways to approach the problem. A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

(a) In how many ways may we pair up all the members of the club? Give at least two solutions different from the one you gave in Problem 12a. (You may not have done Problem 12a. In that case, see if you can find three solutions.)

Solution: Choose people in pairs. There are $\binom{2n}{2}$ ways to choose one pair, $\binom{2n-2}{2}$ ways to choose a second pair, and once k pairs have been chosen, there are $\binom{2n-2k}{2}$ ways to choose the next pair. The number of *lists* of pairs we get in this way is $\prod_{i=0}^{n-1}\binom{2n-2i}{2} = \frac{(2n)!}{2^n}$. However, each way of pairing people gets listed $n!$ times since we see all possible length n lists of pairs. Therefore the number of actual pairings is

$$\frac{(2n)!}{2^n n!} = \frac{(2n)!}{2n \cdot 2n-2 \cdot 2n-4 \cdot \cdots \cdot 2} = \prod_{i=0}^{n-1} 2n-2i-1.$$

Notice how this combinatorial solution gives the formula that we found algebraically in Problem 12a, which then turns out to be algebraically equivalent to the formula we first saw in the solution to Problem 12a.

For yet another solution, we can list the $2n$ members in $(2n)!$ ways. Then we can take the first two as a tennis pair, the next two, and so on. There are $n!$ ways that a given set of tennis pairings could be arranged, and each of the n pairs could appear in 2 ways, so the tennis pairings partition the set of all permutations of the $2n$ members into blocks of size $n!2^n$. Thus we have $\frac{(2n)!}{n!2^n}$ tennis pairings once again. ▮

(b) Suppose that in addition to specifying who plays whom, for each pairing we say who serves first. Now in how many ways may we specify our pairs? Try to find as many solutions as you can.

Solution: Choose people in ordered pairs. The first person in an ordered pair serves first. There are $2n(2n-1)$ ways to choose one pair, $(2n-2)(2n-3)$ ways to choose a second pair, and once k pairs have been chosen, there are $(2n-2k)(2n-2k-1)$ ways to choose the next pair. The number of *lists* of pairs we get in this way is $\prod_{i=0}^{n-1}(2n-2i)(2n-2i-1) = (2n)!$. However, each way of pairing people gets listed $n!$ times since we see all possible length n lists of pairs. Therefore the number of actual pairings is $\frac{(2n)!}{n!} = (2n)^{\underline{n}}$.

For yet another solution, we can list the $2n$ members in $(2n)!$ ways. Then we can take the first two as a tennis pair, with the first person serving first, the next two, and so on. There are $n!$ ways that a given set of tennis pairings could be arranged, so the tennis pairings partition the set of all permutations of the $2n$ members into blocks of size $n!$. Thus we have $\frac{(2n)!}{n!}$ tennis pairings once again. ▮

➔ ·45. (This becomes especially relevant in Chapter 6, though it makes an important point here.) In how many ways may we attach two identical red beads and two identical blue beads to the corners of a square (with one bead per corner) free to move around in (three-dimensional) space?

Solution: Two ways; either the red beads are side-by-side or diagonally opposite. If we think about partitioning lists of 2 Rs and 2 Bs so that two are in the same block if we get one from the other by moving the square, we get two blocks, $\{RRBB, BRRB, BBRR, RBBR\}$ and $\{RBRB, BRBR\}$. This is an example of a problem with a good deal of symmetry in which the blocks of the relevant partition have different sizes. ▮

➔ 46. While the formula you proved in Problem 35 and Problem 39d is very useful, it doesn't give us a sense of how big the binomial coefficients are. We can get a very rough idea, for example, of the size of $\binom{2n}{n}$ by recognizing that we can write $(2n)^{\underline{n}}/n!$ as $\frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdots \frac{n+1}{1}$, and each quotient is at least 2, so the product is at least 2^n. If this were an accurate estimate, it would mean the fraction of n-element subsets of a $2n$-element set would be about $2^n/2^{2n} = 1/2^n$, which becomes very small as n becomes large. However, it is pretty clear the approximation will not be a very good one, because some of the terms in that product are much larger than 2. In fact, if $\binom{2n}{k}$ were the same for every k, then each would be the fraction $\frac{1}{2n+1}$ of 2^{2n}. This is much larger than the fraction $\frac{1}{2^n}$. But our intuition suggests that $\binom{2n}{n}$ is much larger than $\binom{2n}{1}$ and is likely larger than $\binom{2n}{n-1}$ so we can be sure our approximation is a bad one. For estimates like this, James Stirling developed a formula to approximate $n!$ when n is large, namely $n!$ is about $\left(\sqrt{2\pi n}\right) n^n/e^n$. In fact the ratio of $n!$ to this expression approaches 1 as n becomes infinite.[2] We write this as

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}.$$

We read this notation as $n!$ is asymptotic to $\sqrt{2\pi n}\frac{n^n}{e^n}$. Use Stirling's formula to show that the fraction of subsets of size n in an $2n$-element set is approximately $1/\sqrt{\pi n}$. This is a much bigger fraction than $\frac{1}{2^n}$!

Solution:

$$\frac{(2n)!}{n!n!} \sim \frac{\sqrt{4\pi n}\frac{(2n)^{2n}}{e^{2n}}}{\sqrt{2\pi n}\frac{n^n}{e^n}\sqrt{2\pi n}\frac{n^n}{e^n}} = \frac{2^{2n}}{\sqrt{\pi n}}$$

∎

1.3 Some Applications of Basic Counting Principles

1.3.1 Lattice paths and Catalan Numbers

∘47. In a part of a city, all streets run either north-south or east-west, and there are no dead ends. Suppose we are standing on a street corner. In how many ways may we walk to a corner that is four blocks north and six blocks east, using as few blocks as possible?

Solution: The shortest possible walk is going to be ten blocks. To plan a walk, we must choose which four of those ten blocks go north; the other six blocks we will have to go east. There are $\binom{10}{4}$ ways to make this selection. ∎

·48. Problem 47 has a geometric interpretation in a coordinate plane. A *lattice path* in the plane is a "curve" made up of line segments that either go from a point (i, j) to the point $(i+1, j)$ or from a point (i, j) to the point $(i, j+1)$, where i and j are integers. (Thus lattice paths

[2]Proving this takes more of a detour than is advisable here; however there is an elementary proof which you can work through in the problems of the end of Section 1 of Chapter 1 of *Introductory Combinatorics* by Kenneth P. Bogart, Harcourt Academic Press, (2000).

always move either up or to the right.) The length of the path is the number of such line segments.

(a) What is the length of a lattice path from $(0,0)$ to (m,n)?
Solution: The length of a lattice path from $(0,0)$ to (m,n) is $m+n$. ▮

(b) How many such lattice paths of that length are there?
Solution: The number of such paths is $\binom{m+n}{n}$. ▮

(c) How many lattice paths are there from (i,j) to (m,n), assuming i, j, m, and n are integers?
Solution: Since lattice paths move up and to the right, there are no paths from (i,j) to (m,n) unless $i \le m$ and $j \le n$. In that case, the number of paths is $\binom{m+n-i-j}{n-j}$, which is the same as $\binom{m+n-i-j}{m-i}$. ▮

·49. Another kind of geometric path in the plane is a *diagonal lattice path*. Such a path is a path made up of line segments that go from a point (i,j) to $(i+1,j+1)$ (this is often called an *upstep*) or $(i+1,j-1)$ (this is often called a *downstep*), again where i and j are integers. (Thus diagonal lattice paths always move towards the right but may move up or down.)

(a) Describe which points are connected to $(0,0)$ by diagonal lattice paths.
Solution: The points (m,n) connected to $(0,0)$ by diagonal lattice paths will have $m+n$ even, because each upstep adds two to the sum of i and j while each downstep does not change the sum. Further, since we go one step to the right each time we go up or down, we cannot get above the line $y=x$ or below the line $y=-x$. However, for any point (m,n) with m and n nonnegative integers such that $m+n$ is even and $-m \le n \le m$, we can get to (m,n) by making $\frac{m-n}{2}$ downsteps and $\frac{m+n}{2}$ upsteps. ▮

(b) What is the length of a diagonal lattice path from $(0,0)$ to (m,n)?
Solution: From the previous part of this problem, we will make a total of $\frac{m-n}{2}+\frac{m+n}{2}=m$ steps, and our total motion parallel to the y axis will be $\frac{m+n}{2}-\frac{m-n}{2}=n$. The length of such a path is $m\sqrt{2}$; we might informally just call it m steps. ▮

(c) Assuming that (m,n) is a point you can get to from $(0,0)$, how many diagonal lattice paths are there from $(0,0)$ to (m,n)?

Solution: The number of possible paths is the number of ways we can choose which of the m steps are upsteps (or equivalently downsteps). This number is $\binom{m}{\frac{m+n}{2}}$. ▮

∘50. A school play requires a ten dollar donation per person; the donation goes into the student activity fund. Assume that each person who comes to the play pays with a ten dollar bill or a twenty dollar bill. The teacher who is collecting the money forgot to get change before the event. If there are always at least as many people who have paid with a ten as a twenty as they arrive the teacher won't have to give anyone an IOU for change. Suppose $2n$ people come to the play, and exactly half of them pay with ten dollar bills.

(a) Describe a bijection between the set of sequences of tens and twenties people give the teacher and the set of lattice paths from $(0, 0)$ to (n, n).

Solution: For each ten dollar bill take a rightstep and for each twenty dollar bill take an upstep (where rightstep and upstep have the hopefully natural meaning). The assumption that there are an equal number of ten and twenty dollar bills means that the path will end up at (n, n). Each sequence of tens and twenties gives a lattice path and each lattice path corresponds to such a sequence, so we have a bijection. ▮

(b) Describe a bijection between the set of sequences of tens and twenties that people give the teacher and the set of diagonal lattice paths between $(0, 0)$ and $(2n, 0)$.

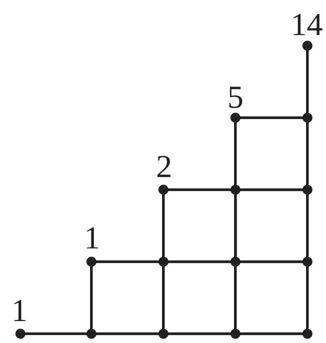
Solution: For each ten dollar bill take an upstep and for each twenty dollar bill take a downstep. Each sequence of tens and twenties will give us a diagonal lattice path from $(0, 0)$, and each diagonal lattice path from $(0, 0)$ to $(2n, 0)$ will give us a sequence of tens and twenties with an equal number of tens and twenties, so we have a bijection. ▮

(c) In each of the previous parts, what is the geometric interpretation of a sequence that does not require the teacher to give any IOUs?

Solution: In the first case a sequence that does not require the teacher to give any IOUs will correspond to a lattice path that stays on or below the line $y = x$, and in the second case such a sequence will correspond to a diagonal lattice path that stays on or above the x-axis. ▮

➔ ·51. Notice that a lattice path from $(0,0)$ to (n,n) stays inside (or on the edges of) the square whose sides are the x-axis, the y-axis, the line $x = n$ and the line $y = n$. In this problem we will compute the number of lattice paths from (0,0) to (n,n) that stay inside (or on the edges of) the triangle whose sides are the x-axis, the line $x = n$ and the line $y = x$. Such lattice paths are called *Catalan paths*. For example, in Figure 1.7 we show the grid of points with integer coordinates for the triangle whose sides are the x-axis, the line $x = 4$ and the line $y = x$.

Figure 1.7: The Catalan paths from $(0,0)$ to (i,i) for $i = 0,1,2,3,4$. The number of paths to the point (i,i) is shown just above that point.



(a) Explain why the number of lattice paths from $(0,0)$ to (n,n) that go outside the triangle described previously is the number of lattice paths from $(0,0)$ to (n,n) that either touch or cross the line $y = x + 1$.

Solution: If a lattice path between $(0,0)$ and (n,n) goes outside the triangle, it can only do so on an upstep. (A step from (i,j) to $(i,j+1)$.) And an upstep must originate at a point with integer coordinates. If $j < i$, an upstep from $(i,j))$ cannot leave the triangle. Thus to leave the triangle, the upstep must leave from a point of the form (i,i), and go to $(i,i+1)$, which is on the line $y = x + 1$. ▮

(b) Find a bijection between lattice paths from $(0,0)$ to (n,n) that touch (or cross) the line $y = x + 1$ and lattice paths from $(-1,1)$ to (n,n).

Solution: Suppose we have a lattice path from $(0,0)$ to (n,n) which touches or crosses the line $y = x+1$. Let $(k, k+1)$ be the first point on the line $y = x+1$ that the lattice path touches. From that point, work backwards, replacing every upstep with a step one unit to the left and every rightstep with a step one unit down. The segment of the path you just changed will have moved left $k+1$ times, so its leftmost x coordinate will be -1, and it will have moved down k times, so its lowest y coordinate will be 1. Thus we now have a lattice path from $(-1,1)$ to (n,n). Further, given a lattice path from $(-1,1)$ to (n,n), it must cross the line $y = x+1$ at least once, because it starts above the line and ends below it. At the first point where such a path touches the line $y = x+1$, say $(k', k'+1)$, work backwards, replacing every upstep with a step to the left and every rightstep with a step downwards. The leftmost point on this path will have x coordinate 0, and the lowest point will have y coordinate 0, so the new path will be a lattice path from $(0,0)$ to (n,n) that touches the line $y = x+1$. Clearly these two processes reverse each other, and so they give us a bijection between paths from $(0,0)$ to (n,n) that touch or cross the line $y = x+1$ and lattice lattice paths from $(-1,1)$ to (n,n). Notice that geometrically what we are doing to get the bijection is to take the portion of a lattice path that goes from the initial point till the first touch of the line $y = x+1$ and reflecting it around that line. This idea of reflection was introduced by Feller, and is called Feller's reflection principle. ▮

(c) Find a formula for the number of lattice paths from $(0,0)$ to (n,n) that do not go above the line $y = x$. The number of such paths is called a *Catalan Number* and is usually denoted by C_n.
Solution: $C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$. ▮

➔ 52. Your formula for the Catalan Number can be expressed as a binomial coefficient divided by an integer. Whenever we have a formula that calls for division by an integer, an ideal combinatorial explanation of the formula is one that uses the quotient principle. The purpose of this problem is to find such an explanation using diagonal lattice paths.[3] A diagonal lattice path that never goes below the y-coordinate of its first point is called a *Dyck Path*. We will call a Dyck Path from $(0,0)$

[3] The result we will derive is called the Chung-Feller Theorem; this approach is based on a paper of Wen-jin Woan "Uniform Partitions of Lattice Paths and Chung-Feller Generalizations," **American Mathematics Monthly 58** June/July 2001, p556.

to $(2n, 0)$ a (diagonal) *Catalan Path* of length $2n$. Thus the number of (diagonal) Catalan Paths of length $2n$ is the Catalan Number C_n. We normally can decide from context whether the phrase Catalan Path refers to a diagonal path, so we normally leave out the word diagonal.

(a) If a Dyck Path has n steps (each an upstep or downstep), why do the first k steps form a Dyck Path for each nonnegative $k \leq n$?

Solution: If no points on the path are lower than the first point, then no points among the first k steps are lower than the first point. ▮

(b) Thought of as a curve in the plane, a diagonal lattice path can have many local maxima and minima, and can have several absolute maxima and minima, that is, several highest points and several lowest points. What is the y-coordinate of an absolute minimum point of a Dyck Path starting at $(0, 0)$? Explain why a Dyck Path whose rightmost absolute minimum point is its last point is a Catalan Path.

Solution: Since the path starts at $(0, 0)$ and can't go below it, the y coordinate of an absolute minimum must be zero. If the last point is an absolute minimum, then (because it ends with the same y coordinate with which it starts) the path has an even number $2k$ of steps and ends at $(2k, 0)$, so it is a Catalan path. ▮

(c) Let D be the set of all diagonal lattice paths from $(0, 0)$ to $(2n, 0)$. (Thus these paths can go below the x-axis.) Suppose we partition D by letting B_i be the set of lattice paths in D that have i upsteps (perhaps mixed with some downsteps) following the last absolute minimum. How many blocks does this partition have? Give a succinct description of the block B_0.

Solution: The path must have n upsteps total, and so can have any number between 0 and n upsteps after the rightmost absolute minimum. Thus the partition has $n+1$ blocks. Block B_0 consists of the Catalan Paths. ▮

(d) How many upsteps are in a Catalan Path?

Solution: n. ▮

$*$(e) We are going to give a bijection between the set of Catalan Paths and the block B_i for each i between 1 and n. For now, suppose the value of i, while unknown, is fixed. We take a Catalan path and break it into three pieces. The piece F (for "front") consists of all steps before the ith upstep in the Catalan path. The piece

U (for "up") consists of the ith upstep. The piece B (for "back") is the portion of the path that follows the ith upstep. Thus we can think of the path as FUB. Show that the function that takes FUB to BUF is a bijection from the set of Catalan Paths onto the block B_i of the partition. (Notice that BUF can go below the x axis.)

Solution: Since we are starting with a Catalan path, the point on the path at the beginning of the ith upstep must have y coordinate greater or equal to than zero. Thus wherever we start the sequence F of upsteps and downsteps, a path constructed by this sequence never goes lower than its starting point. Thus in BUF the last absolute minimum is either right before the U or earlier. But B is the final segment of a Catalan Path, so its final point is at least as low as its starting point. Thus the point at the beginning of the U in BUF is an absolute minimum, and there are i upsteps after that absolute minimum. If we take two different sequences and rearrange them in the same way, we get two different sequences, so the function we just described is a one-to-one function. If we take an arbitrary diagonal lattice path from $(0,0)$ to $(2n,0)$, let U' be the first upstep after the last absolute minimum, F' be the portion of the path that follows U', and B' be the portion that precedes U', then $F'U'B'$ is a Catalan Path, and U' is its ith upstep if and only if in $B'U'F'$ there are i upsteps after the last absolute minimum. Thus the mapping from FUB to BUF is a bijection. ∎

(f) Explain how you have just given another proof of the formula for the Catalan Numbers.

Solution: We have taken the set of all $\binom{2n}{n}$ diagonal lattice paths of length $2n$ from $(0,0)$ to $(2n,0)$ and partitioned it into $n+1$ blocks all of size C_n. Thus by the quotient principle, $C_n = \frac{1}{n+1}\binom{2n}{n}$. ∎

1.3.2 The Binomial Theorem

∘53. We know that $(x+y)^2 = x^2+2xy+y^2$. Multiply both sides by $(x+y)$ to get a formula for $(x+y)^3$ and repeat to get a formula for $(x+y)^4$. Do you see a pattern? If so, what is it? If not, repeat the process to get a formula for $(x+y)^5$ and look back at Figure 1.4 to see the pattern. Conjecture a formula for $(x+y)^n$.

Solution: $(x+y)^3 = x^3+2x^2y+xy^2+x^2y+2xy^2+y^3 = x^3+3x^2y+3xy^2+y^3$.
Similarly, $(x+4)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$,
and $(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$. The pattern is that the coefficient of x^iy^j is $\binom{i+j}{i}$ which is the same as $\binom{i+j}{j}$. Said differently, the coefficient of $x^{n-i}y^i$ is $\binom{n}{i}$ or the coefficient of x^iy^{n-i} is $\binom{n}{i}$. We conjecture that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i.$$

(The reason for putting $x^{n-i}y^i$ into the sum is so that as i goes from 0 to n, the powers of x decrease from n to 0.) ▮

•54. When we apply the distributive law n times to $(x+y)^n$, we get a sum of terms of the form x^iy^{n-i} for various values of the integer i.

(a) If it is clear to you that each term of the form x^iy^{n-i} that we get comes from choosing an x from i of the $(x+y)$ factors and a y from the remaining $n-i$ of the factors and multiplying these choices together, then answer this part of the problem and skip the next part. Otherwise, do the next part instead of this one. In how many ways can we choose an x from i terms and a y from $n-i$ terms?

Solution: The number of ways to choose an x from i of the factors and a y from the remaining ones is the way to choose the i factors from the n factors; that is, $\binom{n}{i}$. ▮

i. Expand the product $(x_1+y_1)(x_2+y_2)(x_3+y_3)$.
Solution:

$$\begin{aligned}(x_1+y_1)(x_2+y_2)(x_3+y_3) = x_1x_2x_3 \;&+\; x_1x_2y_3 + x_1y_2x_3 + \\ y_1x_2x_3 + x_1y_2y_3 + y_1x_2y_3 \;&+\; y_1y_2x_3 + y_1y_2y_3.\end{aligned}$$

▮

ii. What do you get when you substitute x for each x_i and y for each y_i?
Solution: When you substitute x for each x_i and y for each y_i, you get $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$. ▮

iii. Now imagine expanding

$$(x_1+y_1)(x_2+y_2)\cdots(x_n+y_n).$$

Once you apply the commutative law to the individual terms you get, you will have a sum of terms of the form

$$x_{k_1}x_{k_2}\cdots x_{k_i}\cdot y_{j_1}y_{j_2}\cdots y_{j_{n-i}}.$$

What is the set $\{k_1,k_2,\ldots,k_i\}\cup\{j_1,j_2,\ldots,j_{n-i}\}$?
Solution: $\{k_1,k_2,\ldots,k_i\}\cup\{j_1,j_2,\ldots,j_{n-i}\}=\{1,2,\ldots,n\}$. ▮

iv. In how many ways can you choose the set $\{k_1,k_2,\ldots,k_i\}$?
Solution: You can choose the set $\{k_1,k_2,\ldots k_i\}$ in $\binom{n}{i}$ ways. ▮

v. Once you have chosen this set, how many choices do you have for $\{j_1,j_2,\ldots,j_{n-i}\}$?
Solution: Once you have chosen the set of ks, there is just one way to choose the set of js. ▮

vi. If you substitute x for each x_i and y for each y_i, how many terms of the form x^iy^{n-i} will you have in the expanded product

$$(x_1+y_1)(x_2+y_2)\cdots(x_n+y_n)=(x+y)^n?$$

Solution: If you substitute x for x_i and substitute y for y_i, you will get $\binom{n}{i}$ terms of the form x^iy^{n-i}. ▮

vii. How many terms of the form $x^{n-i}y^i$ will you have?
Solution: You will also get $\binom{n}{i}$ terms of the form $x^{n-i}y^i$. ▮

(b) Explain how you have just proved your conjecture from Problem 53. The theorem you have proved is called the **Binomial Theorem**.

Solution: We have proved that the coefficient of x^iy^{n-i} in $(x+y)^n$ is $\binom{n}{i}$, or equivalently that the coefficient of $x^{n-i}y^i$ in $(x+y)^n$ is $\binom{n}{i}$. ▮

55. What is $\sum_{i=1}^{10}\binom{10}{i}3^i$?

Solution: $\sum_{i=1}^{10}\binom{10}{i}3^i=\sum_{i=0}^{10}\binom{10}{i}3^i-\binom{10}{0}3^0=(1+3)^{10}-1=4^{10}-1$ ▮

56. What is $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots\pm\binom{n}{n}$ if n is an integer bigger than zero?

Solution: The sum is 0 because it is $(-1+1)^n$. ▮

➔ •57. Explain why

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}=\binom{m+n}{k}.$$

Find two different explanations.

Solution: When we expand both sides of

$$(x+y)^m(x+y)^n = (x+y)^{m+n}$$

by the binomial theorem we get $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$ as the coefficient of $x^{m+n-k}y^k$ on the left hand side and $\binom{m+n}{k}$ on the right hand side.

For a second explanation, to choose k elements out of the union of an m-element set and a disjoint n-element set, choose some number $i \le m$ of them from the m-element set and the remaining $k-i$ of them from the n-element set. The sum on the left hand side of the equation simply sums the number of such choices over all possible i, and the binomial coefficient on the right hand side of the equation says we will end up choosing k elements from among our $m+n$ elements. ▮

➔ 58. From the symmetry of the binomial coefficients, it is not too hard to see that when n is an odd number, the number of subsets of $\{1,2,\ldots,n\}$ of odd size equals the number of subsets of $\{1,2,\ldots,n\}$ of even size. Is it true that when n is even the number of subsets of $\{1,2,\ldots,n\}$ of even size equals the number of subsets of odd size? Why or why not?

Solution: It is true, because if $n>0$, when you expand $(1-1)^n$ by the binomial theorem, you get an alternating sum of binomial coefficients equal to 0, and so the sum of the binomial coefficients $\binom{n}{i}$ with i even must equal the sum of the binomial coefficients $\binom{n}{i}$ with i odd. ▮

➔ 59. What is $\sum_{i=0}^{n} i\binom{n}{i}$? (Hint: think about how you might use calculus.)

Solution: $\sum_{i=0}^{n}\binom{n}{i}x^i = (1+x)^n$. Taking derivatives of both sides gives us $\sum_{i=0}^{n} i\binom{n}{i}x^{i-1} = n(1+x)^{n-1}$. Now substitute 1 for x and you get $\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$. ▮

Notice how the proof you gave of the binomial theorem was a counting argument. It is interesting that an apparently algebraic theorem that tells us how to expand a power of a binomial is proved by an argument that amounts to counting the individual terms of the expansion. Part of the reason that combinatorial mathematics turns out to be so useful is that counting arguments often underlie important results of algebra. As the algebra becomes more sophisticated, so do the families of objects we have to count, but nonetheless we can develop a great deal of algebra on the basis of counting.

1.3.3 The pigeonhole principle

∘60. American coins are all marked with the year in which they were made. How many coins do you need to have in your hand to guarantee that on two (at least) of them, the date has the same last digit? (When we say "to guarantee that on two (at least) of them,..." we mean that you can find two with the same last digit. You might be able to find three with that last digit, or you might be able to find one pair with the last digit 1 and one pair with the last digit 9, or any combination of equal last digits, as long as there is at least one pair with the same last digit.)

Solution: Since there are ten possible last digits, you need at least 11 coins, and with 11 coins, at least two last digits must be the same. ∎

There are many ways in which you might explain your answer to Problem 60. For example, you can partition the coins according to the last digit of their date; that is, you put all the coins with a given last digit in a block together, and put no other coins in that block; repeating until all coins are in some block. Then you have a partition of your set of coins. If no two coins have the same last digit, then each block has exactly one coin. Since there are only ten digits, there are at most ten blocks and so by the sum principle there are at most ten coins. In fact with ten coins it is possible to have no two with the same last digit, but with 11 coins some block must have at least two coins in order for the sum of the sizes of at most ten blocks to be 11. This is one explanation of why we need 11 coins in Problem 60. This kind of situation arises often in combinatorial situations, and so rather than always using the sum principle to explain our reasoning, we enunciate another principle which we can think of as yet another variant of the sum principle. The **pigeonhole principle** states that

> If we partition a set with more than n elements into n parts,
> then at least one part has more than one element.

The pigeonhole principle gets its name from the idea of a grid of little boxes that might be used, for example, to sort mail, or as mailboxes for a group of people in an office. The boxes in such grids are sometimes called pigeonholes in analogy with stacks of boxes used to house homing pigeons when homing pigeons were used to carry messages. People will sometimes state the principle in a more colorful way as "if we put more than n pigeons into n pigeonholes, then some pigeonhole has more than one pigeon."

61. Show that if we have a function from a set of size n to a set of size less than n, then f is not one-to-one.

    Solution: Let T be the set of size less than n, and S be the set of size n. Let $B_j = \{i|f(i) = j\}$ for each j in T. Then the nonempty sets among the B_js form a partition of S and the number of blocks is less than the size of S. Therefore by the pigeonhole principle, there is at least one block with at least two elements, so there are two elements i_1 and i_2 such that $f(i_1) = f(i_2)$. ▮

•62. Show that if S and T are finite sets of the same size, then a function f from S to T is one-to-one if and only if it is onto.

    Solution: First suppose that f is a one-to-one function from S to T, sets which have the same size. Let $B_j = \{i|f(i) = j\}$ for each j in T. If f is not onto, then the number of nonempty sets B_j is smaller than the number of elements of T and thus is smaller than the size of S. The nonempty sets B_j are a partition of S. But then by the pigeonhole principle, some nonempty B_j has two or more elements, contradicting the assumption that f is one-to-one. Therefore if f is one-to-one, then it is onto. Now suppose that f is an onto function from S to T, sets of the same size. Again let $B_j = \{i|f(i) = j\}$ for each j in T. The size of the union of the sets B_j is, by the sum principle, the sum of their sizes. Since f is onto, each B_j has at least one element. Since the number of sets B_j is the number of elements of S, if one of those sets has more than one element, the the size of their union is more than the size of S, which is a contradiction since they are subsets of S. Therefore each set B_j has exactly one element and therefore f is one-to-one. ▮

·63. There is a *generalized pigeonhole principle* which says that if we partition a set with more than kn elements into n blocks, then at least one block has at least $k + 1$ elements. Prove the generalized pigeonhole principle.

    Solution: Suppose we partition a set S of more than kn elements into n blocks. If each block has at most k elements, the by the sum principle the size of S is at most kn. But this is a contradiction, so some block has at least $k + 1$ elements. ▮

64. All the powers of five end in a five, and all the powers of two are even. Show that for some integer n, if you take the first n powers of a prime other than two or five, one must have "01" as the last two digits.

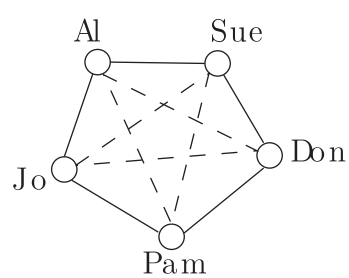
Solution: If we take 40 powers of such a prime, either one will end in "01" or some two, say p^i and p^j with $i > j$ must have the same last two digits by the pigeon hole principle. Then $p^i - p^j = 100k$ for some integer k. Thus $p^j(p^{i-j} - 1)$ must be a multiple of 100, and since neither 2 nor 5 divide p, $p^{i-j} - 1 = 100k'$ for some integer k'. Then $p^{i-j} = 100k' + 1$, so the last two digits of p^{i-j} must be "01." ▮

➔ ·65. Show that in a set of six people, there is a set of at least three people who all know each other, or a set of at least three people none of whom know each other. (We assume that if person 1 knows person 2, then person 2 knows person 1.)

Solution: By the generalized pigeonhole principle, person 1 either knows at least three people or doesn't know at least three people. Suppose person 1 knows three people. Then either two of these people know each other, giving us, with person 1, three mutual acquaintances, or no two of these people know each other, giving us three mutual strangers. On the other hand if there are three people person 1 does not know, then either two of these people don't know each other, giving us, with person 1, three mutual strangers, or all three of these people know each other, giving us three mutual acquaintances. ▮

·66. Draw five circles labeled Al, Sue, Don, Pam, and Jo. Find a way to draw red and green lines between people so that every pair of people is joined by a line and there is neither a triangle consisting entirely of red lines or a triangle consisting of green lines. What does Problem 65 tell you about the possibility of doing this with six people's names? What does this problem say about the conclusion of Problem 65 holding when there are five people in our set rather than six?

Solution: In the figure that follows, we use solid lines for red and dashed lines for green. Clearly there is no solid triangle and no dashed triangle.

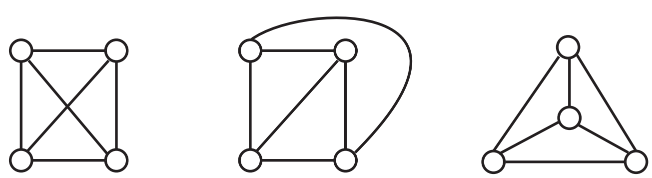


Problem 65 says you can't do this with six people's names. This problem says that the conclusion of Problem 65 does not hold when you have five people. ∎

1.3.4 Ramsey Numbers

Problems 65 and 66 together show that six is the smallest number R with the property that if we have R people in a room, then there is either a set of (at least) three mutual acquaintances or a set of (at least) three mutual strangers. Another way to say the same thing is to say that six is the smallest number so that no matter how we connect six points in the plane (no three on a line) with red and green lines, we can find either a red triangle or a green triangle. There is a name for this property. The **Ramsey Number** $R(m, n)$ is the smallest number R so that if we have R people in a room, then there is a set of at least m mutual acquaintances or at least n mutual strangers. There is also a geometric description of Ramsey Numbers; it uses the idea of a *complete graph* on R vertices. A **complete graph** on R vertices consists of R points in the plane, together with line segments (or curves) connecting each two of the R vertices.[4] The points are called *vertices* and the line segments are called *edges*. In Figure 1.8 we show three different ways to draw a complete graph on four vertices. We use K_n to stand for a complete graph on n vertices.

Figure 1.8: Three ways to draw a complete graph on four vertices



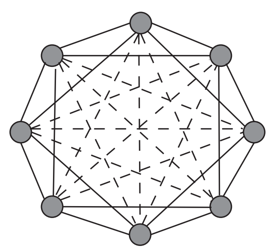
Our geometric description of $R(3, 3)$ may be translated into the language of graph theory (which is the subject that includes complete graphs) by saying $R(3, 3)$ is the smallest number R so that if we color the edges of a K_R with two colors, then we can find in our picture a K_3 all of whose edges have the same color. The graph theory description of $R(m, n)$ is that

[4] As you may have guessed, a complete graph is a special case of something called a graph. The word graph will be defined in Section 2.3.1.

$R(m,n)$ is the smallest number R so that if we color the edges of a K_R with red and green, then we can find in our picture either a K_m all of whose edges are red or a K_n all of whose edges are green. Because we could have said our colors in the opposite order, we may conclude that $R(m,n) = R(n,m)$. In particular $R(n,n)$ is the smallest number R such that if we color the edges of a K_R with two colors, then our picture contains a K_n all of whose edges have the same color.

∘67. Since $R(3,3) = 6$, an uneducated guess might be that $R(4,4) = 8$. Show that this is not the case.

Solution: In the graph



each vertex has three dashed lines emanating from it, and there are no dashed lines connecting any of the three vertices adjacent to it by dashed lines. Each vertex has four solid lines emanating from it, and no three of the four vertices adjacent to it by solid lines are all adjacent by solid lines. Thus there is no solid line K_4 and there is no dashed line K_4. ▮

•68. Show that among ten people, there are either four mutual acquaintances or three mutual strangers. What does this say about $R(4,3)$?

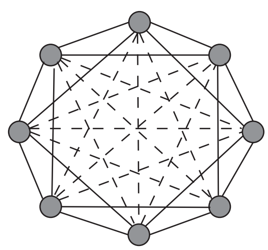
Solution: Take a person, say person 1. If person has six acquaintances, then by Problem 65 among them there are either three mutual strangers, in which case we are done, or three mutual acquaintances. These three acquaintances, together with person 1 form a set of 4 mutual acquaintances in which case we are again done. Thus we may assume Person 1 has at most 5 acquaintances, and so has four nonacquaintances. Now either all four of these people are acquainted, in which case we are done, or else two of them are not acquainted. Then these two people, together with person 1 make three mutual nonacquaintances. Therefore in every possible case, we have either four mutual acquaintances or three mutual strangers. This means that $R(4,3) \leq 10$. ▮

·69. Show that among an odd number of people there is at least one person who is an acquaintance of an even number of people and therefore also a stranger to an even number of people.

Solution: Suppose we add, for each person, the number of people with whom he or she is acquainted. Then we get twice the number of acquaintance edges in the graph of acquaintance and non-acquaintance relationships. Thus the sum must be even. But if each person among an odd number of people were acquainted with an odd number of people, then the sum would be odd. Since this is a contradiction, among an odd number of people, there must be at least one who is acquainted with an even number of people. Since the number of people different from this person is even, the number of people with whom this person is not acquainted is also even. ▮

·70. Find a way to color the edges of a K_8 with red and green so that there is no red K_4 and no green K_3.

Solution: In the graph



there is no K_3 whose edges are dashed, and no K_4 whose edges are solid. By symmetry, to verify this you need only look at vertex 1 and vertices connected to it by either dashed lines or by solid lines. ▮

➔ ·71. Find $R(4,3)$.

Solution: $R(4,3) = 9$. In Problem 70 we showed that $R(4,3)$ is more than 8. So we must show that if we have nine people, we either have 4 mutual acquaintances or three mutual strangers. By Problem 69 there is at least one person (say person A) who is acquainted with an even number of people. If person A is acquainted with six or more people, then among these six people, there are either three mutual acquaintances or three mutual strangers. If there are three mutual strangers, we are done; if there are three mutual acquaintances, they, together with Person A are four mutual acquaintances. Thus we may assume

Person A is acquainted with at most four people. Thus person A is a stranger to at least four people. If two of these people are strangers, then they, together with person A form three mutual strangers and we are done. Otherwise all of these people know each other and we have at least four mutual acquaintances, and so in every possible situation, we have either four mutual acquaintances or three mutual strangers. ▮

As of this writing, relatively few Ramsey Numbers are known. $R(3, n)$ is known for $n < 10$, $R(4, 4) = 18$, and $R(5, 4) = R(4, 5) = 25$.

1.4 Supplementary Chapter Problems

➔ 1. Remember that we can write n as a sum of n ones. How many plus signs do we use? In how many ways may we write n as a sum of a list of k positive numbers? Such a list is called a *composition* of n into k parts.

Solution: We use $n - 1$ plus signs. Write down such a sum and choose $k - 1$ of the plus signs. Then each string of ones and plusses between two chosen plus signs, before the first chosen plus sign or after the last chosen one corresponds to a part of a composition of n. Thus the number of compositions of n with k parts is the number of ways to choose the $k - 1$ places, which is $\binom{n-1}{k-1}$. ▮

2. In Problem 1 we defined a composition of n into k parts. What is the total number of compositions of n (into any number of parts)?

Solution: The total number of compositions is the number of ways to choose a subset of the plus signs which is 2^{n-1}. ▮

·3. Write down a list of all 16 zero-one sequences of length four starting with 0000 in such a way that each entry differs from the previous one by changing just one digit. This is called a Gray Code. That is, a *Gray Code* for 0-1 sequences of length n is a list of the sequences so that each entry differs from the previous one in exactly one place. Can you describe how to get a Gray Code for 0-1 sequences of length five from the one you found for sequences of length 4? Can you describe how to prove that there is a Gray code for sequences of length n?

Solution: (One of many) 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000. To get a code for sequences of length 5, put a zero at the end of each of the sequences we have. Follow that revised sequence by 10001, and write the remainder of the original sequence in reverse order with a 1 at the end of each term. (Don't reverse the individual length four sequences, just the sequence of sequences!) If we do this with a Gray Code for sequences of length n, we get a Gray code for sequences of length $n+1$. Thus we can get a Gray code for sequences of any length we wish. In the terminology of Chapter 2, we just described the inductive step of an inductive proof that Gray Codes exist for sequences of any length. ▮

➔ 4. Use the idea of a Gray Code from Problem 3 to prove bijectively that the number of even-sized subsets of an n-element set equals the number of odd-sized subsets of an n-element set.

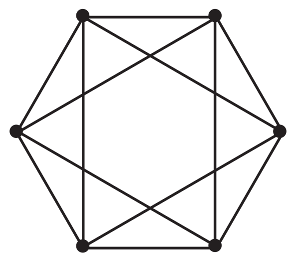
Solution: Each sequence in the Gray Code is the characteristic function of a set, and the number of elements of the set is the number of ones in the sequence. Since each sequence differs in just one place from the preceding one, the sequences alternate between having an even number of ones and an odd number of ones. Since the first sequence is all zeros and there are 2^n sequences, the last one has an odd number of zeros. Thus the map that takes each sequence except the last to the next one, and takes the last to the first is a bijection between the characteristic functions of sets with an even number of elements and sets with an odd number of elements. ▮

➔ 5. A list of parentheses is said to be balanced if there are the same number of left parentheses as right, and as we count from left to right we always find at least as many left parentheses as right parentheses. For example, ((((()()))()) is balanced and ((()) and (()()))(() are not. How many balanced lists of n left and n right parentheses are there?

Solution: The number is the Catalan Number: we get a bijection between balanced lists of parentheses and Catalan paths by sending each left parenthesis to an upstep and each right parenthesis to a downstep. The condition that there are always as many left parentheses as right ensures we never go below the x axis. ▮

$*$6. Suppose we plan to put six distinct computers in a network as shown in Figure 1.9. The lines show which computers can communicate directly with which others. Consider two ways of assigning computers to the nodes of the network different if there are two computers that communicate directly in one assignment and that don't communicate directly in the other. In how many different ways can we assign computers to the network?

Figure 1.9: A computer network.



Solution: We consider two assignments of computers to be equivalent if in both assignments, each computer communicates directly with exactly the same computers. This partitions the set of all 6! computer assignments into blocks of 48 computer assignments each. Thus we have $720/48 = 15$ ways to assign the computers to the network. ∎

➔ 7. In a circular ice cream dish we are going to put four scoops of ice cream of four distinct flavors chosen from among twelve flavors. Assuming we place four scoops of the same size as if they were at the corners of a square, and recognizing that moving the dish doesn't change the way in which we have put the ice cream into the dish, in how many ways may we choose the ice cream and put it into the dish?

Solution: Each ice cream arrangement is equivalent to three others, the ones we get by rotating the dish. This divides the arrangements of four flavors of ice cream into blocks of size 4. Thus we may arrange the ice cream we have chosen in the dish in $4!/4 = 6$ ways. We may choose the ice cream in $\binom{12}{4} = 495$ ways, and so we may choose it and put it into the dish in 2970 ways. ∎

➔ 8. In as many ways as you can, show that $\binom{n}{k}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{k}$.

Solution: You can prove this by plugging in the formula for $\binom{n}{k}$ on both sides and cancelling stuff until you get the same thing on both sides.

However, a much more interesting proof is that the left hand side counts the number of ways to choose a k-element set from an n-element set and then choose an m-element set from what remains. The right hand side counts the number of ways to first choose an m-element subset from the n-element set and then choose a k-element subset from what remains. Thus in both cases you are counting the number of ways to choose an ordered pair consisting of an m-element subset and a disjoint k-element subset from an n-element set.

You can also base a proof on the observation that $(x+y+z)^n = \sum_{k=0}^{n}\binom{n}{k}(x+y)^k z^{n-k}$ and $(x+y+z)^n = \sum_{m=0}^{n}\binom{n}{m}x^m(y+z)^{n-m}$ and asking for the coefficient of $x^m y^{n-m-k} z^k$. You do have to use the binomial theorem with an eye to the result you are looking for, however. ∎

➔ 9. A tennis club has $4n$ members. To specify a doubles match, we choose two teams of two people. In how many ways may we arrange the members into doubles matches so that each player is in one doubles match? In how many ways may we do it if we specify in addition who serves first on each team?

Solution: We now have many methods for solving this problem. Perhaps the easiest is to list all $4n$ people and take them in groups of four for doubles matches, with the first two in a group of four as one team and the second two as another team. We note that interchanging the n blocks of 4 does not change the matches, nor does interchanging the two people on a team nor interchanging the two teams. Thus we have $(4n)!/n!2^{3n}$ ways to arrange the matches. If we are to say who serves first on each team, we might as well say it is the first of the two listed, so now we have $(4n)!/n!2^n$ ways to arrange the matches. It is an excellent exercise to look for more solutions. ∎

10. A town has n streetlights running along the north side of Main Street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green?

Solution: We can think of first choosing the set of even size of poles to be painted green, and the painting the remaining poles red, white, and blue. We may do this in $\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}3^{n-2k}$ ways. ∎

$*$11. We have n identical ping-pong balls. In how many ways may we paint them red, white, blue, and green?

Solution: We can line up the identical ping-pong balls and break them into four groups, those of each color, by inserting dividers. If we want to paint at least one in each color, we can choose three of the spaces between the balls in which to insert dividers, so we can paint them in $\binom{n-1}{3}$ ways. But the problem didn't require us to use each color, so we can put two dividers adjacent to each other. Thus there are $n+1$ places where we can put the first divider (putting it before all the balls means we use no red, and putting it after all of them means we use no green. Now there are $n+2$ places where we can put the second divider, including before or after the first, and $n+3$ places where we can put the third divider. However, if we interchange two dividers we still paint the balls before the first divider red, those between then next two white, and so on. Thus $3! = 6$ of these arrangements of balls and dividers correspond to the same paint job, so the number of ways to paint the balls is $\frac{(n+1)(n+2)(n+3)}{6} = \binom{n+3}{3}$. This suggests that another way to think of the problem is to consider $n+3$ slots in a row, and fill n of them with balls and 3 of them with dividers; since the balls are identical and the dividers might as well be identical, the number of ways to do this is the number of ways to choose the slots that get dividers. ∎

$*$12. We have n identical ping-pong balls. In how many ways may we paint them red, white, blue, and green if we use green paint on an even number of them?

Solution: We first decide how many balls to paint green, then paint the remainder with the other three colors as in Problem 11. This gives us

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-2k+2}{2}$$

ways to paint the balls. ∎

Chapter 2

Applications of Induction and Recursion in Combinatorics and Graph Theory

2.1 Some Examples of Mathematical Induction

If you are unfamiliar with the Principle of Mathematical Induction, you should read Appendix B (a portion of which is repeated here).

2.1.1 Mathematical induction

The **principle of mathematical induction** states that

> In order to prove a statement about an integer n, if we can
>
> 1. Prove the statement when $n = b$, for some fixed integer b, and
> 2. Show that the truth of the statement for $n = k - 1$ implies the truth of the statement for $n = k$ whenever $k > b$,
>
> then we can conclude the statement is true for all integers $n \geq b$.

As an example, let us give yet another proof that a set with n elements has 2^n subsets. This proof uses essentially the the same bijections we used in proving the Pascal Equation. The statement we wish to prove is the statement that "A set of size n has 2^n subsets."

> Our statement is true when $n = 0$, because a set of size 0 is the empty set and the empty set has $1 = 2^0$ subsets. (This step of our proof is called a *base step*.)
>
> Now suppose that $k > 0$ and every set with $k-1$ elements has 2^{k-1} subsets. Suppose $S = \{a_1, a_2, \ldots a_k\}$ is a set with k elements. We partition the subsets of S into two blocks. Block B_1 consists of the subsets that do not contain a_n and block B_2 consists of the subsets that do contain a_n. Each set in B_1 is a subset of $\{a_1, a_2, \ldots a_{k-1}\}$, and each subset of $\{a_1, a_2, \ldots a_{k-1}\}$ is in B_1. Thus B_1 is the set of all subsets of $\{a_1, a_2, \ldots a_{k-1}\}$. Therefore by our assumption in the first sentence of this paragraph, the size of B_1 is 2^{k-1}. Consider the function from B_2 to B_1 which takes a subset of S including a_k and removes a_k from it. This function is defined on B_2, because every set in B_2 contains a_k. This function is onto, because if T is a set in B_1, then $T \cup \{a_k\}$ is a set in B_2 which the function sends to T. This function is one-to-one because if V and W are two different sets in B_2, then removing a_k from them gives two different sets in B_1. Thus we have a bijection between B_1 and B_2, so B_1 and B_2 have the same size. Therefore by the sum principle the size of $B_1 \cup B_2$ is $2^{k-1} + 2^{k-1} = 2^k$. Therefore S has 2^k subsets. This shows that if a set of size $k-1$ has 2^{k-1} subsets, then a set of size k has 2^k subsets. Therefore by the principle of mathematical induction, a set of size n has 2^n subsets for every nonnegative integer n.

The first sentence of the last paragraph is called the *inductive hypothesis.* In an inductive proof we always make an inductive hypothesis as part of proving that the truth of our statement when $n = k-1$ implies the truth of our statement when $n = k$. The last paragraph itself is called the *inductive step* of our proof. In an inductive step we derive the statement for $n = k$ from the statement for $n = k-1$, thus proving that the truth of our statement when $n = k-1$ implies the truth of our statement when $n = k$. The last sentence in the last paragraph is called the *inductive conclusion.* All inductive proofs should have a base step, an inductive hypothesis, an inductive step, and an inductive conclusion.

There are a couple details worth noticing. First, in this problem, our base step was the case $n = 0$, or in other words, we had $b = 0$. However, in other proofs, b could be any integer, positive, negative, or 0. Second, our proof that the truth of our statement for $n = k-1$ implies the truth of our statement for $n = k$ required that k be at least 1, so that there would be an element a_k we could take away in order to describe our bijection. However, condition (2) of the principle of mathematical induction only requires that we be able to prove the implication for $k > 0$, so we were allowed to assume $k > 0$.

Strong Mathematical Induction

One way of looking at the principle of mathematical induction is that it tells us that if we know the "first" case of a theorem and we can derive each other case of the theorem from a smaller case, then the theorem is true in all cases. However, the particular way in which we stated the theorem is rather restrictive in that it requires us to derive each case from the immediately preceding case. This restriction is not necessary, and removing it leads us to a more general statement of the principal of mathematical induction which people often call the **strong principle of mathematical induction**. It states:

> In order to prove a statement about an integer n if we can
>
> 1. Prove our statement when $n = b$, and
> 2. Prove that the statements we get with $n = b$, $n = b + 1$, $\ldots n = k - 1$ imply the statement with $n = k$,
>
> then our statement is true for all integers $n \geq b$.

You will find some explicit examples of the use of the strong principle of mathematical induction in Appendix B and will find some uses for it in this chapter.

2.1.2 Binomial Coefficients and the Binomial Theorem

•72. When we studied the Pascal Equation and subsets in Chapter 1, it may have appeared that there is no connection between the Pascal relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ and the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Of course you probably realize you can prove the Pascal relation by substituting the values the formula gives you into the right-hand side of the equation and simplifying to give you the left hand side. In fact, from the Pascal Relation and the facts that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, you can actually prove the formula for $\binom{n}{k}$ by induction on n. Do so.

Solution: We wish to prove that $\binom{n}{i} = \frac{n!}{i!(n-i)!}$. We note that since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ are in agreement with this formula, we only have to consider the cases in which $0 < i < n$, which by the way, requires that $n \geq 2$. We will prove that our formula holds by induction on n for $n \geq 2$. If $n = 2$, the only i we need to consider is $i = 1$, and we know that $\binom{2}{1} = 2$, the number of one-element subsets of a two-element set. But $\frac{2!}{1!(2-1)!}$ is 2 also, so our formula holds when $n = 2$. Now suppose our formula holds when $n = k-1$, so that for every i with $0 < i < n-1$, $\binom{k-1}{i} = \frac{(k-1)!}{i!(k-1-i)!}$. Then by the Pascal Equation

$$\begin{aligned}
\binom{k}{i} &= \binom{k-1}{i-1} + \binom{k-1}{i} \\
&= \frac{(k-1)!}{(i-1)!(k-1-i+1)!} + \frac{(k-1)!}{i!(k-1-i)!} \\
&= \frac{(k-1)!i + (k-1)!(k-i)}{i!(k-i)!} \quad = \quad \frac{k!}{i!(k-i)!}.
\end{aligned}$$

Thus the truth of our formula for $n = k-1$ implies its truth for $n = k$. Therefore by the principle of mathematical induction, our formula is true for all integers $n \geq 2$. We have already seen is is true when $i = 0$ or $i = 1$, so it is true for all nonnegative n and all numbers i with $0 \leq i \leq n$. ▮

➔ 73. Use the fact that $(x+y)^n = (x+y)(x+y)^{n-1}$ to give an inductive proof of the binomial theorem.

Solution: We prove the binomial theorem by induction on n. When $n = 0$, $(x+y)^n = (x+y)^0 = 1 = \sum_{i=0}^{0} \binom{n}{i} x^{0-i} y^i$ since that last summation consists of the one term $\binom{0}{0} x^0 y^0$.

Now suppose that when $n = k-1$, $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$. This gives us

$$\begin{aligned}
(x+y)^k &= (x+y)(x+y)^{k-1} = (x+y) \sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-1-i} y^i \\
&= \sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-i} y^i + \sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-1-i} y^{i+1} \\
&= \sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-i} y^i + \sum_{i=1}^{k} \binom{k-1}{i-1} x^{k-i} y^i \\
&= x^k + \left(\sum_{i=1}^{k-1} \binom{k-1}{i} x^{k-i} y^i + \binom{k-1}{i-1} x^{k-i} y^i \right) + y^k
\end{aligned}$$

$$= \quad x^k + \left(\sum_{i=1}^{k-1}\binom{k}{i}x^{k-i}y^i\right) + y^k \;=\; \sum_{i=0}^{k}\binom{k}{i}x^{k-i}y^i.$$

Thus the truth of the binomial theorem for $n = k-1$ implies its truth for $n = k$. Then by the principle of mathematical induction, the binomial theorem must be true for all integers $n \geq 0$. ▮

74. Suppose that f is a function defined on the nonnegative integers such that $f(0) = 3$ and $f(n) = 2f(n-1)$. Find a formula for $f(n)$ and prove your formula is correct.

Solution: $f(n) = 3 \cdot 2^n$. We prove our formula is correct by induction. When $n = 0$ our formula gives $f(0) = 3$, which is what we were given. Now suppose that when $n = k-1$, $f(n) = 3 \cdot 2^n$. Then $f(k) = 2 \cdot f(k-1) = 2 \cdot 3 \cdot 2^{k-1} = 3 \cdot 2^k$. Therefore the truth of our formula when $n = k-1$ implies its truth when $n = k$ and so by the principle of mathematical induction, $f(n) = 3 \cdot 2^n$ for all nonnegative integers n. ▮

+75. Prove the conjecture in Problem 13b for an arbitrary positive integer m without appealing to the general product principle.

Solution: In Problem 13b we proved that that the conjecture follows from the general product principle. Now we prove by induction on m that there are n^m functions from $[m]$ to $[n]$. When $m = 0$ there is one function (the so-called "empty function" from $[m]$ to $[n]$). Now assume inductively that when $m = k-1$ there are n^{k-1} functions from $[m]$ to $[n]$. For each value i between 1 and n there is a bijection between the functions from $[k-1]$ to $[n]$ and the functions f from k to n with $f(k) = i$. Thus the set of all functions from $[k]$ to $[n]$ is a union of n sets of size n^{k-1} and so by the ordinary product principle this set has size n^k. Thus by the principle of mathematical induction, the number of functions from m to n is m^n for all nonnegative integers m. ▮

2.1.3 Inductive definition

You may have seen $n!$ described by the two equations $0! = 1$ and $n! = n(n-1)!$ for $n > 0$. By the principle of mathematical induction we know that this pair of equations defines $n!$ for all nonnegative numbers n. For this reason we call such a definition an **inductive definition**. An inductive definition is sometimes called a *recursive definition*. Often we can get very easy proofs of useful facts by using inductive definitions.

➔ 76. An inductive definition of a^n for nonnegative n is given by $a^0 = 1$ and $a^n = aa^{n-1}$. (Notice the similarity to the inductive definition of $n!$.) We remarked above that inductive definitions often give us easy proofs of useful facts. Here we apply this inductive definition to prove two useful facts about exponents that you have been using almost since you learned the meaning of exponents.

(a) Use this definition to prove the rule of exponents $a^{m+n} = a^m a^n$ for nonnegative m and n.

Solution: We use induction on n to prove this. When $n = 0$, the formula gives us $a^{m+0} = a^m a^0 = a^m \cdot 1 = a^m$, so the rule of exponents holds when $n = 0$. Now assume it holds when $n = k-1$ so that $a^{m+k-1} = a^m a^{k-1}$. Then, starting and ending with our inductive definition, we may write

$$a^{m+n} = aa^{m+n-1} = aa^m a^{k-1} = a^m \cdot a \cdot a^{k-1} = a^m a^k.$$

Thus the truth of our law for $n = k - 1$ implies its truth for $n = k$. Therefore, by the principle of mathematical induction, $a^{m+n} = a^m a^n$ for all nonnegative integers n. ∎

(b) Use this definition to prove the rule of exponents $a^{mn} = (a^m)^n$.

Solution: We will use induction on n and part (a) of this problem to prove that $a^{mn} = (a^m)^n$. First, when $n = 0$ the left and right hand sides of the equation are both 1, so $a^{mn} = (a^m)^n$ holds when $n = 0$. Now assume that $a^{m(k-1)} = (a^m)^{k-1}$. This may be rewritten as $a^{mk-m} = (a^m)^{k-1}$. Multiply both sides by a^m and apply part (a) of the problem and then the inductive definition (with a^m replacing a) to get

$$\begin{aligned} a^{mk-m}a^m &= (a^m)^{k-1}a^m \\ a^{mk} &= (a^m)^{k-1}a^m \\ a^{mk} &= (a^m)^k. \end{aligned}$$

Thus the truth of our formula when $n = k - 1$ implies its truth when $n = k$. Therefore by the principle of mathematical induction, the formula is true for all nonnegative integers n. ∎

+ 77. Suppose that f is a function on the nonnegative integers such that $f(0) = 0$ and $f(n) = n + f(n - 1)$. Prove that $f(n) = n(n + 1)/2$. Notice that this gives a third proof that $1 + 2 + \cdots + n = n(n + 1)/2$, because this sum satisfies the two conditions for f. (The sum has no terms and is thus 0 when $n = 0$.)

Solution: We prove the formula for f by induction on n. If $n = 0$, then $n(n+1)/2 = 0$ which is what we were given. Now assume that $f(k-1) = (k-1)k/2$. Then $f(k) = k + f(k-1) = k + (k-1)k/2 = (k^2 + 2k - k)/2 = k(k+1)/2$. Therefore the truth of the formula for $n = k-1$ implies its truth for $n = k$, and thus by the principle of mathematical induction, the formula for f holds for all nonnegative integers n. ▮

➔ 78. Give an inductive definition of the summation notation $\sum_{i=1}^{n} a_i$. Use it and the distributive law $b(a+c) = ba + bc$ to prove the distributive law

$$b\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ba_i.$$

Solution: We define $\sum_{i=1}^{1} a_i = a_1$ and for $n > 1$, $\sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n$. When $n = 1$, $b\sum_{i=1}^{1} a_i = ba_1$ by the base step of our inductive definition. Assume that $k > 1$ and $b\sum_{i=1}^{k-1} a_i = \sum_{i=1}^{k-1} ba_i$. Now we can write

$$b\sum_{i=1}^{k} a_i = b\left[\left(\sum_{i=1}^{k-1} a_i\right) + a_k\right] = \left(b\sum_{i=1}^{k-1} a_i\right) + ba_k = \left(\sum_{i=1}^{k-1} ba_i\right) + ba_k = \sum_{i=1}^{k} ba_i,$$

where the last step is justified by the inductive step of our inductive definition with a_i replaced by ba_i. Thus the truth of our statement for $k-1$ implies its truth for $i = k$, and therefore by the principle of mathematical induction, for all positive integers n, $b\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ba_i$. ▮

2.1.4 Proving the general product principle (Optional)

We stated the sum principle as

> If we have a partition of a finite set S, then the size of S is the sum of the sizes of the blocks of the partition.

In fact, the simplest form of the sum principle says that the size of the sum of two disjoint (finite) sets is the sum of their sizes.

79. Prove the sum principle we stated for partitions of a set from the simplest form of the sum principle.

Solution: We prove by induction on n that the size of the union of n disjoint sets is the sum of their sizes. We assume that the size of the union of two disjoint sets is the sum of their sizes. Now assume $k > 2$ and the size of the union of $k-1$ disjoint sets is the sum of their sizes. Then we may write

$$|\cup_{i=1}^{k} S_i| = |\left(\cup_{i=1}^{k-1} S_i\right) \cup S_k| = \left(\sum_{i=1}^{k-1} |S_i|\right) + |S_k| = \sum_{i=1}^{k} |S_i|.$$

Thus whenever the size of the union of $k-1$ disjoint sets is the sum of their sizes, then the size of a union of k disjoint sets is the sum of their sizes. Thus by the principle of mathematical induction, the size of the union of n disjoint sets is the sum of their sizes for all $n > 1$. The statement holds trivially when $n = 1$ as well. ▮

We stated the partition form of the product principle as

> If we have a partition of a finite set S into m blocks, each of size n, then S has size mn.

In Problem 11 we gave a more general form of the product principle which can be stated as

> If we make a sequence of m choices for which
>
> - there are k_1 possible first choices, and
> - for each way of making the first $i-1$ choices, there are k_i ways to make the ith choice,
>
> then we may make our sequence of choices in $k_1 \cdot k_2 \cdot \cdots \cdot k_m = \prod_{i=1}^{m} k_i$ ways.

In Problem 14 we stated the general product principle as follows.

> Let S be a set of functions f from $[n]$ to some set X. Suppose that
>
> - there are k_1 choices for $f(1)$, and
> - for each choice of $f(1)$, $f(2)$, $\ldots f(i-1)$, there are k_i choices for $f(i)$.
>
> Then the number of functions in the set S is $k_1 k_2 \cdots k_n$.

You may use either way of stating the general product principle in the following Problem.

+80. Prove the general form of the product principle from the partition form of the product principle.

Solution: We prove by induction that if S is a set of functions defined on $[m]$ such that

- there are k_1 choices for $f(1)$ and
- when $2 \leq i \leq m$, for each choice of $f(1)$, $f(2)$, $\ldots f(i-1)$, there are k_i choices for $f(i)$,

then there are $\prod_{i=1}^{m} k_i$ functions in S. When $m = 1$, the product is k_1 and there are k_1 functions in S. Now assume inductively that when S' is a set of functions defined on $[m-1]$ such that

- there are k_1 choices for $f(1)$ and
- when $2 \leq i \leq m-1$, for each choice of $f(1)$, $f(2)$, $\ldots f(i-1)$, there are k_i choices for $f(i)$,

then there are $\prod_{i=1}^{m-1} k_i$ functions in S'. Now partition S into k_1 sets S_j, where S_j is the set of functions f in S with $f(n) = y_j$ for each of the k_n values y_j that are possible for $f(1)$. Thus S is a union of k_n sets S_j each of size $\prod_{i=1}^{m-1} k_i$ (by the inductive hypothesis), and so by the product principle for unions of sets, S has size $\prod_{i=1}^{m} k_i$. Therefore, by the principle of mathematical induction, we have proved the general product principle. ▮

2.1.5 Double Induction and Ramsey Numbers

In Section 1.3.4 we gave two different descriptions of the Ramsey number $R(m,n)$. However, if you look carefully, you will see that we never showed that Ramsey numbers actually exist; we merely described what they were and showed that $R(3,3)$ and $R(3,4)$ exist by computing them directly. As long as we can show that there is some number R such that when there are R people together, there are either m mutual acquaintances or n mutual strangers, this shows that the Ramsey Number $R(m,n)$ exists, because it is the smallest such R. Mathematical induction allows us to show that one such R is $\binom{m+n-2}{m-1}$. The question is, what should we induct on, m or n? In other words, do we use the fact that with $\binom{m+n-3}{m-2}$ people in a room there are at least $m-1$ mutual acquaintances or n mutual strangers, or do we use the fact that with at least $\binom{m+n-3}{n-2}$ people in a room there are at least m mutual acquaintances or at least $n-1$ mutual strangers? It turns out that we use both. Thus we want to be able to simultaneously induct on m and n. One way to do that is to use yet another variation on the principle of mathematical induction, the *Principle of Double Mathematical Induction*. This principle (which can be derived from one of our earlier ones) states that

> In order to prove a statement about integers m and n, if we can
>
> 1. Prove the statement when $m = a$ and $n = b$, for fixed integers a and b
> 2. Prove the statement when $m = a$ and $n > b$ and when $m > a$ and $n = b$ (for the same fixed integers a and b),
> 3. Show that the truth of the statement for $m = j$ and $n = k - 1$ (with $j \geq a$ and $k > j$) and the truth of the statement for $m = j - 1$ and $n = k$ (with $j > a$ and $k \geq b$) imply the truth of the statement for $m = j$ and $n = k$,
>
> then we can conclude the statement is true for all pairs of integers $m \geq a$ and $n \geq b$.

There is a strong version of double induction, and it is actually easier to state. The principle of *strong double mathematical induction* says the following.

> In order to prove a statement about integers m and n, if we can
>
> 1. Prove the statement when $m = a$ and $n = b$, for fixed integers a and b
> 2. Show that the truth of the statement for values of m and n with $a + b \leq m + n < k$ implies the truth of the statement for $m + n = k$,
>
> then we can conclude that the statement is true for all pairs of integers $m \geq a$ and $n \geq b$.

➔ ·81. Prove that $R(m,n)$ exists by proving that if there are $\binom{m+n-2}{m-1}$ people in a room, then there are either at least m mutual acquaintances or at least n mutual strangers.

Solution: We use double induction on m and n to prove this for $m, n \geq 2$. Note first that $R(m,2) = m = \binom{m+2-2}{m-1}$ and $R(2,n) = n = \binom{2+n-2}{1}$. (In words, if there are m people in a room, then either all m people know each other or there are two mutual nonacquaintances, and if there are n people in a room, then either there are two people who know each other or they are all mutual strangers.) Note that we have covered both steps 1 and 2 of a double induction proof now. Now assume that whenever there are $\binom{m+n-3}{m-1}$ people in a room there are either at least m mutual acquaintances or $n-1$ mutual strangers, and that whenever there are at least $\binom{m+n-3}{m-2}$ people in a room there are either at least $m-1$ mutual acquaintances or n mutual strangers. Suppose that we have $\binom{m+n-2}{m-1}$ people in a room. Choose a person, say P. Then since $\binom{m+n-2}{m-1} = \binom{m-n-3}{m-1} + \binom{m+n-3}{m-2}$, person P is, by the generalized pigeonhole principle, either acquainted with $\binom{m+n-3}{m-2}$ people or unacquainted with $\binom{m+n-3}{m-1}$ people. In the first case, among the people with whom P is acquainted, either $m-1$ are mutual acquaintances or n are mutual strangers. If m are mutual strangers, we are done, while if $m-1$ are mutual acquaintances, these $m-1$ people, together with person P, are m mutual acquaintances, in which case we are done as well. In the second case, among the $\binom{m+n-3}{m-1}$ people with whom person P is unacquainted, there are either m mutual acquaintances, in which case we are done, or there are $n-1$ mutual strangers. In this event, these $n-1$ mutual strangers, along with person P make up n mutual strangers. Thus in every case, if we know that with $\binom{m+n-3}{m-2}$ people in a room there are either $m-1$ mutual acquaintances or n mutual strangers, and we know that with $\binom{m+n-3}{m-1}$ people in a room there are either m mutual acquaintances or $n-1$ mutual strangers, we can conclude that with $\binom{m+n-2}{m-1}$ people in a room there are either m mutual acquaintances or n mutual strangers. Therefore by the principle of double mathematical induction we know that for all m and n greater than or equal to 2, if there are $\binom{m+n-2}{m-1}$ people in a room, then there are either m mutual acquaintances of n mutual strangers. This shows that $R(m,n)$ exists and is no more than $\binom{m+n-2}{m-1}$. ▮

·82. Prove that $R(m,n) \leq R(m-1,n) + R(m,n-1)$.

Solution: If there are $R(m-1,n) + R(m,n-1)$ people in a room, choose one person, say person P. By the generalized pigeonhole principle, there are either $R(m-1, n)$ people with whom P is acquainted or $R(m, n-1)$ people with whom person P is unacquainted. In the first case, among the people with whom person P is acquainted, there are either n mutual strangers, in which case we are done, or there are $m-1$ people with whom person P is acquainted. These $m-1$ people and person P form m people who are mutually acquainted, and so we have m mutual acquaintances. On the other hand, if P is unacquainted with $R(m, n-1)$ people, then among these people, there are either m mutually acquainted people, in which case we are done, or among these people there are $m-1$ mutually unacquainted people, and these $m-1$ people together with P make m mutual strangers. Thus in every case, if there are $R(m-1, n) + R(m, n-1)$ people in a room, there are either at least m mutual acquaintances or at least n mutual strangers. Therefore $R(m, n) \leq R(m-1, n) + R(m, n-1)$. ▮

➔ ·83. (a) What does the equation in Problem 82 tell us about $R(4, 4)$?

Solution: $R(4, 4) \leq R(3, 4) + R(4, 3) = 9 + 9 = 18$. ▮

∗(b) Consider 17 people arranged in a circle such that each person is acquainted with the first, second, fourth, and eighth person to the right and the first, second, fourth, and eighth person to the left. Can you find a set of four mutual acquaintances? Can you find a set of four mutual strangers?

Solution: You cannot find either. If there were a set of four mutual acquaintances, you could assume by symmetry that it includes person 1, and two people from among those one, two, four, and eight places to the right. Thus you can assume your set of four acquaintances contains person 1 and two from among persons 2, 3, 5, and 9. However, persons 2 and 5, 2 and 9 and 3 and 9 are not acquainted. Thus three of the mutually acquainted people are either persons 1, 2, and 3, persons 1, 5, and 9 or persons 1, 3, and 5. However, person 5 is not acquainted with the person one, two, or eight places to the left of person 1, so if person 5 is in the set of mutual acquaintances, then person 14 must be as well. However, person 3 and person 9 are not acquainted with person 14. Thus our set must contain persons 1, 2, and 3. However, person 3 is not acquainted with the person one, two, four, or eight persons to the left of person 1, so there is no set of four mutual acquaintances. A similar argument holds for nonacquaintances. ▮

(c) What is $R(4,4)$?

Solution: 18. ▮

84. (Optional) Prove the inequality of Problem 81 by induction on $m+n$.

Solution: We want to prove that if $m \geq 2$ and $n \geq 2$, then when there are $\binom{m+n-2}{m-1}$ people in a room, there are either m mutual acquaintances or n mutual strangers. If $m+n=4$, then $m=2$ and $n=2$, and if there are $\binom{2+2-2}{1}=2$ people in a room, there are either two who know each other or two who don't.

Now assume that when $m+n=k-1$, it is the case that if there are $\binom{m+n-2}{m-1}$ people in a room, then there are either m mutual acquaintances or n mutual strangers. Suppose that $m'+n'=k$ and there are $\binom{m'+n'-2}{m'-1}$ people in a room. If $n'=2$, then we know that with $\binom{m'+2-2}{m'-1}=m'$ people in a room, there are either m' mutual acquaintances or two mutual strangers, and similarly if $m'=2$ there are either two mutual acquaintances or n mutual strangers among $\binom{m'+n'-2}{m-1}$ people. Thus we may assume that both m' and n' are greater than two. Since $\binom{m'+n'-2}{m-1} = \binom{(m'-1)+n-2}{m'-2} + \binom{m+(n-1)-2}{m-1}$, if there are $\binom{m'+n'-2}{m'-1}$ people in a room, then a given person, say person P, is either acquainted with $\binom{(m'-1)+n'-2}{m'-1}$ of them (call this case 1) or is a stranger with $\binom{m'+(n-1)-2}{m-1}$ of them (call this case 2). Notice that $m'-1+n'=k-1$ and $m'+n'-1=k-1$. Thus in case 1, our inductive hypothesis tells us that either $m'-1$ of person P's acquaintances are mutually acquainted, in which case they and person P form m' mutual acquaintances, or n' of P's acquaintances are mutual strangers, in which case we have n' mutual strangers. Similarly in case 2 we have either m' mutual acquaintances or n' mutual strangers. Thus by the principle of mathematical induction, for all values of $m+n$ greater than or equal to 4, if we have $\binom{m+n-2}{m-1}$ people in a room, then we have either m mutual acquaintances or n mutual strangers, so that $R(m,n)$ exists and is no more than $\binom{m+n-2}{m-1}$. ▮

85. Use Stirling's approximation (Problem 46) to convert the upper bound for $R(n,n)$ that you get from Problem 81 to a multiple of a power of an integer.

Solution: $R(n,n) \leq \binom{n+n-2}{m-1} = \frac{(2n-2)!}{(n-1)!^2}$. For n sufficiently large, this is approximately

$$
\begin{aligned}
& \frac{\sqrt{2\pi(2n-2)}(2n-2)^{2n-2}/e^{2n-2}}{\sqrt{2\pi(n-1)}(n-1)^{n-1}\sqrt{2\pi(n-1)}(n-1)^{n-1}/e^{n-1}e^{n-1}} \\
= & \frac{2^{2n-2}(n-1)^{2n-2}}{\sqrt{\pi(n-1)}(n-1)^{2n-2}} \\
= & \frac{1}{\sqrt{\pi(n-1)}}2^{2n-2}
\end{aligned}
$$

▮

2.1.6 A bit of asymptotic combinatorics

Problem 85 gives us an upper bound on $R(n,n)$. A very clever technique due to Paul Erdös, called the "probabilistic method," will give a lower bound. Since both bounds are exponential in n, they show that $R(n,n)$ grows exponentially as n gets large. An analysis of what happens to a function of n as n gets large is usually called an *asymptotic analysis*. The *probabilistic method*, at least in its simpler forms, can be expressed in terms of averages, so one does not need to know the language of probability in order to understand it. We will apply it to Ramsey numbers in the next problem. Combined with the result of Problem 85, this problem will give us that $\sqrt{2}^{\,n} < R(n,n) < 2^{2n-2}$, so that we know that the Ramsey number $R(n,n)$ grows exponentially with n.

➔ 86. Suppose we have two numbers n and m. We consider all possible ways to color the edges of the complete graph K_m with two colors, say red and blue. For each coloring, we look at each n-element subset N of the vertex set M of K_m. Then N together with the edges of K_m connecting vertices in N forms a complete graph on n vertices. This graph, which we denote by K_N, has its edges colored by the original coloring of the edges of K_m.

(a) Why is it that, if there is no subset $N \subseteq M$ so that all the edges of K_N are colored the same color for any coloring of the edges of K_m, then $R(n,n) > m$?

Solution: Another way to say there is no such subset is to say that it is not possible to find a K_n all of whose edges are red or a K_n all of whose edges are blue. This means that $R(n,n) > n$. ▮

(b) To apply the probabilistic method, we are going to compute the average, over all colorings of K_m, of the number of sets $N \subseteq M$

with $|N| = n$ such that K_N *does* have all its edges the same color. Explain why it is that if the average is less than 1, then for some coloring there is no set N such that K_N has all its edges colored the same color. Why does this mean that $R(n,n) > m$?

Solution: If the average of n nonnegative integers is less than one, they cannot all be one or more, so one has to be zero. Thus in this context there must be some coloring that has no set N so that K_N has all its edges colored the same color. ▮

(c) We call a K_N *monochromatic* for a coloring c of K_m if the color $c(e)$ assigned to edge e is the same for every edge e of K_N. Let us define $\mathrm{mono}(c,N)$ to be 1 if N is monochromatic for c and to be 0 otherwise. Find a formula for the average number of monochromatic K_Ns over all colorings of K_m that involves a double sum first over all edge colorings c of K_m and then over all n-element subsets $N \subseteq M$ of $\mathrm{mono}(c,N)$.

Solution:

$$\frac{1}{2^{\binom{m}{2}}} \sum_{c:c \text{ is a coloring of } K_m} \ \sum_{N:N\subseteq M,\ |N|=n} \mathrm{mono}(c,N).$$

▮

(d) Show that your formula for the average reduces to $2\binom{m}{n}\cdot 2^{-\binom{n}{2}}$

Solution:

$$\begin{aligned}
& \frac{1}{2^{\binom{m}{2}}} \sum_{c:c \text{ is a coloring of } K_m} \ \sum_{N:N\subseteq M,\ |N|=n} \mathrm{mono}(c,N) \\
= \ & \frac{1}{2^{\binom{m}{2}}} \sum_{N:N\subseteq M,\ |N|=n} \ \sum_{c:c \text{ is a coloring of } K_m} \mathrm{mono}(c,N) \\
= \ & 2^{-\binom{m}{2}} \sum_{N:N\subseteq M,\ |N|=n} 2\cdot 2^{\binom{m}{2}-\binom{n}{2}} \\
= \ & 2\binom{m}{n} 2^{-\binom{n}{2}}
\end{aligned}$$

▮

(e) Explain why $R(n,n) > m$ if $\binom{m}{n} \le 2^{\binom{n}{2}-1}$.

Solution: $R(n,n) > m$ if the average above is less than 1. Thus $R(n,n) > m$ if $2\binom{m}{2}2^{-\binom{n}{2}} < 1$, which is equivalent to $\binom{m}{n} < 2^{\binom{n}{2}-1}$. ▮

$*$(f) Explain why $R(n,n) > \sqrt[n]{n!2^{\binom{n}{2}-1}}$.

Solution: $\binom{m}{n} < 2^{\binom{n}{2}-1}$ is the same as $\frac{m^{\underline{n}}}{n!} < 2^{\binom{n}{2}-1}$. And since $m^{\underline{n}} < m^n$, the inequality $\frac{m^{\underline{n}}}{n!} < 2^{\binom{n}{2}-1}$ holds if the inequality $\frac{m^n}{n!} \leq 2^{\binom{n}{2}-1}$ holds. And this last inequality holds if $m \leq \sqrt[n]{n!2^{\binom{n}{2}-1}}$ holds. Thus $R(n,n) > m$ for any m such that $m \leq \sqrt[n]{n!2^{\binom{n}{2}-1}}$, which implies that $R(n,n) > \sqrt[n]{n!2^{\binom{n}{2}-1}}$. ∎

(g) By using Stirling's formula, show that if n is large enough, then $R(n,n) > \sqrt{2^n} = \sqrt{2}^{\,n}$. (Here large enough means large enough for Stirling's formula to be reasonably accurate.)

Solution: Using Stirling's approximation to $n!$ we get

$$R(n,n) > \sqrt[n]{\frac{n^n}{e^n}\sqrt{2\pi n}2^{\frac{n^2-n-2}{2}}} = \frac{n}{e}2^{\frac{n^2-n-2}{2n}}\sqrt[2n]{2\pi n} > 2^{n/2} = \sqrt{2}^{\,n}.$$

∎

2.2 Recurrence Relations

87. How is the number of subsets of an n-element set related to the number of subsets of an $(n-1)$-element set? Prove that you are correct.

Solution: Suppose that our n-element set is $N = \{a_1, a_2, \ldots, a_n\}$. Then a subset of N either contains a_n or it doesn't. In our discussion of the Pascal recurrence, we showed that the number of k-element subsets of N that contain a_n is the same as the number of $(k-1)$-element subsets of $N - \{a_n\}$. The bijection we used to prove this consists of taking a_n away from a set containing a_n. Thus the number of subsets of N containing a_n is the same as the number of subsets of the $(n-1)$-element set $N - \{a_n\}$. But the subsets of N not containing a_n are exactly the same as the subsets of $N - \{a_n\}$. Thus we can partition the subsets of N into two blocks, each of which has size equal to the number of subsets of $N - \{a_n\}$. Therefore, by the sum principle, the number of subsets of N is twice the number of subsets of $N - \{a_n\}$. ∎

88. Explain why it is that the number of bijections from an n-element set to an n-element set is equal to n times the number of bijections from an $(n-1)$-element subset to an $(n-1)$-element set. What does this have to do with Problem 27?

Solution: To specify a bijection f from an n-element set $\{a_1, a_2, \ldots a_n\}$ to an n-element set, we have n choices for $f(a_1)$, and then b_{n-1} choices for how to define f from the elements $\{a_2, a_3, \ldots, a_n\}$ to the remaining $n-1$ elements of our range. By the product principle this gives us $b_n = nb_{n-1}$ for the number b_n of bijections from an n-element set to an n-element set. ▮

We can summarize these observations as follows. If s_n stands for the number of subsets of an n-element set, then

$$s_n = 2s_{n-1}, \tag{2.1}$$

and if b_n stands for the number of bijections from an n-element set to an n-element set, then

$$b_n = nb_{n-1}. \tag{2.2}$$

Equations 2.1 and 2.2 are examples of *recurrence equations* or *recurrence relations*. A **recurrence relation** or simply a **recurrence** is an equation that expresses the nth term of a sequence a_n in terms of values of a_i for $i < n$. Thus Equations 2.1 and 2.2 are examples of recurrences.

2.2.1 Examples of recurrence relations

Other examples of recurrences are

$$a_n = a_{n-1} + 7, \tag{2.3}$$

$$a_n = 3a_{n-1} + 2^n, \tag{2.4}$$

$$a_n = a_{n-1} + 3a_{n-2}, \text{ and} \tag{2.5}$$

$$a_n = a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1. \tag{2.6}$$

A **solution** to a recurrence relation is a sequence that satisfies the recurrence relation. Thus a solution to Recurrence 2.1 is the sequence given by $s_n = 2^n$. Note that $s_n = 17 \cdot 2^n$ and $s_n = -13 \cdot 2^n$ are also solutions to Recurrence 2.1. What this shows is that a recurrence can have infinitely many solutions. In a given problem, there is generally one solution that is of interest to us. For example, if we are interested in the number of subsets of a set, then the solution to Recurrence 2.1 that we care about is $s_n = 2^n$. Notice this is the only solution we have mentioned that satisfies $s_0 = 1$.

89. Show that there is only one solution to Recurrence 2.1 that satisfies $s_0 = 1$.

    Solution: We prove by induction on n that there is one and only one value s_n that satisfies both $s_n = 2s_{n-1}$ for $n > 0$ and $s_0 = 1$. First, there is clearly one and only one value s_0 that satisfies $s_0 = 1$. Now assume that $k > 0$ and there is one and only one value s_{k-1} that satisfies the two equations. Then $s_k = 2s_{k-1}$ is the one and only one value that satisfies the two equations. Therefore by the principle of mathematical induction, for all nonnegative integers n there is one and only one value s_n that satisfies the equations $s_0 = 1$ and $s_k = 2s_{k-1}$ for all $k > 0$. (Note that since we were making a statement about s_n for all nonnegative integers n it was not appropriate to use n as the dummy variable in the recursive equation $s_k = 2s_{k-1}$.) ▮

90. A first-order recurrence relation is one which expresses a_n in terms of a_{n-1} and other functions of n, but which does not include any of the terms a_i for $i < n - 1$ in the equation.

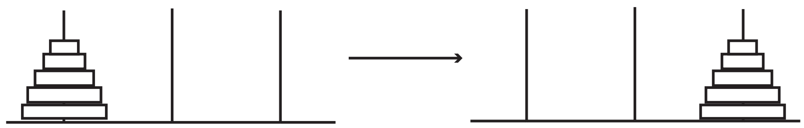
    (a) Which of the recurrences 2.1 through 2.6 are first order recurrences?

    Solution: The recurrences 2.1, 2.2, 2.3, and 2.4 are all examples of first order recurrences. The recurrences 2.5 and 2.6 are not. ▮

    (b) Show that there is one and only one sequence a_n that is defined for every nonnegative integer n, satisfies a given first order recurrence, and satisfies $a_0 = a$ for some fixed constant a.

    Solution: A first order recurrence will give a_n in terms of a_{n-1}, that is, there will be a function f such that $a_n = f(a_{n-1})$ for all $n > 0$. We prove by induction that there is one and only one solution to a first order recurrence that satisfies $a_0 = a$ for some fixed constant a. First, there is one and only one value for a_0. Now suppose that when $n = k - 1$, there is one and only one value possible for a_{n-1}. Then a_k is uniquely determined by $a_k = f(a_{k-1})$. Thus the truth of the statement that a_{k-1} is uniquely determined by the equations $a_0 = a$ and $a_n = f(a_{n-1})$ implies the truth of the statement that a_k is determined uniquely by the equations $a_0 = a$ and $a_n = f(a_{n-1})$. Therefore by the principle of mathematical induction, a_k is uniquely determined by the equations $a_0 = a$ and $a_n = f(a_{n-1})$ for all nonnegative integers k. ▮

Figure 2.1: The Towers of Hanoi Puzzle



➔ 91. The “Towers of Hanoi” puzzle has three rods rising from a rectangular base with n rings of different sizes stacked in decreasing order of size on one rod. A legal move consists of moving a ring from one rod to another so that it does not land on top of a smaller ring. If m_n is the number of moves required to move all the rings from the initial rod to another rod that you choose, give a recurrence for m_n.

Solution: We can solve the puzzle in one step if there is one ring, so $m_1 = 1$. If $n > 0$ and we want to move all the rings from the initial rod to rod 3, then first we solve the problem of moving all but the bottom ring to rod 2; this takes m_{n-1} steps, then we move the bottom ring to rod 3, then we solve the problem of moving all the remaining rings from rod 2 to rod 3. Thus we have $m_n = 2m_{n-1} + 1$. ▮

➔ 92. We draw n mutually intersecting circles in the plane so that each one crosses each other one exactly twice and no three intersect in the same point. (As examples, think of Venn diagrams with two or three mutually intersecting sets.) Find a recurrence for the number r_n of regions into which the plane is divided by n circles. (One circle divides the plane into two regions, the inside and the outside.) Find the number of regions with n circles. For what values of n can you draw a Venn diagram showing all the possible intersections of n sets using circles to represent each of the sets?

Solution: One circle defines two regions, the inside and outside. When we draw a second circle that intersects the first, we can start at one of the intersection points and go inside the first circle, cutting its region into two pieces, and then when we leave it we cut the outside region into two pieces. This suggests the general pattern. If we have drawn $n-1$ circles and start a new one, each time we enter a circle, we start dividing a region into two pieces. Each time we leave a circle, we also start dividing a region into two pieces. Thus if we have r_n regions with n circles, to get the number of regions, we note that in going from $n-1$ circles to n circles, we start with r_{n-1} regions, and divide $2(n-1)$ of them in half, so we get $2n-2$ new regions. This gives us $r_n = r_{n-1} + 2(n-1)$. Notice that by substitution of the formula $r_{n-1} = r_{n-2} + 2(n-2)$, we get $r_n = r_{n-2} + 2(n-2) + 2(n-1)$, and would guess that $r_n = r_{n-3} + 2(n-3) + 2(n-2) + 2(n-1)$. This leads to the conjecture

$$r_n = r_1 + 2 \cdot 1 + 2 \cdot 2 + \cdots + 2 \cdot (n-1) = r_1 + 2\sum_{i=1}^{n-1} i = 2 + n(n-1).$$

We can prove this formula by induction. When $n = 1$ we have $2 + 1 \cdot 0$ regions. Assuming that $n-1$ circles give us $2 + (n-1)(n-2)$ regions, for n circles we have $2 + (n-1)(n-2) + 2(n-1) = 2 + n(n-1)$ regions. Thus the correctness of our formula for $n-1$ circles implies its correctness when we have n circles, so for all positive integers n, we get $2 + n(n-1)$ regions determined by n mutually intersecting circles. Two circles cannot touch more than twice, and if we let some of our n circles touch just once, or not at all, that would reduce the number of regions we would get. Similarly, allowing a circle to go through the intersection point of two other circles could only reduce the number of regions. So with n circles we could never have more than $2 + n(n-1)$ regions. In particular with 4 circles we get just 14 regions, rather than the 16 that would be required in a Venn diagram for four sets. We could prove, again by induction, that $2 + n(n-1) < 2^n$ for all $n > 3$, so it is not possible to draw a Venn diagram using circles to illustrate the intersections of four or more sets. ∎

2.2.2 Arithmetic Series (optional)

93. A child puts away two dollars from her allowance each week. If she starts with twenty dollars, give a recurrence for the amount a_n of money she has after n weeks and find out how much money she has at the end of n weeks.

    Solution: $a_n = a_{n-1} + 2$. Then by substitution $a_n = a_{n-2} + 2 + 2$, and so we conjecture that $a_n = 20 + 2n$. Since she adds two dollars to her savings each week for n weeks, she has added $2n$ dollars to her original 20, which proves the formula. We could have used induction to prove it as well. ∎

94. A sequence that satisfies a recurrence of the form $a_n = a_{n-1} + c$ is called an *arithmetic progression*. Find a formula in terms of the initial

value a_0 and the common difference c for the term a_n in an arithmetic progression and prove you are right.

Solution: $a_n = a_o + cn$. The formula is valid with $n = 0$, and if $a_{n-1} = a_0 + c(n-1)$, then $a_n = a_0 + c(n-1) + c = a_0 + cn$. Therefore the fact that $a_{n-1} = a_0 + ca_{n-1}$ implies the fact that $a_n = a_0 + cn$. Therefore by the principle of mathematical induction, $a_n = a_0 + cn$ for all nonnegative integers n. ∎

95. A person who is earning \$50,000 per year gets a raise of \$3000 a year for n years in a row. Find a recurrence for the amount a_n of money the person earns over $n+1$ years. What is the total amount of money that the person earns over a period of $n+1$ years? (In $n+1$ years, there are n raises.)

    Solution: By Problem 94 we saw that if b_n is the salary in year n, then $b_n = 50,000 + 3000n$. If a_n is the total amount earned over the period of from year 0 through the end of year n, a period of $n+1$ years, then $a_n = a_{n-1} + b_n = a_{n-1} + 50,000 + 3000n$. Further, $a_n = \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} 50,000 + 3000i = 50,000(n+1) + 3000\sum_{i=0}^{n} i = 50,000(n+1) + 1500(n(n+1)$. ∎

96. An *arithmetic series* is a sequence s_n equal to the sum of the terms a_0 through a_n of an arithmetic progression. Find a recurrence for the sum s_n of an arithmetic progression with initial value a_0 and common difference c (using the language of Problem 94). Find a formula for general term s_n of an arithmetic series.

    Solution: $s_n = \sum_{i=0}^{n} a_0 + ci = (n+1)a_0 + c\sum_{i=0}^{n} i = (n+1)a_0 + cn(n+1)/2$. ∎

2.2.3 First order linear recurrences

Recurrences such as those in Equations 2.1 through 2.5 are called *linear recurrences*, as are the recurrences of Problems 91 and 92. A **linear recurrence** is one in which a_n is expressed as a sum of functions of n times values of (some of the terms) a_i for $i < n$ plus (perhaps) another function (called the *driving function*) of n. A linear equation is called *homogeneous* if the driving function is zero (or, in other words, there is no driving function). It is called a *constant coefficient linear recurrence* if the functions that are multiplied by the a_i terms are all constants (but the driving function need not be constant).

97. Classify the recurrences in Equations 2.1 through 2.5 and Problems 91 and 92 according to whether or not they are constant coefficient, and whether or not they are homogeneous.

    Solution: Recurrence 2.1 is first order, linear, constant coefficient, and homogeneous. Recurrence 2.2 is first order, linear, and homogeneous, but not constant coefficient. Recurrence 2.3 is first order, linear, constant coefficient but not homogeneous. Recurrence 2.4 is first order, linear, constant coefficient but not homogeneous. Recurrence 2.5 is not first order (it is second order), is linear, constant coefficient and homogeneous. The recurrence of Problem 91 is first order, linear, and constant coefficient, and that of Problem 92 is first order, linear, and constant coefficient. ∎

•98. As you can see from Problem 97 some interesting sequences satisfy first order linear recurrences, including many that have constant coefficients, have constant driving term, or are homogeneous. Find a formula in terms of b, d, a_0 and n for the general term a_n of a sequence that satisfies a constant coefficient first order linear recurrence $a_n = ba_{n-1} + d$ and prove you are correct. If your formula involves a summation, try to replace the summation by a more compact expression.

    Solution: Note that by the formula, $a_{n-1} = ba_{n-2} + d$. Substituting this into the original equation for a_n gives $a_n = b^2a_{n-2} + bd + d$. Repeating this kind of substitution gives us $a_n = b^3a_{n-3} + b^2d + bd + d$. This suggests that $a_n = a_0b^n + \sum_{i=0}^{n-1} db^i$. We would guess the same formula by writing out the first few values of a_i, namely, a_0, $a_0b + d$, $a_0b^2 + db + d$, $a_0b^3 + b^2d + bd + d$, and so on. We prove our general formula by induction on n. It is clearly true when $n = 0$ as there are no terms in the sum and $b^0 = 1$. If we assume the formula is true when $n = k - 1$, we may write

$$\begin{aligned}
a_k &= ba_{k-1} + d \\
&= b\left(a_0b^{k-1} + \sum_{i=0}^{k-2} db^i\right) + d \\
&= ba_0b^{k-1} + b\sum_{i=0}^{k-2} db^i + d \\
&= a_0b^k + \sum_{i=0}^{k-1} db^i
\end{aligned}$$

Thus the truth of our formula for $n = k-1$ implies its truth for $n = k$ and therefore by the principle of mathematical induction, it is true for all nonnegative integers n.

We can give a more compact expression for the sum $\sum_{i=0}^{n-1} db^i = d\sum_{i=0}^{n-1} b^i$. Recall from algebra that $(1+x)(1-x) = 1-x^2$, $(1+x+x^2)(1-x) = 1-x^3$, and in general $(1+x+x^2+\cdots x^{n-1}))(1-x) = 1-x^n$. If you do not recall this formula, you can prove it by induction, or observe that

$$\begin{aligned}
& (1+x+x^2+\cdots x^{n-1})(1-x) \\
= \;& (1+x+x^2+\cdots x^{n-1})\cdot 1 - (1+x+x^2+\cdots x^{n-1})\cdot x \\
= \;& 1+x+x^2+\cdots x^{n-1} - (x+x^2+\cdots+x^n) \;=\; 1-x^n.
\end{aligned}$$

Dividing the first and last terms by $1-x$ gives us

$$\sum_{i=1}^{n-1} x^i = \frac{1-x^n}{1-x}.$$

Using this in our formula for a_n gives us $a_n = a_o b^n + d\frac{1-b^n}{1-b}$. This is valid except in the case $b = 1$ (in our computation with x above, we would be dividing by 0.) If $b = 1$ we get $a_n = a_0 + nd$ for the sum. ∎

2.2.4 Geometric Series

A sequence that satisfies a recurrence of the form $a_n = ba_{n-1}$ is called a *geometric progression*. Thus the sequence satisfying Equation 2.1, the recurrence for the number of subsets of an n-element set, is an example of a geometric progression. From your solution to Problem 98, a geometric progression has the form $a_n = a_0 b^n$. In your solution to Problem 98 you may have had to deal with the sum of a geometric progression in just slightly different notation, namely $\sum_{i=0}^{n-1} db^i$. A sum of this form is called a **(finite) geometric series**.

99. Do this problem only if your final answer (so far) to Problem 98 contained the sum $\sum_{i=0}^{n-1} db^i$.

    (a) Expand $(1-x)(1+x)$. Expand $(1-x)(1+x+x^2)$. Expand $(1-x)(1+x+x^2+x^3)$.
    Solution: $(1-x)(1+x) = 1-x^2$. $(1-x)(1+x+x^2) = 1-x^3$. $(1-x)(1+x+x^2+x^3) = 1-x^4$. ∎

(b) What do you expect $(1-b)\sum_{i=0}^{n-1} db^i$ to be? What formula for $\sum_{i=0}^{n-1} db^i$ does this give you? Prove that you are correct.

Solution: We expect $(1-b)\sum_{i=0}^{n-1} db^i$ to be $d(1-b^n)$. If $b \neq 1$, this gives us $\sum_{i=0}^{n-1} db^i = d\frac{1-b^n}{1-b}$. We can prove this by induction on n. If $n = 0$ we get 0 for $\frac{1-b^n}{1-b}$, and also for the sum $\sum_{i=0}^{-1} db^i$, since that sum has no terms. Assuming the formula holds when $n = k-1$, we may write

$$\begin{aligned}
&\sum_{i=0}^{k-1} db^i \\
= &\sum_{i=0}^{k-2} db^i + db^{k-1} \\
= &d\frac{1-b^{k-1}}{1-b} + db^{k-1} \\
= &\frac{d - db^{k-1} + db^{k-1} - db^k}{1-b} = d\frac{1-b^k}{1-b}.
\end{aligned}$$

Since the truth of the formula for $n = k-1$ implies its truth for $n = k$, by the principle of mathematical induction the formula is true for all nonnegative integers n. If $b = 1$ we get the formula $\sum_{i=0}^{n-1} db^i = dn$. ∎

In Problem 98 and perhaps 99 you proved an important theorem. While the theorem does not have a name, the formula it states is called the **sum of a finite geometric series**.

Theorem 2 *If $b \neq 1$ and $a_n = ba_{n-1} + d$, then $a_n = a_0 b^n + d\frac{1-b^n}{1-b}$. If $b = 1$, then $a_n = a_0 + nd$.*

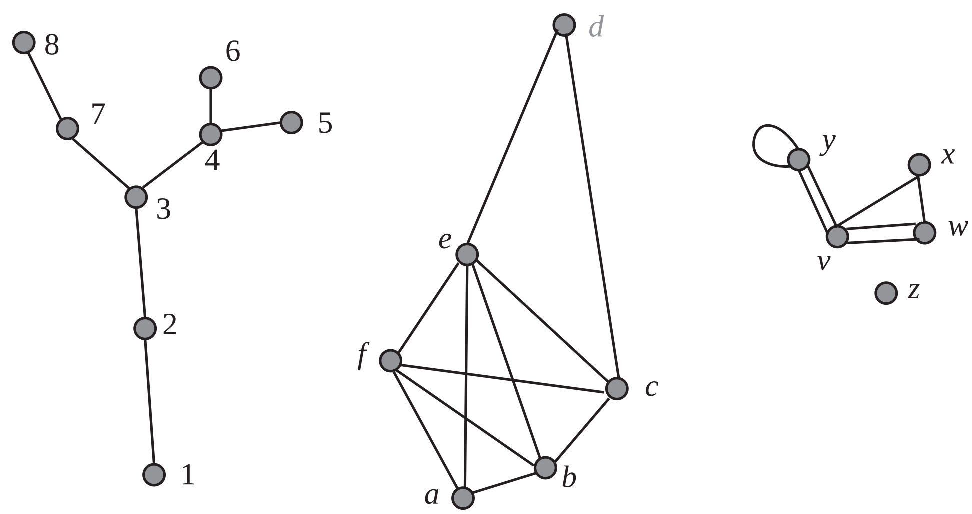
Corollary 1 *If $b \neq 1$, then $\sum_{i=0}^{n-1} b^i = \frac{1-b^n}{1-b}$. If $b = 1$, $\sum_{i=0}^{n-1} b^i = n$.*

2.3 Graphs and Trees

2.3.1 Undirected graphs

In Section 1.3.4 we introduced the idea of a directed graph. Graphs consist of vertices and edges. We describe vertices and edges in much the same way as we describe points and lines in geometry: we don't really say what vertices and edges are, but we say what they do. We just don't have a complicated axiom system the way we do in geometry. A **graph** consists of a set V called a vertex set and a set E called an edge set. Each member of V is called a *vertex* and each member of E is called an *edge*. Associated with each edge are two (not necessarily different) vertices called its endpoints. We draw pictures of graphs by drawing points to represent the vertices and line segments (curved if we choose) whose endpoints are at vertices to represent the edges. In Figure 2.2 we show three pictures of graphs. Each grey circle

Figure 2.2: Three different graphs



in the figure represents a vertex; each line segment represents an edge. You will note that we labelled the vertices; these labels are names we chose to give the vertices. We can choose names or not as we please. The third graph also shows that it is possible to have an edge that connects a vertex (like the one labelled y) to itself or it is possible to have two or more edges (like those between vertices v and y) between two vertices. The *degree* of a vertex is the number of times it appears as the endpoint of edges; thus the degree of y in the third graph in the figure is four.

∘100. In the graph on the left in Figure 2.2, what is the degree of each vertex?

Solution: The degree of vertex 1 is one, of vertex 2 is two, of vertex 3 is three, of vertex 4 is three, of vertex 5 is one, of vertex 6 is one, of vertex 7 is two, of vertex 8 is one. ▮

∘101. For each graph in Figure 2.2 is the number of vertices of odd degree even or odd?

Solution: In all three cases it is even. ▮

➔ ·102. The sum of the degrees of the vertices of a (finite) graph is related in a natural way to the number of edges.

(a) What is the relationship?

Solution: The sum of the degrees of the vertices is twice the number of edges. ▮

(b) Find a proof that what you say is correct that uses induction on the number of edges.

Solution: If a graph has no edges, then the sum of the degrees of the vertices is 0, which is twice the number of edges. Now suppose that whenever a graph has $n-1$ edges, the sum of the degrees of the vertices is twice the number of edges. Let G be a graph with n edges, and delete an edge from G to get G'. The the sum of the degrees of G' is $2(n-1)$, and adding the edge back into G' to get G either increases the degrees of exactly two vertices by one each or increases the degree of one vertex by 2. Thus the sum of the degrees of the vertices of G is $2n$, which is twice the number of edges. Thus by the principle of mathematical induction, for all nonnegative integers n, if a graph has n edges, then the sum of the degrees of the vertices is twice the number of edges. ▮

(c) Find a proof that what you say is correct which uses induction on the number of vertices.

(d) Find a proof that what you say is correct that does not use induction.

Solution: The sum of the degrees of the vertices is the sum over all edges of the number of times that edge touches a vertex, which is twice the number of edges. ▮

·103. What can you say about the number of vertices of odd degree in a graph?

Solution: The number of vertices of odd degree must be even, because otherwise the sum of the degrees of the vertices would be odd. ▮

2.3.2 Walks and paths in graphs

A *walk* in a graph is an alternating sequence $v_0 e_1 v_1 \ldots e_i v_i$ of vertices and edges such that edge e_i connects vertices v_{i-1} and v_i. A graph is called connected if, for any pair of vertices, there is a walk starting at one and ending at the other.

104. Which of the graphs in Figure 2.2 is connected?

  Solution: The first two are connected; the third is not. ▮

∘105. A *path* in a graph is a walk with no repeated vertices. Find the longest path you can in the third graph of Figure 2.2.

  Solution: The path from y to v to x to w is a typical longest path. There are quite a few others. Notice you have two choices for the edge to use to get from y to v. ▮

∘106. A *cycle* in a graph is a walk (with at least one edge) whose first and last vertex are equal but which has no other repeated vertices or edges. Which graphs in Figure 2.2 have cycles? What is the largest number of edges in a cycle in the second graph in Figure 2.2? What is the smallest number of edges in a cycle in the third graph in Figure 2.2?

  Solution: The second and third graphs have cycles. The largest number of edges in a cycle in the second graph is six; the smallest number of edges in a cycle in the third graph is one. ▮

∘107. A connected graph with no cycles is called a **tree**. Which graphs, if any, in Figure 2.2 are trees?

  Solution: The first graph is a tree. ▮

2.3.3 Counting vertices, edges, and paths in trees

➔ ·108. Draw some trees and on the basis of your examples, make a conjecture about the relationship between the number of vertices and edges in a tree. Prove your conjecture.

  Solution: The number of edges of a tree is one less than the number of vertices. We prove this by strong induction on the number of edges. First, if a tree has no edges, it can have only one vertex (otherwise it is not connected). Thus the number of edges is one less than the number of vertices. Now suppose that if a tree T has fewer than n edges, the number of edges is one less than the number of vertices.

Choose an edge e with endpoints x and y in the tree and remove it. The resulting graph is not connected, for if it were, the path remaining between the endpoints of e, together with e, would form a cycle. If we add an edge to the resulting graph, it can reduce the number of connected components by at most one, because it joins vertices in at most two connected components. In particular, since adding e reduces the number of connected components to one, the graph we got by deleting e must have exactly two connected components. Therefore when we remove e, the graph that remains consists of two trees, because neither connected component can have a cycle, for then T would have a cycle. Each of these trees has fewer edges than the original tree, so if they have m_1 and m_2 vertices respectively, they have, by the inductive hypothesis, $m_1 - 1$ and $m_2 - 1$ edges respectively. But together they have all the vertices of the original tree, so the original tree has $m_1 + m_2$ vertices, and has $m_1 - 1 + m_2 - 1 + 1 = m_1 + m_2 - 1$ edges, the edges of each of the two smaller trees as well as the edge e. Therefore the number of edges of the original tree is one less than the number of vertices. Therefore by the strong principle of mathematical induction, the number of edges of a tree is one less than the number of vertices. ∎

·109. What is the minimum number of vertices of degree one in a finite tree? What is it if the number of vertices is bigger than one? Prove that you are correct. See if you can find (and give) more than one proof.

Solution: The minimum is zero, which happens with a tree with one vertex. If the tree has more than one vertex, the minimum number of vertices of degree one is two. To prove this, we prove that every tree with two or more vertices has at least two vertices of degree two. Note that a tree with two vertices has exactly two vertices of degree 2. Now take a tree with more than two vertices. Remove an edge e without removing its endpoints. As in the solution to Problem 108 this gives two trees. We may assume inductively that each has at least two vertices of degree 1, or else is a single vertex. When we put e back in, it connects one vertex in one tree to one in the other. If both these vertices have degree 1 in their trees, there will be at least one vertex of degree 1 remaining in each tree, so there will be at least two vertices of degree 1 in the tree we get. If exactly one of these vertices is a tree with one vertex after the removal of e, when we connect it to the other tree, we will increase the degree of at most one vertex of degree 1 and will create a new vertex of degree 1, so the tree that results still has at least two vertices of degree 1. Therefore by the strong principle of mathematical induction, every tree with more than two vertices has at least two vertices of degree 1. Since a two-vertex tree has two vertices of degree 1, the minimum number of vertices of degree 1 in a tree with two or more vertices is two. (In fact a path with n vertices is a tree and it has exactly two vertices of degree one also.)

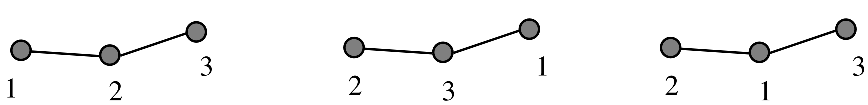
Alternately, the number of edges in a n vertex tree is $n-1$, and so the sum of the degrees of the vertices is $2n-2$. If we have more than one vertex, we can have no vertices of degree zero, and if all or all but one vertex had degree at least two, the sum of the degrees would have to be more than $2n-2$. ▮

➔ · 110. In a tree on any number of vertices, given two vertices, how many paths can you find between them? Prove that you are correct.

Solution: Exactly one. Suppose there were two distinct paths P_1 and P_2 from x to y. As they leave x, they might leave on the same edge or on different edges. However, since they are different, there must be some first vertex x' on both paths so that when leave x' (as we go from x to y), they leave on different edges. Then since they must both enter y, there must be some first vertex y', following x' on both paths as we go from x to y, such that the two paths enter y' on two different edges. Then the portion of path 1 from x' to y' followed by the portion of path 2 from y' to x' will be a cycle. This is impossible in a tree, so the supposition that there were two distinct paths is impossible. ▮

➔ 111. How many trees are there on the vertex set $\{1, 2\}$? On the vertex set $\{1, 2, 3\}$? When we label the vertices of our tree, we consider the tree which has edges between vertices 1 and 2 and between vertices 2 and 3 different from the tree that has edges between vertices 1 and 3 and between 2 and 3. See Figure 2.3. How many (labelled) trees are

Figure 2.3: The three labelled trees on three vertices



there on four vertices? How many (labelled) trees are there with five vertices? You don't have a lot of data to guess from, but try to guess a formula for the number of labelled trees with vertex set $\{1, 2, \cdots, n\}$.

Solution: There is one labelled tree on two vertices. We know there are three labelled trees on three vertices, and they all are paths. The difference among them is which vertex is the central vertex on the path. On four vertices a tree either has a vertex of degree 3 (there are four such trees) or it is a path, in which case there are six choices for the two vertices of degree 2, and for each choice of these two vertices, there are two different ways to attach the remaining vertices to them as vertices of degree 1. Thus there are $12 + 4 = 16$ trees on four vertices. On five vertices, we either have a vertex of degree 4 (there are five such trees), or we have a vertex of degree three which must be adjacent to a vertex of degree two in order to have five vertices. There are $5 \cdot 4 = 20$ ways to choose these two vertices; then there are three more choices we can make for the degree one vertex attached to the degree 2 vertex. Thus we have 60 trees with a vertex of degree three. If we have neither a vertex of degree four nor a vertex of degree three, then the tree is a path. We have $\binom{5}{2} = 10$ ways to choose the two vertices of degree one, and then there are $3! = 6$ ways to arrange the remaining vertices along the path, so we have 60 paths. Thus we have 125 trees on five vertices. These computations suggest there are n^{n-2} labelled trees on n vertices. ▮

We are now going to introduce a method to prove the formula you guessed. Given a tree with two or more vertices, labelled with positive integers, we define a sequence $b_1, b_2, \ldots$ of integers inductively as follows: If the tree has two vertices, the sequence consists of one entry, namely the label of the vertex with the larger label. Otherwise, let a_1 be the lowest numbered vertex of degree 1 in the tree. Let b_1 be the label of the unique vertex in the tree adjacent to a_1 and write down b_1. For example, in the first graph in Figure 2.2, a_1 is 1 and b_1 is 2. Given a_1 through a_{i-1}, let a_i be the lowest numbered vertex of degree 1 in the tree you get by deleting a_1 through a_{i-1} and let b_i be the unique vertex in this new tree adjacent to a_i. For example, in the first graph in Figure 2.2, $a_2 = 2$ and $b_2 = 3$. Then $a_3 = 5$ and $b_3 = 4$. We use b to stand for the sequence of b_is we get in this way. In the tree (the first graph) in Figure 2.2, the sequence b is 2344378. (If you are unfamiliar with inductive (recursive) definition, you might want to write down some other labelled trees on eight vertices and construct the sequence of b_is.)

112. (a) How long will the sequence of b_is be if it is computed from a tree with n vertices (labelled with 1 through n)?

Solution: On a tree with n vertices, the sequence b will have length $n-1$. ▮

(b) What can you say about the last member of the sequence of b_is?

Solution: The last member of the sequence b will be n. To see why, note that vertex n can not be in the sequence a, because the tree that remains after we delete an a_i will have at least two vertices of degree 1, so the one of smaller degree will be a_{i+1}. Thus we never delete the vertex n from the tree. Therefore when we choose the last b, we have vertex n and one other vertex, so the other vertex is our a-vertex and n is the vertex adjacent to it. ▮

(c) Can you tell from the sequence of b_is what a_1 is?

Solution: a_1 will be the smallest number that is not in the sequence of b's. ▮

(d) Find a bijection between labelled trees and something you can "count" that will tell you how many labelled trees there are on n labelled vertices.

Solution: Once we know a_1, we know one edge of the tree, namely the edge between a_1 and b_1. In general, when we know a_i, this will tell us that the edge form a_i to b_i is in the tree. The vertex a_2 will be the smallest number different from a_1 not in the sequence b_2 through b_{n-1}. In general, a_i will be the smallest vertex different from a_1 through a_{i-1} not in the sequence b_i through b_{n-1}, which gives us all $n-1$ edges of the tree (edge i goes from a_i to b_i). Thus there is a bijection between trees and the sequences b_1 through b_{n-1}. But since $b_{n-1} = n$, there is also a bijection between trees and the sequences b_1 through b_{n-2}. But given a sequence of numbers $c_1, c_2, \ldots, c_{n-2}, c_{n-1}$, all between 1 and n and with $c_{n-1} = n$, there is always a smallest number a_1 not in the sequence, and given $a_1, a_2, \ldots a_{i-1}$, there is always a smallest number not in the sequence c_i through c_{n-1} and different from the a_is already chosen, so we can construct the edges from a_i to c_i. Further, if we start with the edge from a_{n-1} to c_{n-1} and work backwards, we will always have a connected graph and will always be adding a vertex of degree 1 to it, so we will have no cycles. Therefore we will get a tree. Thus we have a bijection between labelled trees on n vertices and sequences of length $n-2$ consisting of members of $[n]$. There are n^{n-2} such sequences, and thus n^{n-2} labelled trees on n vertices. ▮

The sequence $b_1, b_2, \ldots, b_{n-2}$ in Problem 112 is called a *Prüfer coding* or *Prüfer code* for the tree. Thus the Prüfer code for the tree of Figure 2.2 is 234437. Notice that we do not include the term b_{n-1} in the Prüfer code because we know it is n. There is a good bit of interesting information encoded into the Prüfer code for a tree.

113. What can you say about the vertices of degree one from thePrüfer code for a tree labeled with the integers from 1 to n?

 Solution: They are exactly the numbers between 1 and n *not* listed in the Prüfer code to the tree. ▮

114. What can you say about the Prüfer code for a tree with exactly two vertices of degree 1 (and perhaps some vertices with other degrees as well)? Does this characterize such trees?

 Solution: It consists of $n-2$ distinct numbers between 1 and n. Any such Prüfer code is the code of a tree with exactly two vertices of degree one. ▮

➔ 115. What can you determine about the degree of the vertex labelled i from the Prüfer code of the tree?

 Solution: The degree of a vertex in a tree is one more than the number of times the vertex appears in the Prüfer code of the tree. ▮

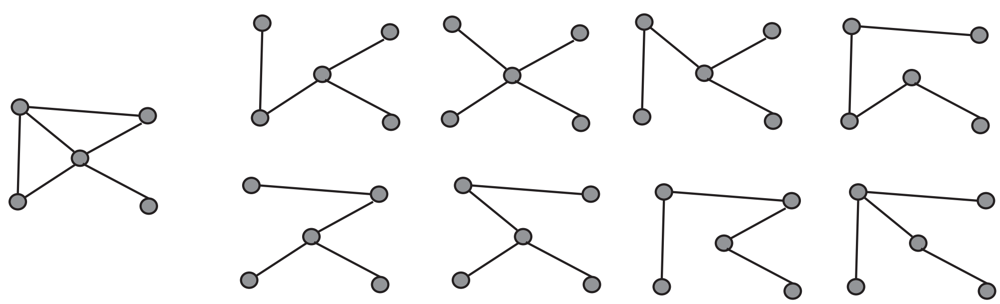
➔ 116. What is the number of (labelled) trees on n vertices with three vertices of degree 1? (Assume they are labelled with the integers 1 through n.) This problem will appear again in the next chapter after some material that will make it easier.

 Solution: There are $\binom{n}{3}$ ways to choose the three vertices of degree one. Each of the other $n-3$ vertices must appear in the Prüfer Code, so exactly one must appear twice. We have $n-3$ ways to choose that one vertex and $\binom{n-2}{2}\binom{n-4}{1}\binom{n-5}{1}\cdots\binom{1}{1} = \frac{(n-2)!}{2!}$ ways to choose which of the $n-2$ places to use for which vertices in the Prüfer code. Thus there are $\binom{n}{3}(n-3)\frac{(n-2)!}{2} = \frac{n!(n-2)(n-3)}{12}$ labelled trees with three vertices of degree one. ▮

2.3.4 Spanning trees

Many of the applications of trees arise from trying to find an efficient way to connect all the vertices of a graph. For example, in a telephone network, at any given time we have a certain number of wires (or microwave channels, or cellular channels) available for use. These wires or channels go from a specific place to a specific place. Thus the wires or channels may be thought of as edges of a graph and the places the wires connect may be thought of as vertices of that graph. A tree whose edges are some of the edges of a graph G and whose vertices are all of the vertices of the graph G is called a **spanning tree** of G. A spanning tree for a telephone network will give us a way to route calls between any two vertices in the network. In Figure 2.4 we show a graph and all its spanning trees.

Figure 2.4: A graph and all its spanning trees.



117. Show that every connected graph has a spanning tree. It is possible to find a proof that starts with the graph and works "down" towards the spanning tree and to find a proof that starts with just the vertices and works "up" towards the spanning tree. Can you find both kinds of proof?

    Solution: Here are three proofs:

    Start with a connected graph, and if you can find a cycle, remove one edge of that cycle. Repeat this process until you get a tree. You will get a tree, because removing an edge of a cycle reduces the number of cycles but leaves the graph connected. This tree will be a spanning tree of the original graph.

    Start with the original vertex set and no edges for a graph H. Go through the edges of the original graph G one at a time, and if you can add an edge of G to the graph H without creating a cycle, do so. Otherwise discard the edge and go on to the next one. This will give you a graph H with no cycles, and if it were not connected, there would be an edge in G between two vertices not yet connected in H. (If there weren't, the graph you just constructed would be connected.) Thus you get a spanning tree.

Start with vertex 1 and no edges as a graph H. Take an edge from it to another vertex. Add that edge and vertex to H. Now take an edge from one of the vertices you currently have to yet another vertex of the original graph. Add that vertex and edge to H. Repeat this process until you can find no such edge. You will get a tree because each edge you add connects a vertex of degree 1 to a tree you have already constructed. Since the original graph is connected, there must always be an edge from the current set of vertices you are considering to something not in that set. ▮

2.3.5 Minimum cost spanning trees

Our motivation for talking about spanning trees was the idea of finding a minimum number of edges we need to connect all the edges of a communication network together. In many cases edges of a communication network come with costs associated with them. For example, one cell-phone operator charges another one when a customer of the first uses an antenna of the other. Suppose a company has offices in a number of cities and wants to put together a communication network connecting its various locations with high-speed computer communications, but to do so at minimum cost. Then it wants to take a graph whose vertices are the cities in which it has offices and whose edges represent possible communications lines between the cities. Of course there will not necessarily be lines between each pair of cities, and the company will not want to pay for a line connecting city i and city j if it can already connect them indirectly by using other lines it has chosen. Thus it will want to choose a spanning tree of minimum cost among all spanning trees of the communications graph. For reasons of this application, if we have a graph with numbers assigned to its edges, the sum of the numbers on the edges of a spanning tree of G will be called the *cost* of the spanning tree.

➔ 118. Describe a method (or better, two methods different in at least one aspect) for finding a spanning tree of minimum cost in a graph whose edges are labelled with costs, the cost on an edge being the cost for including that edge in a spanning tree. Prove that your method(s) work.

Solution: Start with a vertex, number it vertex 1, and choose the least costly edge leaving it. Number the new vertex 2. Given your current set of vertices and edges, choose the least costly edge leaving that set, and if your set has $i-1$ vertices, label the new vertex i. You will always have a tree as you go along since you are adding a vertex of degree 1 to a tree you already have. You will get a spanning tree because the graph you start with is connected. If your tree did not have the lowest cost among all spanning trees, there would be some smallest i such that there is an edge from vertex i in a least cost spanning tree that is not in your tree. Since we chose the least i, the edge we just chose from vertex i goes to a higher-numbered vertex. Thus you could have chosen that edge as you were constructing your tree, so there cannot be a tree of lower total cost than the one you chose.

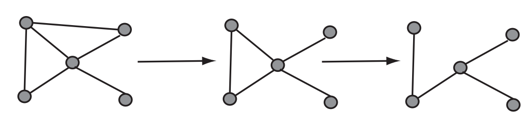
Alternatively, start with the vertex set of the graph and no edges. Choose an edge of least cost. Repeat the following until it cannot be repeated. Given the set of edges you have so far, choose an edge of least cost among all edges that do not form a cycle with edges you already have. The graph you get will have no cycles, and it will have to be connected, because otherwise there would be an edge in the original graph that connects two vertices in the graph you just constructed. Thus you will have a tree. Suppose there is another tree with lower total cost. Choose such a tree with as many edges as common with your tree as possible. Then there is some edge e of this tree of lowest cost among all edges connecting two vertices that are not connected by an edge in your tree. Suppose the cost of this edge is c. Since these vertices are connected by some path in your tree, when you were considering edges of cost c, these two vertices were already connected by a path in your tree. There must be some edge f on that path not in the least cost tree. The edge f was already in your tree while you were considering edges of cost c, so its cost is no more than c. Adding f to the tree of least cost gives a cycle. All edges on that cycle that are not in your tree have cost at least c by our choice of e. But since f was from your tree, there must be some edge g of the cycle that is in the least cost tree that but not in your tree. Removing g from the least cost tree and adding f cannot increase the cost of the tree, but it gives a tree that has one more edge in common with your tree. This contradicts the choice of the least cost tree, so there must have been no tree of lower total cost than the one you constructed. ∎

The method you used in Problem 118 is called a *greedy method*, because each time you made a choice of an edge, you chose the least costly edge available to you.

2.3.6 The deletion/contraction recurrence for spanning trees

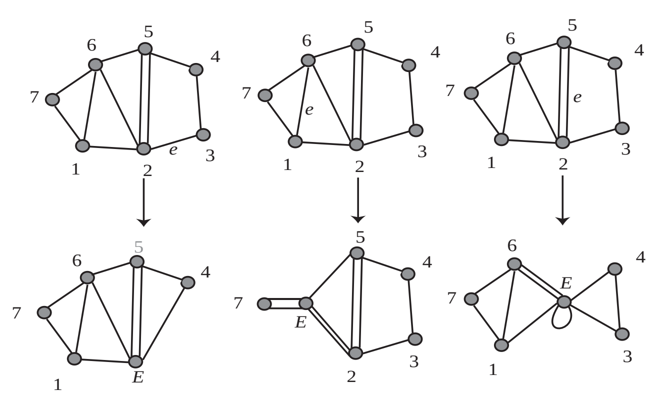
There are two operations on graphs that we can apply to get a recurrence (though a more general kind than those we have studied for sequences) which will let us compute the number of spanning trees of a graph. The operations each apply to an edge e of a graph G. The first is called *deletion*; we *delete* the edge e from the graph by removing it from the edge set. Figure 2.5 shows how we can delete edges from a graph to get a spanning tree.

Figure 2.5: Deleting two appropriate edges from this graph gives a spanning tree.



The second operation is called *contraction*. Contractions of three differ-

Figure 2.6: The results of contracting three different edges in a graph.



ent edges in the same graph are shown in Figure 2.6. Intuitively, we contract an edge by shrinking it in length until its endpoints coincide; we let the rest of the graph "go along for the ride." To be more precise, we *contract* the edge e with endpoints v and w as follows:

1. remove all edges having either v or w or both as an endpoint from the edge set,
2. remove v and w from the vertex set,
3. add a new vertex E to the vertex set,
4. add an edge from E to each remaining vertex that used to be an endpoint of an edge whose other endpoint was v or w, and add an edge from E to E for any edge other than e whose endpoints were in the set $\{v, w\}$.

We use $G-e$ (read as G minus e) to stand for the result of deleting e from G, and we use G/e (read as G contract e) to stand for the result of contracting e from G.

➔ ·119. (a) How do the number of spanning trees of G not containing the edge e and the number of spanning trees of G containing e relate to the number of spanning trees of $G - e$ and G/e?

Solution: The number of spanning trees of $G-e$ is the number of spanning trees of G not containing e. The number of spanning trees of G/e is the number of spanning trees of G containing e. ▮

(b) Use $\#(G)$ to stand for the number of spanning trees of G (so that, for example, $\#(G/e)$ stands for the number of spanning trees of G/e). Find an expression for $\#(G)$ in terms of $\#(G/e)$ and $\#(G - e)$. This expression is called the *deletion-contraction recurrence.*

Solution: The number of spanning trees of G not containing e is the number of spanning trees of $G-e$. The number of spanning trees of G containing the edge e is the number of spanning trees of G/e. Therefore $\#(G) = \#(G - e) + \#(G/e)$. ▮

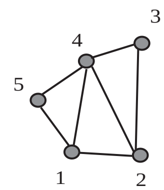
(c) Use the recurrence of the previous part to compute the number of spanning trees of the graph in Figure 2.7.

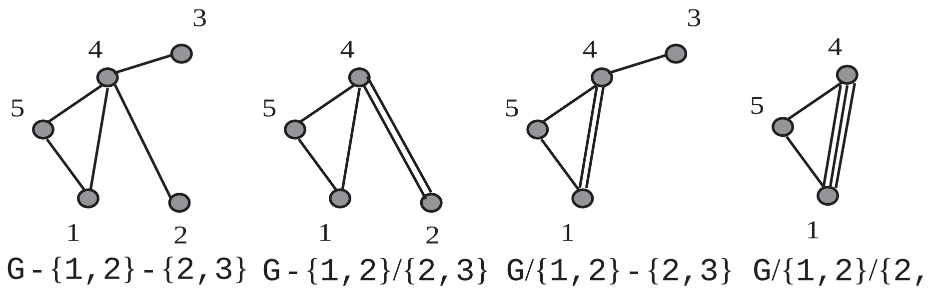
Solution: Applying the formula twice to G gives

$$\begin{aligned}\#(G) &= \#(G - \{1,2\} - \{2,3\}) + \#((G - \{1,2\})/\{2,3\}) \\ &+ \#((G/\{1,2\}) - \{2,3\}) + \#(G/\{1,2\}/\{2,3\}).\end{aligned}$$

We now show the four graphs on the right hand side of the equation.

Figure 2.7: A graph.





We could now convert each of these graphs to trees with multiple edges by deleting and contracting one more edge, say edge $\{1, 5\}$, which would make the analysis easier but the picture twice as big. Since we can easily count spanning trees of a triangle, we can also stop here, noting that the first graph has three spanning trees, the second has six, the third has five, and the fourth has seven, so the total number of spanning trees is $3+6+5+7 = 21$. ▮

2.3.7 Shortest paths in graphs

Suppose that a company has a main office in one city and regional offices in other cities. Most of the communication in the company is between the main office and the regional offices, so the company wants to find a spanning tree that minimizes not the total cost of all the edges, but rather the cost of communication between the main office and each of the regional offices. It is not clear that such a spanning tree even exists. This problem is a special case of the following. We have a connected graph with nonnegative numbers assigned to its edges. (In this situation these numbers are often called weights.) The *(weighted) length* of a path in the graph is the sum of the weights of its edges. The *distance* between two vertices is the least (weighted) length of any path between the two vertices. Given a vertex v, we would like to know the distance between v and each other vertex, and we would like to know if there is a spanning tree in G such that the length of the path in the spanning tree from v to each vertex x is the distance from v to x in G.

120. Show that the following algorithm (known as Dijkstra's algorithm) applied to a weighted graph whose vertices are labelled 1 to n gives, for each i, the distance from vertex 1 to i as $d(i)$.

   (a) Let $d(1) = 0$. Let $d(i) = \infty$ for all other i. Let $v(1)$=1. Let $v(j) = 0$ for all other j. For each i and j, let $w(i, j)$ be the minimum weight of an edge between i and j, or ∞ if there are no such edges. Let $k = 1$. Let $t = 1$.

   (b) For each i, if $d(i) > d(k) + w(k, i)$ let $d(i) = d(k) + w(k, i)$.

   (c) Among those i with $v(i) = 0$, choose one with $d(i)$ a minimum, and let $k = i$. Increase t by 1. Let $v(i) = 1$.

   (d) Repeat the previous two steps until $t = n$.

   Solution: We prove that the distance from vertex 1 of a vertex i with $v(i) = 1$ is $d(i)$. We use induction on the number t of vertices with $v(i) = 1$. If $t = 1$, then $d(1) = 0$ is the distance from vertex 1 to vertex 1. Now when $t = s-1$, we have vertices i_1, i_2, $\ldots i_{s-1}$ such that $v(i_p) = 1$. Suppose inductively that $d(i_p)$ is the distance from vertex i_p to vertex 1 for $p = 1, 2, \ldots, s-1$. When $t = s$, we choose a vertex i with $d(i)$ a minimum. Suppose $u_1, u_2, \ldots, u_r$ is the sequence of vertices of a shortest (least total weight) path from vertex $u_1 = 1$ to vertex $u_r = i$ and that the length (total weight) of this path is less than $d(i)$. Suppose that some vertex u_p has $v(u_p) = 0$, and choose the smallest p such that this is so. Then $d(u_{p-1})$ is the distance from vertex 1 to vertex u_{p-1}, and $d(u_{p-1}) + w(u_{p-1}, u_p)$ is less than $d(i)$ because the length of the path from u_1 to u_r is less than $d(i)$. Then we would not have chosen the vertex i after all, but rather the vertex u_p, which is a contradiction, so all the vertices u_p with $p < r$ must have $v(u_p) = 1$, and, by our inductive hypothesis, $d(u_p)$ must be the distance from vertex 1 to vertex u_p for $p < r$. Thus when we were computing $d(i)$, we would have found that $d(i) \leq d(u_{r-1}) + w(u_{r-1}, i)$. Thus $d(i)$ must be the distance from vertex 1 to vertex i after all. Therefore the fact that Dijkstra's algorithm works when $t = s - 1$ implies that it works when $t = s$, so that, by the principle of mathematical induction, it works for all nonnegative integers t. ▮

121. Is there a spanning tree such that the distance from vertex 1 to vertex i given by the algorithm in Problem 120 is the distance from vertex 1 to vertex i in the tree (using the same weights on the edges, of course)?

Solution: Yes, when we choose the i with $d(i)$ a minimum, before we change k to i, we add an edge from vertex k to vertex i to the edge set of a graph on the vertex set $[n]$. We get a tree each time we do this step, because we are adding a vertex of degree 1 to a smaller tree. We essentially proved in the solution to Problem 120 that the path from vertex 1 to vertex i in this tree has length (total weight) $d(i)$. Using the same approach we could prove it directly by induction on the number of vertices in our tree. ∎

2.4 Supplementary Problems

1. Use the inductive definition of a^n to prove that $(ab)^n = a^n b^n$ for all nonnegative integers n.

   Solution: If $n = 0$ we get $(ab)^0 = 1$ and $a^0 b^0 = 1$. Assume inductively that $(ab)^{n-1} = a^{n-1}b^{n-1}$. Then by the inductive definition, inductive hypothesis, and commutative law,

   $$(ab)^n = (ab)^{n-1}ab = a^{n-1}b^{n-1}ab = a^{n-1}ab^{n-1}b = a^n b^n.$$

   Thus the fact that $(ab)^{n-1} = a^{n-1}b^{n-1}$ implies the fact that $(ab)^n = a^n b^n$. Therefore by the principle of mathematical induction, $(ab)^n = a^n b^n$ for all nonnegative integers n. ▮

2. Give an inductive definition of $\bigcup_{i=1}^{n} S_i$ and use it and the two set distributive law to prove the distributive law $A \cap \bigcup_{i=1}^{n} S_i = \bigcup_{i=1}^{n} A \cap S_i$.

   Solution: We define $\bigcup_{i=1}^{1} S_i = S_i$ and $\bigcup_{i=1}^{n} S_i = \bigcup_{i=1}^{n-1} S_i \cup S_n$. Then

   $$A \cap \bigcup_{i=1}^{1} S_i = A \cap S_1 = \bigcup_{i=1}^{1} A \cap S_i.$$

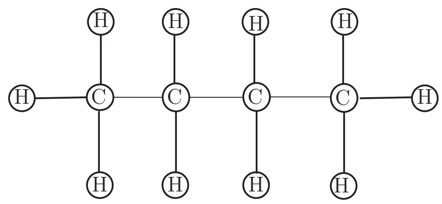
   Assume that $n > 1$ and $A \cap \bigcup_{i=1}^{n-1} = \bigcup_{i=1}^{n-1} A \cap S_i$. Now

   $$\begin{aligned} A \cap \bigcup_{i=1}^{n} S_i &= A \cap \left(\bigcup_{i=1}^{n-1} S_i \cup S_n \right) = \left(A \cap \bigcup_{i=1}^{n-1} S_i \right) \cup (A \cap S_n) \\ &= \left(\bigcup_{i=1}^{n-1} A \cap S_i \right) \cup (A \cap S_n) = \bigcup_{i=1}^{n} A \cap S_i. \end{aligned}$$

   Thus the truth of the distributive law for distributing an intersection over a union of $n-1$ sets implies its truth for distributing an intersection over a union of n sets. Therefore by the principle of mathematical induction, the distributive law $A \cap \bigcup_{i=1}^{n} S_i = \bigcup_{i=1}^{n} A \cap S_i$ holds for all positive integers n. ▮

➔ 3. A hydrocarbon molecule is a molecule whose only atoms are either carbon atoms or hydrogen atoms. In a simple molecular model of a hydrocarbon, a carbon atom will bond to exactly four other atoms and hydrogen atom will bond to exactly one other atom. Such a model is shown in Figure 2.8. We represent a hydrocarbon compound

Figure 2.8: A model of a butane molecule



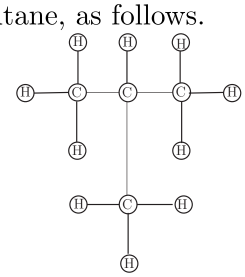
with a graph whose vertices are labelled with C's and H's so that each C vertex has degree four and each H vertex has degree one. A hydrocarbon is called an "alkane" if the graph is a tree. Common examples are methane (natural gas), butane (one version of which is shown in Figure 2.8), propane, hexane (ordinary gasoline), octane (to make gasoline burn more slowly), etc.

(a) How many vertices are labelled H in the graph of an alkane with exactly n vertices labelled C?

Solution: We have n vertices of degree four, and so if we have m vertices of degree 1, we get $4n + m = 2(m + n - 1)$ from the fact that the sum of the degrees of the vertices must be twice the number of edges. Thus we have $m = 2n + 2$ hydrogen atoms. ▮

(b) An alkane is called butane if it has four carbon atoms. Why do we say one version of butane is shown in Figure 2.8?

Solution: There is another tree with four carbon atoms, sometimes called isobutane, as follows.



▮

4. (a) Give a recurrence for the number of ways to divide $2n$ people into sets of two for tennis games. (Don't worry about who serves first.)

   Solution: $t_{2n} = (2n-1)t_{2n-2}$. ▮

   (b) Give a recurrence for the number of ways to divide $2n$ people into sets of two for tennis games and to determine who serves first.)

   Solution: $t_{2n} = 2(2n-1)t_{2n-2}$. ▮

➔ 5. Give a recurrence for the number of ways to divide $4n$ people into sets of four for games of bridge. (Don't worry about how they sit around the bridge table or who is the first dealer.)

   Solution: $b_{4n} = \binom{4n-1}{3} b_{4n-4}$. ▮

6. Use induction to prove your result in Supplementary Problem 2 at the end of Chapter 1.

   Solution: A composition of n is an ordered list of positive numbers that adds to n. We wish to show that there are 2^{n-1} compositions of n. There is one composition of the number 1, and $2^{1-1} = 1$. Now assume inductively that there are 2^{n-2} compositions of the number $n-1$. From a composition of $n-1$, we can get a composition of n either by making a new last part of size 1, or by adding one to the last part. Clearly these two operations give different partitions of n; what is not so clear is that they give all partitions of n, but they do: Either the last part of a partition of n is 1, in which case it comes from the first kind of operation, or it is larger than one, in which case it comes from the second operation. Thus the number of compositions of n is twice the number of compositions of $n-1$, and so is $2 \cdot 2^{n-2} = 2^{n-1}$. Therefore the statement that there are 2^{n-2} compositions of $n-1$ implies the statement that there are 2^{n-1} compositions of n. Thus by the principle of mathematical induction, there are 2^{n-1} compositions of n for every positive integer n. ▮

7. Give an inductive definition of the product notation $\prod_{i=1}^{n} a_i$.

   Solution: $\prod_{i=1}^{1} a_i = a_1$, and $\prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) \cdot a_n$. ▮

8. Using the fact that $(ab)^k = a^k b^k$, use your inductive definition of

product notation in Problem 7 to prove that $\left(\prod_{i=1}^{n} a_i\right)^k = \prod_{i=1}^{n} a_i^k$.

Solution: When $n = 1$ we get $\left(\prod_{i=1}^{1} a_i\right)^k = a_i^k = \prod_{i=1}^{1} a_i^k$. Now assume inductively that $\left(\prod_{i=1}^{m-1} a_i\right)^k = \prod_{i=1}^{m-1} a_i^k$. Then we may write

$$\left(\prod_{i=1}^{m} a_i\right)^k = \left(\left(\prod_{i=1}^{m-1} a_i\right) \cdot a_m\right)^k = \left(\prod_{i=1}^{m-1} a_i^k\right) \cdot a_m^k = \prod_{i=1}^{m} a_i^k.$$

Thus the correctness of the formula for $n = m-1$ implies its correctness for $n = m$. Therefore by the principle of mathematical induction, the formula holds for all positive integers n. ▮

➔ $*$9. How many labelled trees on n vertices have exactly four vertices of degree 1? (This problem also appears in the next chapter since some ideas in that chapter make it more straightforward.)

Solution: The vertices of degree 1 are the vertices that do not appear in the Prüfer code for the tree. So we first choose four vertices out of n in $\binom{n}{4}$ ways to be our vertices of degree 1, and then we use the remaining $n-4$ vertices to fill in our list of $n-2$ vertices, using each of the $n-4$ at least once. Thus we either use one of them 3 times and the rest once, or two of them twice and the rest once. There are $n-4$ ways to choose the one we use three times and $\binom{n-2}{3}\binom{n-5}{1}\binom{n-6}{1}\cdots\binom{1}{1} = \frac{(n-2)!}{3!}$ ways to label the $n-2$ places with the chosen vertices. There are $\binom{n-4}{2}$ ways to choose the two vertices we would use twice, and $\binom{n-2}{2}\binom{n-4}{2}\binom{n-6}{1}\binom{n-7}{1}\cdots\binom{1}{1}/2 = \frac{(n-2)!}{2!2!}$ ways to assign the chosen vertices to the $n-2$ places in the Prüfer Code. Thus we have

$$\begin{aligned} &\binom{n}{4}\left((n-4)\frac{(n-2)!}{3!} + \frac{(n-4)(n-5)}{2}\frac{(n-2)!}{4}\right) \\ = \;& \frac{n!}{24}(n-2)^{\underline{3}}\left(\frac{1}{6} + \frac{n-5}{8}\right) \\ = \;& n!(n-2)(n-3)(n-4)(3n-11)/576 \end{aligned}$$

possible Prüfer codes and therefore the same number of labelled trees. ▮

➔ $*$10. The *degree sequence* of a graph is a list of the degrees of the vertices in nonincreasing order. For example the degree sequence of the first graph in Figure 2.4 is $(4, 3, 2, 2, 1)$. For a graph with vertices labelled 1 through n, the *ordered degree sequence* of the graph is the sequence $d_1, d_2, \dots d_n$ in which d_i is the degree of vertex i. For example the ordered degree sequence of the first graph in Figure 2.2 is $(1, 2, 3, 3, 1, 1, 2, 1)$.

(a) How many labelled trees are there on n vertices with ordered degree sequence $d_1, d_2, \dots d_n$? (This problem appears again in the next chapter since some ideas in that chapter make it more straightforward.)

Solution: We are given that d_i is the degree of vertex i. The number of times i appears in the Prüfer code of a tree is one less than the degree of i, so vertex i appears $d_i - 1$ times. Thus the sum of the $d_i - 1$ should be $2n - 2 - n = n - 2$. Of the $n - 2$ places in the Prüfer code, we want to label $d_1 - 1$ of them with 1, $d_2 - 1$ of them with 2 and in general $d_i - 1$ of them with i. There are

$$\binom{n-2}{d_1-1}\binom{n-2-(d_1-1)}{d_2-1}\binom{n-2-(d_1-1+d_2-1)}{d_3-1}\cdots\binom{d_n-1}{d_n-1}$$

ways to do this, so the number of trees in which vertex i has degree d_i is $\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}$ ∎

$*$(b) How many labelled trees are there on n vertices with with the degree sequence in which the degree d appears i_d times?

Solution: Now we modify the solution of the previous part by observing that to count all graphs with a given degree sequence, the actual vertices which have the given degrees is irrelevant, so we must multiply the result of the easier problem by the number of ways to assign the degrees to the vertices. To assign the degrees, we can list the vertices in $n!$ ways, choose the first i_1 of these vertices to have degree 1, the next i_2 to have degree 2, and so on. But the order in which we list the vertices of a given degree is irrelevant. Thus the number of ways to assign the degrees is $\frac{n!}{i_1!i_2!\cdots i_n!}$. Once the degrees are assigned, there are $\frac{(n-2)!}{\prod_{d=1}^{n}(d-1)!^{i_d}}$, by translating our easier result. Thus the total number of trees with the degree sequence in which there are i_d vertices of degree

d is

$$\frac{n!(n-2)!}{\prod_{j=1}^{n} i_j!(j-1)!^{i_j}}.$$

$\blacksquare$

Chapter 3

Distribution Problems

3.1 The Idea of a Distribution

Many of the problems we solved in Chapter 1 may be thought of as problems of distributing objects (such as pieces of fruit or ping-pong balls) to recipients (such as children). Some of the ways of viewing counting problems as distribution problems are somewhat indirect. For example, in Problem 37 you probably noticed that the number of ways to pass out k ping-pong balls to n children so that no child gets more than one is the number of ways that we may choose a k-element subset of an n-element set. We think of the children as recipients and objects we are distributing as the identical ping-pong balls, distributed so that each recipient gets at most one ball. Those children who receive an object are in our set. It is helpful to have more than one way to think of solutions to problems. In the case of distribution problems, another popular model for distributions is to think of putting balls in boxes rather than distributing objects to recipients. Passing out identical objects is modeled by putting identical balls into boxes. Passing out distinct objects is modeled by putting distinct balls into boxes.

3.1.1 The twenty-fold way

When we are passing out objects to recipients, we may think of the objects as being either identical or distinct. We may also think of the recipients as being either identical (as in the case of putting fruit into plastic bags in the grocery store) or distinct (as in the case of passing fruit out to children). We may restrict the distributions to those that give at least one object to each recipient, or those that give exactly one object to each recipient, or those that give at most one object to each recipient, or we may have no

Table 3.1: An incomplete table of the number of ways to distribute k objects to n recipients, with restrictions on how the objects are received

The Twenty-fold Way: A Table of Distribution Problems		
k objects and conditions on how they are received	n recipients and mathematical model for distribution	
	Distinct	Identical
1. Distinct no conditions	n^k functions	? set partitions ($\leq n$ parts)
2. Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations	1 if $k \leq n$; 0 otherwise
3. Distinct Each gets at least one	? onto functions	? set partitions (n parts)
4. Distinct Each gets exactly one	$k! = n!$ bijections	1 if $k = n$; 0 otherwise
5. Distinct, order matters	? ?	? ?
6. Distinct, order matters Each gets at least one	? ?	? ?
7. Identical no conditions	? ?	? ?
8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$; 0 otherwise
9. Identical Each gets at least one	? ?	? ?
10. Identical Each gets exactly one	1 if $k = n$; 0 otherwise	1 if $k = n$; 0 otherwise

such restrictions. If the objects are distinct, it may be that the order in which the objects are received is relevant (think about putting books onto the shelves in a bookcase) or that the order in which the objects are received is irrelevant (think about dropping a handful of candy into a child's trick or treat bag). If we ignore the possibility that the order in which objects are received matters, we have created $2 \cdot 2 \cdot 4 = 16$ distribution problems. In the cases where a recipient can receive more than one distinct object, we also have four more problems when the order objects are received matters. Thus we have 20 possible distribution problems.

We describe these problems in Table 3.1. Since there are twenty possible distribution problems, we call the table the "Twenty-fold Way," adapting terminology suggested by Joel Spencer for a more restricted class of distribution problems. In the first column of the table we state whether the objects are distinct (like people) or identical (like ping-pong balls) and then give any conditions on how the objects may be received. The conditions we consider are whether each recipient gets at most one object, whether each recipient gets at least one object, whether each recipient gets exactly one object, and whether the order in which the objects are received matters. In the second column we give the solution to the problem and the name of the mathematical model for this kind of distribution problem when the recipients are distinct, and in the third column we give the same information when the recipients are identical. We use question marks as the answers to problems we have not yet solved and models we have not yet studied. We give explicit answers to problems we solved in Chapter 1 and problems whose answers are immediate. The goal of this chapter is to develop methods that will allow us to fill in the table with formulas or at least quantities we know how to compute, and we will give a completed table at the end of the chapter. We will now justify the answers that are not question marks and replace some question marks with answers as we cover relevant material.

If we pass out k distinct objects (say pieces of fruit) to n distinct recipients (say children), we are saying for each object to which recipient it goes. Thus we are defining a function from the set of objects to the recipients. We saw the following theorem in Problem 13b.

Theorem 3 *There are n^k functions from a k-element set to an n-element set.*

We proved it in one way in Problem 13b and in another way in Problem 75. If we pass out k distinct objects (say pieces of fruit) to n indistinguishable recipients (say identical paper bags) then we are dividing the objects up into disjoint sets; that is, we are forming a partition of the objects into some number, certainly no more than the number k of objects, of parts. Later in this chapter (and again in the next chapter) we shall discuss how to compute the number of partitions of a k-element set into n parts. This explains the entries in row one of our table.

If we pass out k distinct objects to n recipients so that each gets at most one, we still determine a function, but the function must be one-to-one. The number of one-to-one functions from a k-element set to an n element set is the same as the number of one-to-one functions from the set $[k] = \{1, 2, \ldots, k\}$ to an n-element set. In Problem 20 we proved the following theorem.

Theorem 4 *If $0 \leq k \leq n$, then the number of k-element permutations of an n-element set is*

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1) = n!/(n-k)!.$$

If $k > n$ there are no one-to-one functions from a k element set to an n element set, so we define $n^{\underline{k}}$ to be zero in this case. Notice that this is what the indicated product in the middle term of our formula gives us. If we are supposed to distribute k distinct objects to n identical recipients so that each gets at most one, we cannot do so if $k > n$, so there are 0 ways to do so. On the other hand, if $k \leq n$, then it doesn't matter which recipient gets which object, so there is only one way to do so. This explains the entries in row two of our table.

If we distribute k distinct objects to n distinct recipients so that each recipient gets at least one, then we are counting functions again, but this time functions from a k-element set *onto* an n-element set. At present we do not know how to compute the number of such functions, but we will discuss how to do so later in this chapter and in the next chapter. If we distribute k identical objects to n recipients, we are again simply partitioning the objects, but the condition that each recipient gets at least one means that we are partitioning the objects into exactly n blocks. Again, we will discuss how to compute the number of ways of partitioning a set of k objects into n blocks later in this chapter. This explains the entries in row three of our table.

If we pass out k distinct objects to n recipients so that each gets exactly one, then $k = n$ and the function that our distribution gives us is a bijection. The number of bijections from an n-element set to an n-element set is $n!$ by Theorem 4. If we pass out k distinct objects to n identical recipients so that each gets exactly 1, then in this case it doesn't matter which recipient gets which object, so the number of ways to do so is 1 if $k = n$. If $k \neq n$, then the number of such distributions is zero. This explains the entries in row four of our table.

We now jump to row eight of our table. We saw in Problem 37 that the number of ways to pass out k identical ping-pong balls to n children is simply the number of k-element subsets of an n-element set. In Problem 39d we proved the following theorem.

Theorem 5 *If $0 \leq k \leq n$, the number of k-element subsets of an n-element set is given by*

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

We define $\binom{n}{k}$ to be 0 if $k > n$, because then there are no k-element subsets of an n-element set. Notice that this is what the middle term of the formula in the theorem gives us. This explains the entries of row 8 of our table. For now we jump over row 9.

In row 10 of our table, if we are passing out k identical objects to n recipients so that each gets exactly one, it doesn't matter whether the recipients are identical or not; there is only one way to pass out the objects if $k = n$ and otherwise it is impossible to make the distribution, so there are no ways of distributing the objects. This explains the entries of row 10 of our table. Several other rows of our table can be computed using the methods of Chapter 1.

3.1.2 Ordered functions

➔ •122. Suppose we wish to place k distinct books onto the shelves of a bookcase with n shelves. For simplicity, assume for now that all of the books would fit on any of the shelves. Also, let's imagine that once we are done putting books on the shelves, we push the books on a shelf as far to the left as we can, so that we are only thinking about how the books sit relative to each other, not about the exact places where we put the books. Since the books are distinct, we can think of the first book, the second book and so on.

(a) How many places are there where we can place the first book?

Solution: There are n places where we can place the first book. ∎

(b) When we place the second book, if we decide to place it on the shelf that already has a book, does it matter if we place it to the left or right of the book that is already there?

Solution: Yes. ∎

(c) How many places are there where we can place the second book?

Solution: Once we have placed it, there are $n+1$ places where we can place the second book, because on the shelf that has one book, we could put the second book to the left or to the right of the book already there. ∎

Solution: O ∎nce we have $i-1$ books placed, if we want to place book i on a shelf that already has some books, is sliding it in to the left of all the books already there different from placing it to the right of all the books already there or between two books already there?

Solution: All of these are different. ∎

(d) In how many ways may we place the ith book into the bookcase?

Solution: Once we have $i-1$ books on the shelves the ith book could go on any shelf to the left of all books there, if any, giving us n places, or it could go to the immediate right of any book already there, giving us another $i-1$ places. Thus there are $n+i-1$ places where we could place book i. ▮

(e) In how many ways may we place all the books?

Solution: From the previous parts, we can see by the product principle that the number of ways to place all the books is

$$\prod_{i=1}^{k}(n+i-1).$$

▮

123. Suppose we wish to place the books in Problem 122e (satisfying the assumptions we made there) so that each shelf gets at least one book. Now in how many ways may we place the books?

Solution: Choose n books from the k books in $\binom{k}{n}$ ways, and assign them to the n places shelves in $n!$ ways, giving us $k!/(k-n)!$ ways to put a book on each shelf. Now leaving these books at the far left of each shelf, place the remaining books in

$$\prod_{i=1}^{k-n}(n+i-1) = \frac{(n+(k-n)-1)!}{(n-1)!} = \frac{(k-1)!}{(n-1)!}$$

ways. Thus we have

$$\frac{k!(k-1)!}{(k-n)!(n-1)!} = k!\binom{k-1}{n-1}$$

ways to place the books. Of course the right hand side of that equation cries out for a combinatorial explanation. Here it is. Imagine lining up the k books in a row. Then there are $k-1$ places in between them. Choose $n-1$ of these places, and slide a piece of paper in there as a divider. Now put the books before the first divider on shelf one, and the books after divider i on shelf $i+1$. This gives an arrangement of the books on the shelves so that every shelf has a book! ▮

The assignment of which books go to which shelves of a bookcase is simply a function from the books to the shelves. But a function doesn't determine which book sits to the left of which others on the shelf, and this information is part of how the books are arranged on the shelves. In other words, the order in which the shelves receive their books matters. Our function must thus assign an ordered list of books to each shelf. We will call such a function an ordered function. More precisely, an **ordered function** from a set S to a set T is a function that assigns an (ordered) list of elements of S to some, but not necessarily all, elements of T in such a way that each element of S appears on one and only one of the lists.[1] (Notice that although it is not the usual definition of a function from S to T, a function can be described as an assignment of subsets of S to some, but not necessarily all, elements of T so that each element of S is in one and only one of these subsets.) Thus the number of ways to place the books into the bookcase is the entry in the middle column of row 5 of our table. If in addition we require each shelf to get at least one book, we are discussing the entry in the middle column of row 6 of our table. An *ordered onto function* is one which assigns a list to each element of T.

In Problem 122e you showed that the number of ordered functions from a k-element set to an n-element set is $\prod_{i=1}^{k}(n+i-1)$. This product occurs frequently enough that it has a name; it is called the kth *rising factorial power* of n and is denoted by $n^{\overline{k}}$. It is read as "n to the k rising." (This notation is due to Don Knuth, who also suggested the notation for falling factorial powers.) We can summarize with a theorem that adds two more formulas for the number of ordered functions.

Theorem 6 *The number of ordered functions from a k-element set to an n-element set is*

$$n^{\overline{k}} = \prod_{i=1}^{k}(n+i-1) = \frac{(n+k-1)!}{(n-1)!} = (n+k-1)^{\underline{k}}.$$

Ordered functions explain the entries in the middle column of rows 5 and 6 of our table of distribution problems.

3.1.3 Multisets

In the middle column of row 7 of our table, we are asking for the number of ways to distribute k identical objects (say ping-pong balls) to n distinct recipients (say children).

[1]The phrase ordered function is not a standard one, because there is as yet no standard name for the result of an ordered distribution problem.

•124. In how many ways may we distribute k identical books on the shelves of a bookcase with n shelves, assuming that any shelf can hold all the books?

Solution: We saw that we could arrange k distinct books on n shelves in $\prod_{i=1}^{k}(n+i-1)$ ways. We partition these arrangements into blocks by putting two arrangements in the same block if we can get one from the other by permuting the books among themselves. Then the number of blocks is the number of ways to place identical books on the shelves. However, there are $k!$ arrangements per block, so there are

$$\frac{\prod_{i=1}^{k}(n+i-1)}{k!} = (n+k-1)^{\underline{k}} = \binom{n+k-1}{k}$$

ways to arrange identical books. ▮

•125. A *multiset* chosen from a set S may be thought of as a subset with repeated elements allowed. To determine a multiset we must say how many times (including, perhaps, zero) each member of S appears in the multiset. The number of times an element appears is called its *multiplicity*. For example if we choose three identical red marbles, six identical blue marbles and four identical green marbles, from a bag of red, blue, green, white and yellow marbles then the multiplicity of a red marble in our multiset is three, while the multiplicity of a yellow marble is zero. The size of a multiset is sum of the multiplicities of its elements. For example if we choose three identical red marbles, six identical blue marbles and four identical green marbles, then the size of our multiset of marbles is 13. What is the number of multisets of size k that can be chosen from an n-element set?

Solution: There is a bijection between arrangements of identical books on n shelves and multisets chosen from an n-element set: the multiplicity of element i is the number of books on shelf i. Thus we have $\binom{n+k-1}{k}$ ways to choose a k-element multiset from an n-element set by Problem 124. ▮

➔126. Your answer in the previous problem should be expressible as a binomial coefficient. Since a binomial coefficient counts subsets, find a bijection between subsets of something and multisets chosen from a set S.

Solution: We will show a bijection between ways of choosing $n-1$ things out of $n+k-1$ things and multisets. Namely, take $n+k-1$

objects and line them up in a row. Choose $n-1$ of them. Now let the multiplicity of element 1 of our multiset be the number of objects before the first object we chose. If $1 < i < n$, let the multiplicity of element i of our multiset be the number of objects between the $i-1$th object we choose and the ith object we choose. Let the multiplicity of the nth element of our multiset be the number of objects after the last one we choose. Another way to say the essentially same thing is to make a list of $n+k-1$ blank spaces. We choose k of them in which we put ones and $n-1$ of them in which we put plus signs. Then the multiplicity of element 1 is the number of ones before the first plus sign, the multiplicity of element n is the number of ones after the last plus sign and if $1 < i < n$, the multiplicity of element i is the number of ones between the $(i-1)$th plus sign and the ith plus sign. Notice that we could have two plus signs in a row if some element has multiplicity 0. ▮

127. How many solutions are there in nonnegative integers to the equation $x_1 + x_2 + \cdots + x_m = r$, where m and r are constants?

Solution: We can think of x_i as the multiplicity of element i of a multiset chosen from among m things. The total number of elements of the multiset will be r. Thus we have $\binom{m+r-1}{r}$ solutions. ▮

128. In how many ways can we distribute k identical objects to n distinct recipients so that each recipient gets at least m?

Solution: First give each recipient m objects. This leaves $k-mn$ identical objects to be distributed among n recipients, so we may do this in the number of ways to choose a $(k-mn)$-element multiset from n things. This is $\binom{n+k-mn+1}{k-mn}$, or $\binom{k-(m-1)n+1}{n+1}$. ▮

Multisets explain the entry in the middle column of row 7 of our table of distribution problems.

3.1.4 Compositions of integers

·129. In how many ways may we put k identical books onto n shelves if each shelf must get at least one book?

Solution: In problem 123 we showed that with k distinct books we could place the books in $k!\binom{k-1}{n-1}$ ways. We can partition these arrangements of distinct books into blocks, where each block consists of all arrangements that we get just by permuting the books among themselves. Thus each block has $k!$ arrangements in it, and each arrangement corresponds to an arrangement of identical books. Thus there are $\binom{k-1}{n-1}$ ways to arrange identical books. Alternatively, as in Problem 128, we could put one book on each shelf and then distribute the remaining $k-n$ books in $\binom{n+k-1-n}{k-n} = \binom{k-1}{n-1}$ ways, using the formula from Problem 124. We could also line up the k identical books in a row and then insert dividers into $n-1$ of the $k-1$ places in between the books. Those before the first divider go on the first shelf, between the first and second on the second shelf, and so on until those after the last divider go onto the nth shelf. ▮

·130. A **composition** of the integer k into n parts is a list of n positive integers that add to k. How many compositions are there of an integer k into n parts?

Solution: There is a bijection between compositions of k into n parts and arrangements of k identical books on n shelves so that each shelf gets a book. Namely, the number of books on shelf i is the ith element of the list. Thus the number of compositions of k into n parts is $\binom{k-1}{n-1}$. ▮

➔ 131. Your answer in Problem 130 can be expressed as a binomial coefficient. This means it should be possible to interpret a composition as a subset of some set. Find a bijection between compositions of k into n parts and certain subsets of some set. Explain explicitly how to get the composition from the subset and the subset from the composition.

Solution: If we line up k identical books, there are $k-1$ places in between two books. If we choose $n-1$ of these places and slip dividers into those places, then we have a first clump of books, a second clump of books, and so on. The ith element of our list is the number of books in the ith clump. Clearly using books is irrelevant; we could line up any k identical objects and make the same argument. Our bijection is between compositions and $n-1$-element subsets of the set of $k-1$ spaces between our objects. ▮

·132. Explain the connection between compositions of k into n parts and the problem of distributing k identical objects to n recipients so that each recipient gets at least one.

Solution: Since the recipients are distinct, we can think of them as a first recipient, a second, and so on. Given a composition of k into n parts, let the ith element of the list be the number of objects given to recipient number i. ▮

The sequence of problems you just completed should explain the entry in the middle column of row 9 of our table of distribution problems.

3.1.5 Broken permutations and Lah numbers

➔ · 133. In how many ways may we stack k distinct books into n identical boxes so that there is a stack in every box?

Solution: We can make a list of the k distinct books in $k!$ ways. Then we have to choose $n-1$ of the $k-1$ places between the lists as the places where we will break the list. However, the order in which we list the boxes is irrelevant, so we have equivalence classes of $n!$ arrangements for each way of putting the books into boxes. Thus we can put the books in boxes in $k!\binom{k-1}{n-1}/n!$ ways.

Alternately, we can take the number of ways to put k books onto n bookshelves so that each shelf gets at least one, and then divide by the number of shelves factorial. That gives us $k!\binom{k-1}{n-1}/n!$ ways to arrange the books. ▮

We can think of stacking books into identical boxes as partitioning the books and then ordering the blocks of the partition. This turns out not to be a useful computational way of visualizing the problem because the number of ways to order the books in the various stacks depends on the sizes of the stacks and not just the number of stacks. However, instead of dividing a set up into non-overlapping parts, we may think of dividing a *permutation* (thought of as a list) of our k objects up into n ordered blocks. We will say that a set of ordered lists of elements of a set S is a **broken permutation** of S if each element of S is in one and only one of these lists.[2] The number of broken permutations of a k-element set with n blocks is denoted by $L(k,n)$. The number $L(k,n)$ is called a *Lah Number* (this is standard) and, from our solution to Problem 133, is equal to $k!\binom{k-1}{n-1}/n!$.

The Lah numbers are the solution to the question "In how many ways may we distribute k distinct objects to n identical recipients if order matters and each recipient must get at least one?" Thus they give the entry in row 6 and column 3 of our table. The entry in row 5 and column 3 of our table will be the number of broken permutations with less than or equal to n parts. Thus it is a sum of Lah numbers.

We have seen that ordered functions and broken permutations explain the entries in rows 5 and 6 of our table.

[2]The phrase broken permutation is not standard, because there is no standard name for the solution to this kind of distribution problem.

In the next two sections we will give ways of computing the remaining entries.

3.2 Partitions and Stirling Numbers

We have seen how the number of partitions of a set of k objects into n blocks corresponds to the distribution of k distinct objects to n identical recipients. While there is a formula that we shall eventually learn for this number, it requires more machinery than we now have available. However there is a good method for computing this number that is similar to Pascal's equation. Now that we have studied recurrences in one variable, we will point out that Pascal's equation is in fact a *recurrence in two variables*; that is, it lets us compute $\binom{n}{k}$ in terms of values of $\binom{m}{i}$ in which either $m < n$ or $i < k$ or both. It was the fact that we had such a recurrence and knew $\binom{n}{0}$ and $\binom{n}{n}$ that let us create Pascal's triangle.

3.2.1 Stirling Numbers of the second kind

We use the notation $S(k, n)$ to stand for the number of partitions of a k element set with n blocks. For historical reasons, $S(k, n)$ is called a *Stirling Number of the second kind*.

➔ •134. In a partition of the set $[k]$, the number k is either in a block by itself, or it is not. How does the number of partitions of $[k]$ with n parts in which k is in a block with other elements of $[k]$ compare to the number of partitions of $[k-1]$ into n blocks? Find a two-variable recurrence for $S(k, n)$, valid for k and n larger than one.

Solution: The number of partitions of $[k]$ into n parts in which k is in a block with other elements of $[k]$ is equal n times the number of partitions of $[k-1]$ into n blocks, because k could be in any of the n parts, and since it is in a block with other elements of $[k-1]$, removing it leaves a partition of $[k-1]$ into n blocks. The number of partitions of $[k]$ into n blocks in which k is in a block by itself is the number of partitions of $[k]$ into $n-1$ blocks, because you can get any such partition by deleting the block containing k from a partition of $[k]$ in which k is in a block by itself. Thus $S(k, n) = S(k-1, n-1) + nS(k-1, n)$. ▮

135. What is $S(k, 1)$? What is $S(k, k)$? Create a table of values of $S(k, n)$ for k between 1 and 5 and n between 1 and k. This table is sometimes

called *Stirling's Triangle (of the second kind)*. How would you define $S(k,0)$ and $S(0,n)$? (Note that the previous question includes $S(0,0)$.) How would you define $S(k,n)$ for $n > k$? Now for what values of k and n is your two variable recurrence valid?

Solution: $S(k,1) = 1$ and $S(k,k) = 1$. We give a table of values of (k,n) for k and n between 0 and 5, a bit more than the problem initially asked for.

$k\backslash n$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	1	3	1	0	0	0
4	0	1	7	6	1	0	0
5	0	1	15	25	10	1	0

As you see in the table, we define $S(0,0) = 1$, and $S(0,n)$ or $S(k,0)$ to be 0 otherwise. This makes sense because for $n > 0$ there is no partition of an empty set into n parts, and for $k > 0$ there is no partition of a k-element set into no parts, but saying there is one partition of the empty set into no parts allows us to use our recurrence to compute $S(1,1)$. Similarly, for $n > k$ there is no way to partition a k element set into n nonempty blocks, giving us $S(k,n) = 0$ if $n > k$. This makes our recurrence valid for all nonnegative values of k and n. ∎

136. Extend Stirling's triangle enough to allow you to answer the following question and answer it. (Don't fill in the rows all the way; the work becomes quite tedious if you do. Only fill in what you need to answer this question.) A caterer is preparing three bag lunches for hikers. The caterer has nine different sandwiches. In how many ways can these nine sandwiches be distributed into three identical lunch bags so that each bag gets at least one?

Solution: We need $S(9,3)$. Thus we need to extend our table for four more rows, but only out to the column labeled 3. These rows are 6,0,1,31,90; 7,0,1,63,301, 8,0,1,127,966; 9,0,1,255,3025. Thus there are 3025 ways to distribute the sandwiches into the lunch bags. If you work backwards from $S(9,3)$, you will see we don't need the first thee entries of row 9, the first two entries of row 8 and the first entry of row 7 (which is zero anyhow). ∎

137. The question in Problem 136 naturally suggests a more realistic question; in how many ways may the caterer distribute the nine sandwiches into three identical bags so that each bag gets exactly three? Answer this question.

Solution: $\binom{9}{3}\binom{6}{3}\binom{3}{3}/3!$. First we choose three sandwiches for bag 1, then three for bag 2, and put the remainder in bag 3. However, it doesn't matter which bags the sandwiches are in so we have counted each partition 3! times. ▮

·138. What is $S(k, k-1)$?

Solution: If a partition has $k-1$ parts, then one part has two elements, so once we choose those two elements from the k elements, we are done. Therefore $S(k, k-1) = \binom{k}{2}$. ▮

•139. In how many ways can we partition k (distinct) items into n blocks so that we have k_i blocks of size i for each i? (Notice that $\sum_{i=1}^{k} k_i = n$ and $\sum_{i=1}^{k} ik_i = k$.) The sequence $k_1, k_2, \ldots, k_n$ is called the *type vector* of the partition.

Solution: $\frac{n!}{\prod_{i=1}^{n}(i!)^{k_i}k_i!}$. We can make a list in $n!$ ways, and then break it into first k_1 blocks of size 1, then k_2 blocks of size 2, k_3 blocks of size 3 up to k_n blocks of size n. But then we realize that we get the same partition if we permute the $i!$ elements of a block of size i and we get the same partition if we permute the k_i blocks of size i so we apply the quotient principle. ▮

+140. Describe how to compute $S(n, k)$ in terms of quantities given by the formula you found in Problem 139.

Solution: We can find $S(n, k)$ by summing $\frac{n!}{\prod_{i=1}^{n}(i!)^{k_i}k_i!}$ over all type vectors $(k_1, k_2, \ldots, k_n)$ such that $k_1 + k_2 + \cdots + k_n = k$. ▮

➔141. Find a recurrence for the Lah numbers $L(k, n)$ similar to the one in Problem 134.

Solution: $L(k, n)$ is the number of broken permutations of a k-element set into n parts. Either k is in an ordered block by itself or it is not. If it is, it can go after any of the $k-1$ other elements, or it can go at the beginning of any of the n blocks. If it is not, deleting it gives a broken permutation of a $k-1$-element set into $n-1$ blocks. Thus $L(k, n) = L(k-1, n-1) + (n+k-1)L(k, n)$. ▮

·142. (Relevant in Appendix C.) The total number of partitions of a k-element set is denoted by $B(k)$ and is called the k-th *Bell number*. Thus $B(1) = 1$ and $B(2) = 2$.

(a) Show, by explicitly exhibiting the partitions, that $B(3) = 5$.

Solution: The five partitions of $[3]$ are the sets $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1\},\{2,3\}\}$, and $\{\{1,2,3\}\}$. ▮

(b) Find a recurrence that expresses $B(k)$ in terms of $B(n)$ for $n < k$ and prove your formula correct in as many ways as you can.

Solution: If we delete the block containing k, we get a partition of a subset of $[k-1]$. Thus $B(k)$ is the sum over all subsets of $[k-1]$ of the number of partitions of that subset. This gives us $B(k) = \sum_{n=0}^{k-1} \binom{k-1}{n} B(n)$.

Alternatively, we can show by the same sort of argument that $S(k,n) = \sum_{i=0}^{k-1} \binom{k-1}{i} S(i, n-1)$ and then use the fact that $B(k) = \sum_{n=0}^{k} S(k,n)$ to get the recurrence for $B(k)$. ▮

(c) Find $B(k)$ for $k = 4, 5, 6$.

Solution:

$$B(4) = \binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 + \binom{3}{3}B_3 = 1+3+3\cdot 2+5 = 15$$

$$B(5) = \sum_{n=0}^{4} \binom{4}{n} B_n = 1 + 4 + 6\cdot 2 + 4\cdot 5 + 15 = 52$$

$$B(6) = \sum_{n=0}^{5} \binom{5}{n} B_n = 1 + 5 + 10\cdot 2 + 10\cdot 5 + 5\cdot 15 + 52 = 203$$

▮

3.2.2 Stirling Numbers and onto functions

∘143. Given a function f from a k-element set K to an n-element set, we can define a partition of K by putting x and y in the same block of the partition if and only if $f(x) = f(y)$. How many blocks does the partition have if f is onto? How is the number of functions from a k-element set onto an n-element set related to a Stirling number? Be as precise in your answer as you can.

Solution: If f is onto, the number of blocks of the partition is n. The number of onto functions from a k-element set onto an n-element set is $S(k,n)n!$, because we have a one-to-one function from the blocks to the n-element set. ▮

➔ 144. How many labeled trees on n vertices have exactly 3 vertices of degree one? Note that this problem has appeared before in Chapter 2.

Solution: There are $\binom{n}{3}$ ways to choose the three vertices of degree 1. The remaining $n-3$ vertices must appear in the Prüfer code for the tree. We can think of the Prüfer code as a function from the $n-2$ places of the code onto the $n-2$ remaining vertices, so that there are $S(n-2,n-3)(n-3)!$ possible Prüfer codes. Thus we have $\binom{n}{3}\binom{n-2}{2}(n-3)! = n!(n-2)(n-3)/12$ labeled trees on n vertices. ▮

• 145. Each function from a k-element set K to an n-element set N is a function from K onto *some* subset of N. If J is a subset of N of size j, you know how to compute the number of functions that map onto J in terms of Stirling numbers. Suppose you add the number of functions mapping onto J over all possible subsets J of N. What simple value should this sum equal? Write the equation this gives you.

Solution: The sum should equal the number of functions, n^k. Thus we get $\sum_{j=0}^{n}\binom{n}{j}S(k,j)j! = n^k$. By using the fact that $\binom{n}{j} = n^{\underline{j}}/j!$, this may be rewritten as $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = n^k$. ▮

∘ 146. In how many ways can the sandwiches of Problem 136 be placed into three distinct bags so that each bag gets at least one?

Solution: $S(9,3)\cdot 3! = 55,980$. ▮

∘ 147. In how many ways can the sandwiches of Problem 137 be placed into distinct bags so that each bag gets exactly three?

Solution: Choose three sandwiches for bag one in $\binom{9}{3}$ ways, three for bag two in $\binom{6}{3}$ ways and put the remainder in bag 3. This gives us $\binom{9}{3}\binom{6}{3} = \frac{9!}{3!3!3!} = 1680$ ways.

The $\frac{9!}{3!3!3!}$ suggests another solution. We can line up the sandwiches in 9! ways. We take the first three for bag one, the second three for bag two and the last three for bag 3. The order of the sandwiches in the bag does not matter though, so each there are 3!3!3! listings corresponding to each way of putting sandwiches in bags, giving us $\frac{9!}{3!3!3!}$ ways to put the sandwiches in bags. ▮

• 148. In how many ways may we label the elements of a k element set with n distinct labels (numbered 1 through n) so that label i is used j_i times?

(If we think of the labels as $y_1, y_2, \ldots, y_n$, then we can rephrase this question as follows. How many functions are there from a k-element set K to a set $N = \{y_1, y_2, \ldots y_n\}$ so that each y_i is the image of j_i elements of K?) This number is called a *multinomial coefficient* and denoted by

$$\binom{k}{j_1, j_2, \ldots, j_n}.$$

Solution: If the j_is don't add to k, it is zero. Otherwise, $\binom{k}{j_1,j_2,\ldots,j_n} = \frac{k!}{j_1!j_2!\cdots j_n!}$. We get this either as the product of binomial coefficients

$$\binom{k}{j_1}\binom{k-j_1}{j_2}\binom{k-j_1-j_2}{j_3}\cdots\binom{j_n}{j_n},$$

or more elegantly, by lining up the elements of the domain in $k!$ ways, taking the first j_1 elements to y_1, the next j_2 elements to y_2 and so on. However the order of the j_i elements that go to y_i is irrelevant, so $j_1!j_2!\cdots j_n!$ lists all correspond to the same function, giving us $\frac{k!}{j_1!j_2!,\cdots j_n!}$ functions. ∎

149. Explain how to compute the number of functions from a k-element set K to an n-element set N by using multinomial coefficients.

  Solution: Add the multinomial coefficients $\binom{k}{j_1,j_2,\ldots,j_n}$ over all possible nonnegative values of the j_is that add to k. To see why, let $N = \{y_1, y_2, \ldots, y_n\}$ and apply the definition of multinomial coefficients. ∎

150. Explain how to compute the number of functions from a k-element set K onto an n-element set N by using multinomial coefficients.

  Solution: Add the multinomial coefficients $\binom{k}{j_1,j_2,\ldots,j_n}$ in which each j_i is positive. To see why, let $N = \{y_1, y_2, \ldots, y_n\}$ and note that we are counting functions that send at least one element of K to each element y_i. ∎

•151. What do multinomial coefficients have to do with expanding the kth power of a multinomial $x_1 + x_2 + \cdots + x_n$? This result is called the *multinomial theorem.*

  Solution: When we use the distributive law to multiply out $(x_1 + x_2 + \cdots + x_n)^k$, we will get a sum of a bunch of terms of the form

$x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ where $i_1+i_2+\cdots+i_n=k$. The terms with a given sequence $i_1, i_2, \ldots, i_n$ of exponents will arise from choosing, as we apply the distributive law over and over again, x_1 from i_1 of the factors, x_2 from i_2 of the factors, and so on. Thus the number of terms $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ will be the number of ways to label i_1 of the factors with a 1, i_2 of the factors with a 2, …, and i_n of the factors with an n. The number of ways to do this is a multinomial coefficient, as we now explain. This labeling gives us a function from $[k]$ to $[n]$ as follows. If factor i is labeled j we let $f(i)=j$. Further each function f from $[k]$ to $[n]$ gives us that maps i_j elements of $[k]$ to j will give us such a labeling. Thus the coefficient of $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ will be the multinomial coefficient $\binom{k}{i_1,i_2,\ldots,i_n}$. ∎

3.2.3 Stirling Numbers and bases for polynomials

·152. (a) Find a way to express n^k in terms of $n^{\underline{j}}$ for appropriate values j. You may use Stirling numbers if they help you.

Solution: In Problem 145, we saw that $\sum_{j=0}^{n}\binom{n}{j}S(k,j)j!=n^k$. Using the relationship between binomial coefficients and falling factorials, we may rewrite this as $\sum_{j=0}^{n}n^{\underline{j}}S(k,j)=n^k$. This expresses n^k in terms of $n^{\underline{j}}$. ∎

(b) Notice that $x^{\underline{j}}$ makes sense for a numerical variable x (that could range over the rational numbers, the real numbers, or even the complex numbers instead of only the nonnegative integers, as we are implicitly assuming n does), just as x^j does. Find a way to express the power x^k in terms of the polynomials $x^{\underline{j}}$ for appropriate values of j and explain why your formula is correct.

Solution: To be precise, we define $x^{\underline{j}}$ to be $x(x-1)\cdots(x-j+1)$. At first glance it looks like we could express $x^{\underline{j}}$ in terms of powers of x by simply substituting x for n in the equation $\sum_{j=0}^{n}n^{\underline{j}}S(k,j)=n^k$. However this gives us $\sum_{j=0}^{x}x^{\underline{j}}S(k,j)=x^k$, and we have never defined what we mean by a sum whose upper limit is a variable x. Thus we need to examine the equation $\sum_{j=0}^{n}n^{\underline{j}}S(k,j)=n^k$ to see if we can replace the n that is the upper limit of the sum with something else. Notice that $S(k,j)=0$ when $j>k$. This means that if $k\le j$, then $\sum_{j=0}^{n}n^{\underline{j}}S(k,j)=\sum_{j=0}^{k}n^{\underline{j}}S(k,j)$. Notice also that $n^{\underline{j}}=n(n-1)\cdots(n-j+1)$ is zero when $j>n$ because one of its factors is zero then. This implies that if $k>j$, then $\sum_{j=0}^{n}n^{\underline{j}}S(k,j)=\sum_{j=0}^{k}n^{\underline{j}}S(k,j)$.

Therefore, regardless of the relative size of k and n, we have that $\sum_{j=0}^{n} n^{\underline{j}} S(k,j) = \sum_{j=0}^{k} n^{\underline{j}} S(k,j)$. Therefore

$$\sum_{j=0}^{k} n^{\underline{j}} S(k,j) = n^k. \qquad (*)$$

It makes sense to write the polynomial $\sum_{j=0}^{k} x^{\underline{j}} S(k,j)$; this is simply a polynomial of degree k in the variable x. The expression $\sum_{j=0}^{k} x^{\underline{j}} S(k,j) - x^k$ is also a polynomial in x, but it might not be of degree k since we are subtracting a degree k term from a degree k polynomial. In fact for every positive integer value n of x, this polynomial is zero. That is, $\sum_{j=0}^{k} n^{\underline{j}} S(k,j) - n^k = 0$, which is just a restatement of the Equation marked $(*)$. But it is a fact of algebra that the number of solutions of a nontrivial polynomial equation is no more than the degree of the polynomial. Since the polynomial equation $\sum_{j=0}^{k} x^{\underline{j}} S(k,j) - x^k$ has infinitely many different solutions, it must be a trivial equation; that is, $\sum_{j=0}^{k} x^{\underline{j}} S(k,j) - x^k$ must be zero for every real (and even every complex) number x. Thus $\sum_{j=0}^{k} x^{\underline{j}} S(k,j) = x^k$, and we have expressed x^k in terms of $x^{\underline{j}}$ for $j \leq k$. ∎

You showed in Problem 152b how to get each power of x in terms of the falling factorial powers $x^{\underline{j}}$. Therefore every polynomial in x is expressible in terms of a sum of numerical multiples of falling factorial powers. Using the language of linear algebra, we say that the ordinary powers of x and the falling factorial powers of x each form a basis for the "space" of polynomials, and that the numbers $S(k,n)$ are "change of basis coefficients." If you are not familiar with linear algebra, a *basis* for the *space of polynomials*[3] is a set of polynomials such that each polynomial, whether in that set or not, can be expressed in one and only one way as a sum of numerical multiples of polynomials in the set.

∘153. Show that every power of $x+1$ is expressible as a sum of numerical multiples of powers of x. Now show that every power of x (and thus every polynomial in x) is a sum of numerical multiples (some of which could be negative) of powers of $x+1$. This means that the powers of $x+1$ are a basis for the space of polynomials as well. Describe the change of basis coefficients that we use to express the binomial powers

[3] The space of polynomials is just another name for the set of all polynomials.

$(x+1)^n$ in terms of the ordinary x^j explicitly. Find the change of basis coefficients we use to express the ordinary powers x^n in terms of the binomial powers $(x+1)^k$.

Solution: We know that

$$(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \tag{3.1}$$

from the binomial theorem. (In the way we stated the binomial theorem, instead of $\binom{n}{i}x^i$ we would have gotten $\binom{n}{i}x^{n-i}$. There are two ways to fix this. One is to observe that the coefficient of x^i in that expansion is $\binom{n}{n-i}$, which equals $\binom{n}{i}$. The other is to observe that when we expand $(1+x)^n$ according to the binomial theorem we get exactly what we wrote on the right hand side in Equation 3.1.) Therefore every power of $x+1$ is expressible in terms of powers of x.

How do we express powers of x in terms of powers of $x+1$? Some experimentation would help us guess how to do so; however there is a really nice trick that also isn't hard to see. Namely, we can write

$$\begin{aligned} x^n = (x+1-1)^n = [(x+1)-1]^n &= \sum_{i=0}^{n} \binom{n}{i} (x+1)^{n-i}(-1)^i \\ &= \sum_{i=0}^{n} \binom{n}{i} (x+1)^{i}(-1)^{n-i} \end{aligned}$$

This means that every power of x is expressible in terms of powers of $x+1$ and the change of basis coefficients to express powers of x in terms of powers of $x+1$ are $(-1)^{n-i}\binom{n}{i}$, while the change of basis coefficients used to express powers of $x+1$ in terms of powers of x are $\binom{n}{i}$. ∎

➔ · 154. By multiplication, we can see that every falling factorial polynomial can be expressed as a sum of numerical multiples of powers of x. In symbols, this means that there are numbers $s(k,n)$ (notice that this s is lower case, not upper case) such that we may write $x^{\underline{k}} = \sum_{n=0}^{k} s(k,n)x^n$. These numbers $s(k,n)$ are called Stirling Numbers of the first kind. By thinking algebraically about what the formula

$$x^{\underline{k}} = x^{\underline{k-1}}(x-k+1) \tag{3.2}$$

means, we can find a recurrence for Stirling numbers of the first kind that gives us another triangular array of numbers called Stirling's triangle of the first kind. Explain why Equation 3.2 is true and use it to derive a recurrence for $s(k,n)$ in terms of $s(k-1,n-1)$ and $s(k-1,n)$.

Solution: Equation 3.2 is effectively the inductive step of an inductive definition of $x^{\underline{k}}$. With this equation we can write

$$\begin{aligned}
\sum_{n=0}^{k} s(k,n)x^n &= x^{\underline{k}} = x^{\underline{k-1}}(x-k+1) \\
&= \left(\sum_{n=0}^{k-1} s(k-1,n)x^n\right)(x-k+1) \\
&= \sum_{n=0}^{k-1} s(k-1,n)x^{n+1} - \sum_{n=0}^{k-1}(k-1)s(k-1,n)x^n \\
&= \sum_{n=1}^{k} s(k-1,n-1)x^n - \sum_{n=0}^{k-1}(k-1)s(k-1,n)x^n.
\end{aligned}$$

Equating the coefficients of x^n in the first and last line of this equation, we get $s(k,n) = s(k-1,n-1) - (k-1)s(k-1,n)$, for n between 1 and $k-1$. ▮

155. Write down the rows of Stirling's triangle of the first kind for $k = 0$ to 6.

Solution:

$k \backslash n$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	-1	1	0	0	0	0
3	0	2	-3	1	0	0	0
4	0	-6	11	-6	1	0	0
5	0	24	-50	35	-10	1	0
6	0	-120	274	-225	85	-15	1

▮

By definition, the Stirling numbers of the first kind are also change of basis coefficients. The Stirling numbers of the first and second kind are change of basis coefficients from the falling factorial powers of x to the ordinary factorial powers, and vice versa.

➔ 156. Explain why every rising factorial polynomial $x^{\overline{k}}$ can be expressed as a sum of multiples of the falling factorial polynomials $x^{\underline{n}}$. Let $b(k,n)$ stand for the change of basis coefficients that allow us to express $x^{\overline{k}}$ in terms of the falling factorial polynomials $x^{\underline{n}}$; that is, define $b(k,n)$ by the equations

$$x^{\overline{k}} = \sum_{n=0}^{k} b(k,n)x^{\underline{n}}.$$

(a) Find a recurrence for $b(k,n)$.

Solution:

$$\begin{aligned}
&\sum_{n=0}^{k} b(k,n)x^{\underline{n}} = x^{\overline{k}} = x^{\overline{k-1}}(x+k-1) \\
= &\left(\sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n}}\right)(x+k-1) \\
= &\sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n}}(x+k-1) \\
= &\sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n}}(x-n+n+k-1) \\
= &\sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n+1}} + (n+k-1)b(k-1,n)x^{\underline{n}} \\
= &\sum_{n=1}^{k} b(k-1,n-1)x^{\underline{n}} + \sum_{n=0}^{k-1}(n+k-1)b(k-1,n)x^{\underline{n}}
\end{aligned}$$

Thus if n is not 0 or k, we equate the coefficient of $x^{\underline{n}}$ in the first line and last line to get

$$b(k,n) = b(k-1,n-1) + (n+k-1)b(k-1,n).$$

The trick of subtracting n and adding n in the middle of the computation was the result of wanting to mimic the way in which we increased the power on x in the solution to Problem 154. ▮

(b) Find a formula for $b(k,n)$ and prove the correctness of what you say in as many ways as you can.

Solution: We will answer the next part of the problem here! The recurrence for $b(k,n)$ is exactly the same as the recurrence for $L(k,n)$. Further, $b(0,0) = 1 = L(0,0)$, $b(0,n) = 0 = L(0,n)$ for $n > 0$, and $b(k,k) = L(k,k) = 1$. Thus $b(k,n)$ and $L(k,n)$ are identical. This and the formula from Problem 133 gives one proof that $b(k,n) = k!\binom{k-1}{n-1}/n!$.

A second proof that the change of basis coefficients are Lah numbers goes as follows. $n^{\overline{k}}$ counts the number of ordered functions from a k-element set to an n-element set. One way to determine such an ordered function is to take a broken permutation of the k-element set into n or fewer parts, and then take a one-to-one function from the parts to the n-element set. More informally we assign the parts of the broken permutation to distinct elements of the n-element set. If the broken permutation has i parts, the number of ways to do this assignment is the number of i-element permutations of an n-element set, $n^{\underline{i}}$. Thus $n^{\overline{k}} = \sum_{i=0}^{n} L(k,i)n^{\underline{i}}$. However we can change the upper limit of the sum to k because $L(k,i)$ is zero when $i > k$ and $n^{\underline{i}}$ is zero when $i > n$. Now we change n to x because we have a polynomial equality which is valid for infinitely many of the values of the variable. This gives us $x^{\overline{k}} = \sum_{i=0}^{k} L(k,i)x^{\underline{i}}$. Thus $b(k,i) = L(k,i)$. ∎

(c) Is $b(k,n)$ the same as any of the other families of numbers (binomial coefficients, Bell numbers, Stirling numbers, Lah numbers, etc.) we have studied?

Solution: As we said in our solution to the previous part, $b(k,n)$ is the Lah number $L(k,n)$. ∎

(d) Say as much as you can (but say it precisely) about the change of basis coefficients for expressing $x^{\underline{k}}$ in terms of $x^{\overline{n}}$.

Solution: There are several ways of finding this relationship, but the most concise way is to observe that $(-x)^{\underline{k}} = (-1)^k x^{\overline{k}}$ and $(-x)^{\overline{k}} = (-1)^k x^{\underline{k}}$. This lets us write

$$\begin{aligned}
(-x)^{\overline{k}} &= \sum_{n=0}^{k} b(k,n)(-x)^{\underline{n}} \\
(-1)^k x^{\underline{k}} &= \sum_{n=0}^{k} (-1)^n b(k,n) x^{\overline{n}} \\
x^{\underline{k}} &= \sum_{n=0}^{k} (-1)^{n-k} b(k,n) x^{\overline{n}}.
\end{aligned}$$

Therefore the change of basis coefficients are $(-1)^{n-k}b(k,n)$. ∎

3.3 Partitions of Integers

We have now completed all our distribution problems except for those in which both the objects and the recipients are identical. For example, we might be putting identical apples into identical paper bags. In this case all that matters is how many bags get one apple (how many recipients get one object), how many get two, how many get three, and so on. Thus for each bag we have a number, and the multiset of numbers of apples in the various bags is what determines our distribution of apples into identical bags. A multiset of positive integers that add to n is called a **partition** of n. Thus the partitions of 3 are 1+1+1, 1+2 (which is the same as 2+1) and 3. The number of partitions of k is denoted by $P(k)$; in computing the partitions of 3 we showed that $P(3) = 3$. It is traditional to use Greek letters like λ (the Greek letter λ is pronounced LAMB duh) to stand for partitions; we might write $\lambda = 1, 1, 1$, $\gamma = 2, 1$ and $\tau = 3$ to stand for the three partitions of three. We also write $\lambda = 1^3$ as a shorthand for $\lambda = 1, 1, 1$, and we write $\lambda \dashv 3$ as a shorthand for "λ is a partition of three."

∘157. Find all partitions of 4 and find all partitions of 5, thereby computing $P(4)$ and $P(5)$.

Solution: $4 = 1+1+1+1$, $4 = 2+1+1$, $4 = 2+1$, $4 = 3+1$, $4 = 4$, so that $P(4) = 5$. $5 = 1+1+1+1+1$, $5 = 2+1+1+1$, $5 = 2+2+1$, $5 = 3+1+1$, $5 = 3+2$, $5 = 4+1$, $5 = 5$, so that $P(5) = 7$. ▮

3.3.1 The number of partitions of k into n parts

A *partition of the integer k into n parts* is a multiset of n positive integers that add to k. We use $P(k, n)$ to denote the number of partitions of k into n parts. Thus $P(k, n)$ is the number of ways to distribute k identical objects to n identical recipients so that each gets at least one.

∘158. Find $P(6, 3)$ by finding all partitions of 6 into 3 parts. What does this say about the number of ways to put six identical apples into three identical bags so that each bag has at least one apple?

Solution: $6 = 4+1+1$, $6 = 3+2+1$, $6 = 2+2+2$, so $P(6, 3) = 3$. This says there are three ways to put six identical apples into three identical bags so that each bag gets at least one apple. ▮

3.3.2 Representations of partitions

∘159. How many solutions are there in the positive integers to the equation $x_1 + x_2 + x_3 = 7$ with $x_1 \geq x_2 \geq x_3$?

Solution: This problem is asking for $P(7, 3)$ and suggests an organized way to go about finding it: list the partitions starting with the largest part and work down. $7 = 5+1+1$, $7 = 4+2+1$, $7 = 3+3+1$, $7 = 3+2+2$, and if we have three numbers that add to seven, one must be larger than two, so there are four such solutions. ▮

160. Explain the relationship between partitions of k into n parts and lists $x_1, x_2, \ldots, x_n$ of positive integers that add to k with $x_1 \geq x_2 \geq \ldots \geq x_n$. Such a representation of a partition is called a *decreasing list* representation of the partition.

Solution: There is a bijection between partitions of k into n parts and lists, in non-increasing order, of n positive integers that add to k, because each multiset of numbers that adds to k can be listed in non-increasing order in exactly one way. ∎

∘161. Describe the relationship between partitions of k and lists or vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + 2x_2 + \ldots kx_k = k$. Such a representation of a partition is called a *type vector* representation of a partition, and it is typical to leave the trailing zeros out of such a representation; for example $(2, 1)$ stands for the same partition as $(2, 1, 0, 0)$. What is the decreasing list representation for this partition, and what number does it partition?

Solution: The type vector of a partition of k is a way of representing the multiplicity function of the multiset of integers that adds to k. Thus there is a bijection between type vectors and partitions. The decreasing list representation of the partition with type vector $(2, 1)$ is 2,1,1. This is a partition of 4 ∎

162. How does the number of partitions of k relate to the number of partitions of $k + 1$ whose smallest part is one?

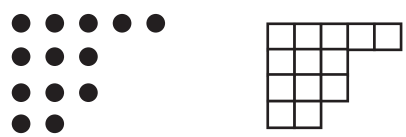
Solution: They are equal, because if we take two different partitions of k and increase the multiplicity of 1 in each (by one), they are still different; also if we take two different partitions of $k + 1$ that have parts of size one, and decrease the multiplicity of 1 in each (by one), they are still different. ∎

When we write a partition as $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$, it is customary to write the list of λ_is as a decreasing list. When we have a type vector $(t_1, t_2, \ldots, t_m)$ for a partition, we write either $\lambda = 1^{t_1} 2^{t_2} \cdots m^{t_m}$ or $\lambda = m^{t_m} (m-1)^{t_{m-1}} \cdots 2^{t_2} 1^{t_1}$. Henceforth we will use the second of these. When we write $\lambda = \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_n^{i_n}$, we will assume that $\lambda_i > \lambda_{i+1}$.

3.3.3 Ferrers and Young Diagrams and the conjugate of a partition

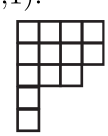
The decreasing list representation of partitions leads us to a handy way to visualize partitions. Given a decreasing list $(\lambda_1, \lambda_2, \ldots \lambda_n)$, we draw a figure made up of rows of dots that has λ_1 equally spaced dots in the first row, λ_2 equally spaced dots in the second row, starting out right below the beginning of the first row and so on. Equivalently, instead of dots, we may use identical squares, drawn so that a square touches each one to its immediate right or immediately below it along an edge. See Figure 3.1 for examples. The figure we draw with dots is called the Ferrers diagram of the partition; sometimes the figure with squares is also called a Ferrers diagram; sometimes it is called a Young diagram. At this stage it is irrelevant which name we choose and which kind of figure we draw; in more advanced work the squares are handy because we can put things like numbers or variables into them. From now on we will use squares and call the diagrams Young diagrams.

Figure 3.1: The Ferrers and Young diagrams of the partition (5,3,3,2)



•163. Draw the Young diagram of the partition (4,4,3,1,1). Describe the geometric relationship between the Young diagram of (5,3,3,2) and the Young diagram of (4,4,3,1,1).

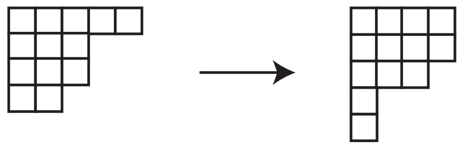
Solution:



We get the Young diagram of $(5, 3, 3, 2)$ by flipping the Young diagram of $(4, 4, 3, 1, 1)$ around a line that includes the diagonal of the upper left box; if we think of the top left corner of the diagram as being at the origin, we flip around the line $y = -x$. ▮

•164. The partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is called the *conjugate* of the partition $(\gamma_1, \gamma_2, \ldots, \gamma_m)$ if we obtain the Young diagram of one from the Young diagram of the other by flipping one around the line with slope -1 that extends the diagonal of the top left square. See Figure 3.2 for an example. What is the conjugate of (4,4,3,1,1)? How is the largest part of a partition related to the number of parts of its conjugate? What does this tell you about the number of partitions of a positive integer k with largest part m?

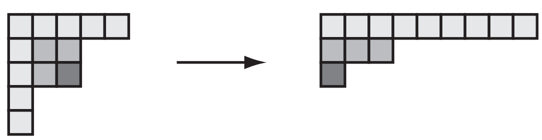
Figure 3.2: The Ferrers diagram the partition (5,3,3,2) and its conjugate.



Solution: $(5, 3, 3, 2)$. The largest part of a partition equals the number of parts of its conjugate. The number of partitions of k with largest part m equals the number of partitions of k with m parts. ▮

➔165. A partition is called *self-conjugate* if it is equal to its conjugate. Find a relationship between the number of self-conjugate partitions of k and the number of partitions of k into distinct odd parts.

Solution: The number of self-conjugate partitions of k equals the number of partitions of k with distinct odd parts. Here is a geometric description of a bijection from self-conjugate partitions of k to partitions into distinct odd parts.



Take the top row and left column of squares of the Young diagram, and make them into one row in a new diagram. (Only include the square that is in both the row and column once.) Now take the remaining squares in the next row and column and make a new row of the Young diagram of the second partition with them. Continue this process with succeeding rows and columns, not using any squares you have already used. Because the first partition is self-conjugate, the diagram has the same number of rows as columns and row i and column i have the same length. Because row i and column i share one square, and we only use that square once when we create a new row, each row we create has odd length. Thus we get a partition with the same number of squares, so it is a partition of k and each part is odd. The parts are distinct because when we take off the squares of a row and column, we reduce the number of squares in each row and column that remains. Given a partition of k into distinct odd parts, we use the fact that each row has a unique middle element, and each is shorter than the one above (by at least two squares) to reverse the process. Thus we have a bijection. ▮

166. Explain the relationship between the number of partitions of k into even parts and the number of partitions of k into parts of even multiplicity, i.e. parts which are each used an even number of times as in (3,3,3,3,2,2,1,1).

Solution: The number of partitions of k into even parts equals the number of partitions of parts of even multiplicity, because if we take the Young diagram of a partition of k into even parts and conjugate it, the resulting diagram has columns of even length. Thus the difference in heights of two successive columns is an even number, but this difference is the multiplicity of one of the parts of the conjugate. Further the height of the last column of a partition is the multiplicity of the first part. Since the multiplicity of any part of a partition is either the difference in height of two successive columns of the Young diagram or the height of the last column, then each part of the conjugate has even multiplicity. This bijection can be reversed, because if all the differences in height of the columns are even and the height of the last column is even, then when we conjugate this partition, the last row will be an even length, and all differences in length of the rows will be even, so all the parts of the resulting partition will be even. ▮

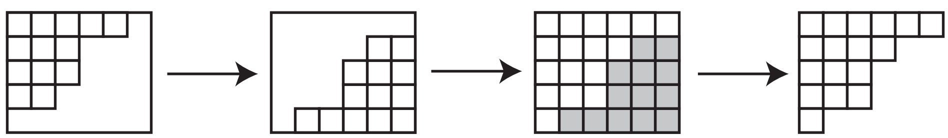
➔ 167. Show that the number of partitions of k into four parts equals the number of partitions of $3k$ into four parts of size at most $k-1$ (or $3k-4$ into four parts of size at most $k-2$ or $3k+4$ into four parts of size at most k).

Solution: Think about putting the Young diagram of the partition into the upper left corner of a rectangle that is k units wide and four units high. Subdivide the rectangle into $4k$ squares of unit area. The Young diagram covers k of these squares. The uncovered squares are in rows of length $r_1 \leq r_2 \leq r_3 \leq r_4$. Thus if we list these lengths in the opposite order, we have a decreasing list representation of a partition of $3k$. Even r_1 will have to be positive, because the first part of the original partition will be at most $k-3$. The part size will be at most $k-1$ because r_4 must be less than k since the smallest part of the original partition is at least 1. To get partitions of $3k+4$, use a rectangle of width $k+1$, and to get partitions of $3k-4$, use a rectangle of width $k-1$. Since the first row of the Young diagram has at most

$k-3$ squares, we will still have four nonzero parts in the partition that results. ▮

168. The idea of conjugation of a partition could be defined without the geometric interpretation of a Young diagram, but it would seem far less natural without the geometric interpretation. Another idea that seems much more natural in a geometric context is this. Suppose we have a partition of k into n parts with largest part m. Then the Young diagram of the partition can fit into a rectangle that is m or more units wide (horizontally) and n or more units deep. Suppose we place the Young diagram of our partition in the top left-hand corner of an m' unit wide and n' unit deep rectangle with $m' \geq m$ and $n' \geq n$, as in Figure 3.3.

Figure 3.3: To complement the partition (5,3,3,2) in a 6 by 5 rectangle: enclose it in the rectangle, rotate, and cut out the original Young diagram.



(a) Why can we interpret the part of the rectangle not occupied by our Young diagram, rotated in the plane, as the Young diagram of another partition? This is called the *complement* of our partition in the rectangle.

Solution: If we fill the rectangle with unit squares, those not in the Young diagram of the original partition λ will fall into rows. The lengths of the rows are nonnegative, and are nondecreasing as we move down. Therefore, after we rotate through 180 degrees, these same rows will be listed in the opposite order, lined up along the left sides, and will have non-increasing length. Thus they will be the Young diagram of a partition. ▮

(b) What integer is being partitioned by the complement?

Solution: The integer being partitioned will be $m'n' - k$. ▮

(c) What conditions on m' and n' guarantee that the complement has the same number of parts as the original one?

Solution: If $m' > m$ and $n' = n$, then the two partitions will have the same number of parts, because we will have a nonzero number of empty squares at the end of each row of the Young diagram of λ. If $m' = m$ and $n' - n$ is the multiplicity of the largest part of λ, they will have the same number of parts. Otherwise, their numbers of parts will differ. ∎

(d) What conditions on m' and n' guarantee that the complement has the same largest part as the original one?

Solution: If $n' > n$ and $m = m'$, then the two partitions will have the same largest part. If $n' = n$ and $m' - m$ is the smallest part of λ, then they will have the same largest part. Otherwise, their largest parts will differ. ∎

(e) Is it possible for the complement to have both the same number of parts and the same largest part as the original one?

Solution: For the two partitions to have the same number of parts, either $m' = m$ or $n' = n$. If $m' = m$ and they have the same largest part, then $n' > n$. But this is consistent with $n' - n$ being the multiplicity of the largest part of λ. Thus they can have the same number of parts and the same largest part if $m' = m$ and $n' - n$ is the multiplicity of the largest part of λ, or similarly if $n = n'$ and $m' - m$ is the smallest part of λ. ∎

(f) If we complement a partition in an m' by n' box and then complement that partition in an m' by n' box again, do we get the same partition that we started with?

Solution: If we complement a partition in an m' by n' box and then complement that partition in *the same rectangle*, then we get the original partition back. ∎

➔ 169. Suppose we take a partition of k into n parts with largest part m, complement it in the smallest rectangle it will fit into, complement the result in the smallest rectangle it will fit into, and continue the process until we get the partition 1 of one into one part. What can you say about the partition with which we started?

Solution: Let us call the process of enclosing λ in the smallest rectangle possible and then forming the complement in that rectangle *encomplementation* (this is short for *en*closure and *complementation* and is not a standard term—there is no standard term for this operation) and call the result of it the *encomplement* of λ. The result of two encomplementations on the Young diagram of a partition is to remove all rows of maximum length and all columns of maximum length from the Young diagram. Thus the description of the result of an even number

$2j$ of encomplementations is straightforward; we remove all the rows of the j largest distinct lengths and all columns of the j largest distinct lengths. So if an even number of encomplementations brings us to a partition with one block of size one, we should be able to describe the original partition fairly easily. To deal with the result of an odd number of encomplementations, we ask what happens if we encomplement just once. If the complement of λ in the smallest rectangle in which if fits has one square, then $\lambda = \lambda_1^{n_1}\lambda_1 - 1$. Thus we are asking for the partitions which, after an even number of encomplementations, give us either the partition with one block or a partition of the form $\lambda_1^{n_1}(\lambda_1 - 1)$. First we ask what kind of partition results in the second one after two encomplementations. If we get $\lambda_1^{n_1}(\lambda_1 - 1)$ from two encomplementations, the partition we started with had the form

$$\lambda_0^{n_0}(\lambda_1 + \lambda_2)^{n_1}(\lambda_1 + \lambda_2 - 1)\lambda_2^{n_2}.$$

If we get $\lambda_1^{n_1}(\lambda_1 - 1)$ from four encomplementations, then we started with a partition of the form

$$\lambda_{-1}^{n_{-1}}(\lambda_0 + \lambda_3)^{n_0}(\lambda_1 + \lambda_2 + \lambda_3)^{n_1}(\lambda_1 + \lambda_2 + \lambda_3 - 1)(\lambda_2 + \lambda_3)^{n_3}\lambda_3^{n_3}.$$

From this pattern we see that a partition that results in $\lambda_1^{n_1}(\lambda_1 - 1)$ after $2j$ encomplementations has the form

$$\lambda_{1-j}^{n_{1-j}}\lambda_{2-j}^{n_{2-j}} \cdots \lambda_0^{n_0}{\lambda'_1}^{n_1}(\lambda'_1 - 1)\lambda_2^{n_2} \cdots \lambda_{j+1}^{n_{j+1}}, \tag{3.3}$$

where $\lambda_i > \lambda_{i+1}$ and $\lambda_0 > \lambda'_1 > \lambda_2 + 1$.

On the other hand, a partition λ that results in 1 after two encomplementations has the form $\lambda_0^{n_0}(\lambda_1 + 1)\lambda_1^{n_1}$, and so a partition that results in 1 after j encomplementations is of the form

$$\lambda_{1-j}^{n_{1-j}}\lambda_{2-j}^{n_{2-j}} \cdots \lambda_0^{n_0}(\lambda_1 + 1)\lambda_1^{n_1}\lambda_2^{n_2} \cdots \lambda_j^{n_j}, \tag{3.4}$$

where $\lambda_i > \lambda_{i+1}$ and $\lambda_0 > \lambda_1 + 1$. Thus a partition results in a single part of size 1 after some number of encomplementations if and only if it has the form of Equation 3.3 or Equation 3.4. ∎

170. Show that $P(k, n)$ is at least $\frac{1}{n!}\binom{k-1}{n-1}$.

Solution: The number of compositions of k into n parts is $\binom{k-1}{n-1}$. We can divide the compositions into blocks, where two compositions are in the same block if and only if one is a rearrangement of the other.

Then the blocks correspond bijectively to partitions of k into n parts. However we cannot compute the number of blocks by dividing by the number of compositions per block since the number of compositions per block ranges from 1 to $n!$. But then if we divide the number of compositions by $n!$ we will get a number less than the number of blocks because $n!$ times the number of blocks would be, by the sum principle, greater than the number of partitions. ▮

With the binomial coefficients, with Stirling numbers of the second kind, and with the Lah numbers, we were able to find a recurrence by asking what happens to our subset, partition, or broken permutation of a set S of numbers if we remove the largest element of S. Thus it is natural to look for a recurrence to count the number of partitions of k into n parts by doing something similar. Unfortunately, since we are counting distributions in which all the objects are identical, there is no way for us to identify a largest element. However if we think geometrically, we can ask what we could remove from a Young diagram to get a Young diagram. Two natural ways to get a partition of a smaller integer from a partition of n would be to remove the top row of the Young diagram of the partition and to remove the left column of the Young diagram of the partition. These two operations correspond to removing the largest part from the partition and to subtracting 1 from each part of the partition respectively. Even though they are symmetric with respect to conjugation, they aren't symmetric with respect to the number of parts. Thus one might be much more useful than the other for finding a recurrence for the number of partitions of k into n parts.

➔ ·171. In this problem we will study the two operations and see which one seems more useful for getting a recurrence for $P(k,n)$. Part of the reason

(a) How many parts does the remaining partition have when we remove the largest part (more precisely, we reduce its multiplicity by one) from a partition of k into n parts? (A geometric way to describe this is that we remove the first row from the Young diagram of the partition.) What can you say about the number of parts of the remaining partition if we remove one from each part?

Solution: Reducing the multiplicity of the largest part by one reduces the number of parts by one. Removing 1 from each part reduces the number of parts by the multiplicity of the smallest part, so it strictly reduces the number of parts, perhaps even to one. ∎

(b) If we remove the largest part from a partition, what can we say about the integer that is being partitioned by the remaining parts of the partition? If we remove one from each part of a partition of k into n parts, what integer is being partitioned by the remaining parts? (Another way to describe this is that we remove the first column from the Young diagram of the partition.)

Solution: If we remove the largest part, the integer being partitioned is k minus the largest part. Thus it is a number less than k and at least $n-1$. If we remove one from each part of the partition, the integer being partitioned is $k-n$. ∎

(c) The last two questions are designed to get you thinking about how we can get a bijection between the set of partitions of k into n parts and some other set of partitions that are partitions of a smaller number. These questions describe two different strategies for getting that set of partitions of a smaller number or of smaller numbers. Each strategy leads to a bijection between partitions of k into n parts and a set of partitions of a smaller number or numbers. For each strategy, use the answers to the last two questions to find and describe this set of partitions into a smaller number and a bijection between partitions of k into n parts and partitions of the smaller integer or integers into appropriate numbers of parts. (In one case the set of partitions and bijection are relatively straightforward to describe and in the other case not so easy.)

Solution: Removing the largest part of a partition of k into n parts gives us a bijection between partitions of k into n parts and and partitions of numbers k' between $n-1$ and $k-1$ into $n-1$ parts of size at most $k-k'$. (To see that this is a bijection, note that removing the largest part gives us such a partition, and adjoining a part of size $k-k'$ to such a partition gives us a partition of k with n parts.)

Removing one from each part of a partition of k into n parts gives us a bijection between partitions of k into n parts and and partitions $k-n$ into n or fewer parts. (To see that this is a bijection, note that removing one from each part of a partition of k into n parts gives us such a partition, and, given such a partition, we get a partition of k into n parts by adding one to each part and then creating enough parts of size 1 to have n parts.) ▮

(d) Find a recurrence (which need not have just two terms on the right hand side) that describes how to compute $P(k,n)$ in terms of the number of partitions of smaller integers into a smaller number of parts.

Solution: The second bijection is to the set of partitions of $k-1$ into n or fewer parts, and this makes the second bijection sound easier to work with. We get $P(k,n) = \sum_{i=1}^{n} P(k-n,i)$. The proof is the bijection we already described; in particular a partition of $k-n$ into i parts corresponds to the partition of k we get by adding one to each of the i parts and then creating $n-i$ parts of size one. ▮

(e) What is $P(k,1)$ for a positive integer k?

Solution: $P(k,1) = 1$. ▮

(f) What is $P(k,k)$ for a positive integer k?

Solution: $P(k,k) = 1$. ▮

(g) Use your recurrence to compute a table with the values of $P(k,n)$ for values of k between 1 and 7.

Solution:

$k\backslash n$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1	1	1	0	0	0	0
4	1	2	1	1	0	0	0
5	1	2	2	1	1	0	0
6	1	3	3	2	1	1	0
7	1	3	4	3	2	1	1

▮

(h) What would you want to fill into row 0 and column 0 of your table in order to make it consistent with your recurrence? What does this say $P(0,0)$ should be? We usually define a sum with no terms in it to be zero. Is that consistent with the way the recurrence says we should define $P(0,0)$?

Solution: We would want to have $P(0,0) = 1$ and $P(k,0) = P(0,n) = 0$ for positive integer k or n. Since the sum of the empty multiset of positive integers is zero, this gives us one partition of the number zero, namely the empty multiset of positive integers. ▮

It is remarkable that there is no known formula for $P(k,n)$, nor is there one for $P(k)$. This section is devoted to developing methods for computing values of $P(n,k)$ and finding properties of $P(n,k)$ that we can prove even without knowing a formula. Some future sections will attempt to develop other methods.

We have seen that the number of partitions of k into n parts is equal to the number of ways to distribute k identical objects to n recipients so that each receives at least one. If we relax the condition that each recipient receives at least one, then we see that the number of distributions of k identical objects to n recipients is $\sum_{i=1}^{n} P(k,i)$ because if some recipients receive nothing, it does not matter which recipients these are. This completes rows 7 and 8 of our table of distribution problems. The completed table is shown in Figure 3.2. Every entry in that table tells us how to count something. There are quite a few theorems that you have proved which are summarized by Table 3.2. It would be worthwhile to try to write them all down! The methods we used to complete Figure 3.2 are extensions of the basic counting principles we learned in Chapter 1. The remaining chapters of this book develop more sophisticated kinds of tools that let us solve more sophisticated kinds of counting problems.

3.3.4 Partitions into distinct parts

Often $Q(k,n)$ is used to denote the number of partitions of k into distinct parts, that is, parts that are different from each other.

172. Show that

$$Q(k,n) \leq \frac{1}{n!}\binom{k-1}{n-1}.$$

Solution: The number of compositions of k into n parts is $\binom{k-1}{n-1}$. Thus the number of compositions of k into n distinct parts is less than $\binom{k-1}{n-1}$. Divide the compositions of k into n distinct parts into blocks with two compositions in the same block if one is a rearrangement of the other. Because the parts are distinct, each block has $n!$ members. Further, there is a bijection between the blocks of this partition and the partitions of k into n distinct parts. Since the number of compositions of k into n distinct parts is less than $\binom{k-1}{n-1}$, the number of partitions of k into n distinct parts is less than $\frac{1}{n!}\binom{k-1}{n-1}$. ∎

➔ 173. Show that the number of partitions of seven into three parts equals the number of partitions of 10 into three distinct parts.

Table 3.2: The number of ways to distribute k objects to n recipients, with restrictions on how the objects are received

The Twenty-fold Way: A Table of Distribution Problems		
k objects and conditions on how they are received	n recipients and mathematical model for distribution	
	Distinct	Identical
1. Distinct no conditions	n^k functions	$\sum_{i=1}^{k} S(n,i)$ set partitions ($\leq n$ parts)
2. Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations	1 if $k \leq n$; 0 otherwise
3. Distinct Each gets at least one	$S(k,n)n!$ onto functions	$S(k,n)$ set partitions (n parts)
4. Distinct Each gets exactly one	$k! = n!$ permutations	1 if $k = n$; 0 otherwise
5. Distinct, order matters	$(k+n-1)^{\underline{k}}$ ordered functions	$\sum_{i=1}^{n} L(k,i)$ broken permutations ($\leq n$ parts)
6. Distinct, order matters Each gets at least one	$(k)^{\underline{n}}(k-1)^{\underline{k-n}}$ ordered onto functions	$L(k,n) = \binom{k}{n}(k-1)^{\underline{k-n}}$ broken permutations (n parts)
7. Identical no conditions	$\binom{n+k-1}{k}$ multisets	$\sum_{i=1}^{n} P(k,i)$ number partitions ($\leq n$ parts)
8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$; 0 otherwise
9. Identical Each gets at least one	$\binom{k-1}{n-1}$ compositions (n parts)	$P(k,n)$ number partitions (n parts)
10. Identical Each gets exactly one	1 if $k = n$; 0 otherwise	1 if $k = n$; 0 otherwise

Solution: Given a partition λ of 7 in decreasing list form $\lambda_1, \lambda_2, \lambda_3$, if we add 0 to λ_3, 1 to λ_2 and 2 to λ_1 the resulting partition of 10 has distinct parts. If we take a partition λ' of 10 with distinct parts, then $\lambda'_1 \geq \lambda'_2 + 1$, $\lambda'_1 \geq \lambda'_2 + 2$, and $\lambda'_2 \geq \lambda'_3 + 1$. Therefore if we subtract 2 from λ'_1 to get λ_1, subtract 1 from λ'_2 to get λ_2 and let $\lambda_3 = \lambda'_3$, then $\lambda_1, \lambda_2, \lambda_3$ is the decreasing list representation of a partition of $10 - 3 = 7$. Thus there is a bijection between partitions of 7 into three parts and partitions of 10 into three distinct parts. ∎

➔ ·174. There is a relationship between $P(k,n)$ and $Q(m,n)$ for some other number m. Find the number m that gives you the nicest possible relationship.

Solution: The number of partitions of k into n parts is equal to the number of partitions of $k+\binom{n}{2}$ into n distinct parts. The bijection from partitions of k with n parts to partitions of $k+\binom{n}{2}$ with n distinct parts that proves this is the one that takes a partition $\lambda_n\lambda_{n-1}\cdots\lambda_1$ of k with $\lambda_i > \lambda_{i+1}$ and adds $i-1$ to λ_i to get λ'_i. Then λ' is a partition into distinct parts, and the number it partitions is $k+1+2+\cdots+n-1 = k+\binom{n}{2}$. The proof that it is a bijection is the fact that subtracting $n-i$ from the ith part of a partition of k into distinct parts yields a partition of k, because part $i+j$ is at least j smaller than part i. ▮

·175. Find a recurrence that expresses $Q(k,n)$ as a sum of $Q(k-n,m)$ for appropriate values of m.

Solution: Suppose λ is a partition of k into n distinct parts. Either 1 is one of those parts or not. Thus if we subtract 1 from each part, we either get a partition of $k-n$ into $n-1$ parts or a partition of $k-n$ into n parts. If λ and λ' are different partitions of k into n distinct parts, they go to different partitions. Each partition of $k-n$ into $n-1$ parts or n parts can be gotten in this way from a corresponding partition of k into n parts. Thus we have a bijective correspondence and $Q(k,n) = Q(k-n,n-1) + Q(k-n,n)$. ▮

➔ ∗176. Show that the number of partitions of k into distinct parts equals the number of partitions of k into odd parts.

Solution: We start by giving a function from the set of partitions of k to the set of partitions of k with (only) odd parts. Clearly such a function cannot be one to one. Then we show that when restricted to the partitions with distinct parts it is one-to-one and onto by constructing an inverse. Given a partition $\lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_n^{i_n}$, write $\lambda_i = \gamma_i 2^{k_i}$, where γ_i is odd. (Thus 2^{k_i} is the highest power of 2 that is a factor of λ_i, so it is 1 if λ_i is odd.). It is possible that $\gamma_i = \gamma_j$, for example if $\lambda_i = 36$ and $\lambda_j = 18$, then $\gamma_i = \gamma_j = 9$. We construct a new partition π whose parts are the numbers γ_j as follows: Given an odd number p, let the multiplicity $m(p)$ of p in π be $\sum_{j:\gamma_j=m} 2^{k_j}$. Thus $\sum_{p:m(p)\neq 0} m(p)p = k$. Therefore, π is a partition of k whose parts are all odd.

Now consider a partition π of k whose parts are all odd. Let $\pi = \pi_1^{r_1}\pi_2^{r_2}\cdots\pi_t^{r_t}$, with $\pi_i > \pi_{i+1}$. (In terms of the multiplicity function m, $m(\pi_i) = r_i$, and $\sum_{i=1}^{t} r_i\pi_i = k$.) We are going to write the binary expansion of each r_i as $r_i = \sum_{j=0}^{\lfloor \log_2 r_i \rfloor} 2^{j a_{ij}}$, where a_{ij} is 1 if 2^j appears in the binary expansion of r_i, and 0 otherwise. All of the numbers $\pi_i 2^{j a_{ij}}$ are distinct, because a power of two times one odd number cannot equal a power of two times another odd number. The numbers $\pi_i 2^{j a_{ij}}$ add to k, so they are the parts of a partition π' of k into distinct parts. When we apply the function constructed in the first part of the solution to π', we get π, so the correspondence between π and π' is a bijection. ▮

➔ ∗177. Euler showed that if $k \neq \frac{3j^2+j}{2}$, then the number of partitions of k into an even number of distinct parts is the same as the number of partitions of k into an odd number of distinct parts. Prove this, and in the exceptional case find out how the two numbers relate to each other.

Solution: This solution is taken largely from the book Introduction to Combinatorics by Ioan Tomescu (published in London by Collet's in 1975). Tomescu calls a collection of rows in a Young diagram a "trapezoid" if each row contains one less cell than the row above and the number of cells in the rows above and below the trapezoid differ by two or more from the number of cells in rows of the trapezoid. Thus in (8,6,5,4,2,1) we have 3 trapezoids, the first row, the next three rows, and the last two. Since we are dealing with partitions with distinct parts, we don't have to worry about how two equal rows affect the definition of a trapezoid. We will describe a way to transform a partition with an even number of distinct parts into a partition with an odd number of distinct parts and vice versa.

First we describe a transformation on Young diagrams. Here is the first part of the description. Suppose the smallest part m of λ is less than or equal to the number j of rows in the top trapezoid. Suppose further that if we have only one trapezoid, then $j > m$. Then we construct a partition with one less part by adding 1 to each of the m largest parts and discarding the part m. We still have a diagram for a partition of the same integer, but now the parity of the number of parts has changed, and we *may* have increased the number of trapezoids by 1. The smallest part will now be larger than the number (now m) of rows in the top trapezoid. (Notice that the construction would not work if we had only one trapezoid and $j = m$ because we would first remove one row of the trapezoid and thus have no row to which to attach one of our squares.)

Here is the second part of the description of the transformation. Suppose now that m is larger than the number j of rows of the top trapezoid in the Young diagram. Suppose also that the Young diagram has at least two trapezoids or it has one trapezoid and $j \geq m - 2$. Take one square from each of the j rows of the top trapezoid (which is the whole diagram if there is only one trapezoid) and also add a row of j squares at the bottom of the diagram. (Since $m > j$, this gives us a Young diagram of a partition of the same integer into distinct parts.) The parity of the number of rows has changed, and now the number of rows of the top trapezoid is at least as large as the smallest part of the partition. (Note, two previously distinct trapezoids may have joined together to form one on top.) (Notice that if we have one trapezoid and $j = m+1$, then the construction yields a partition with two equal parts, which is why we made the special assumption above.) Now let T be the transformation described by the two constructions above. Its domain is all Young diagrams except those with one trapezoid and $m \leq j \leq m + 1$. T^2 is the identity, and so T is a bijection. When restricted to partitions with an odd number of parts, T gives partitions with an even number of parts, so on its domain it gives a bijection between partitions with an even number of parts and partitions with an odd number of parts.

If $m = j$ and the diagram has just one trapezoid, then the diagram has $\frac{3j^2-j}{2}$ squares, and if $m = j + 1$ and the diagram has just one trapezoid, then the diagram has $\frac{3j^2+j}{2}$ squares. Thus if $k \neq \frac{3j^2 \pm j}{2}$, the number of partitions of k into distinct even parts equals the number of partitions of k into distinct odd parts.

If $k = \frac{3j^2 \pm j}{2}$ and j is even, then there is one diagram of a partition of k that is not in the domain of the bijection and has an even number of rows, so in this case there will be one more partition with an even number of parts than with an odd number. If $k = \frac{3j^2 \pm j}{2}$ and j is odd, there is one diagram with an odd number of rows not in the domain and so in this case there is one more partition with an odd number of parts than with an even number. This completes the exceptional cases of the problem. ∎

3.3.5 Supplementary Problems

1. Answer each of the following questions with n^k, k^n, $n!$, $k!$, $\binom{n}{k}$, $\binom{k}{n}$, $n^{\underline{k}}$, $k^{\underline{n}}$, $n^{\overline{k}}$, $k^{\overline{n}}$, $\binom{n+k-1}{k}$, $\binom{n+k-1}{n}$, $\binom{n-1}{k-1}$, $\binom{k-1}{n-1}$, or "none of the above."

(a) In how many ways may we pass out k identical pieces of candy to n children?
Solution: $\binom{n+k-1}{k}$ $\blacksquare$

(b) In how many ways may we pass out k distinct pieces of candy to n children?
Solution: n^k $\blacksquare$

(c) In how many ways may we pass out k identical pieces of candy to n children so that each gets at most one? (Assume $k \leq n$.)
Solution: $\binom{n}{k}$. $\blacksquare$

(d) In how many ways may we pass out k distinct pieces of candy to n children so that each gets at most one? (Assume $k \leq n$.)
Solution: $n^{\underline{k}}$ $\blacksquare$

(e) In how many ways may we pass out k distinct pieces of candy to n children so that each gets at least one? (Assume $k \geq n$.)
Solution: None of the above. $\blacksquare$

(f) In how many ways may we pass out k identical pieces of candy to n children so that each gets at least one? (Assume $k \geq n$.)
Solution: $\binom{k-1}{n-1}$ $\blacksquare$

2. The neighborhood betterment committee has been given r trees to distribute to s families living along one side of a street. Unless otherwise specified, it doesn't matter where a family plants the trees it gets.

(a) In how many ways can they distribute all of them if the trees are distinct, there are more families than trees, and each family can get at most one?
Solution: $s^{\underline{r}}$ $\blacksquare$

(b) In how many ways can they distribute all of them if the trees are distinct and any family can get any number?
Solution: s^r $\blacksquare$

(c) In how many ways can they distribute all the trees if the trees are identical, there are no more trees than families, and any family receives at most one?
Solution: $\binom{s}{r}$ $\blacksquare$

(d) In how many ways can they distribute them if the trees are distinct, there are more trees than families, and each family receives at most one (so there could be some leftover trees)?
Solution: $\sum_{k=0}^{s} \binom{s}{k} r^{\underline{k}}$ or $\sum_{k=0}^{s} s^{\underline{k}} \binom{r}{k}$ $\blacksquare$

(e) In how many ways can they distribute all the trees if they are identical and anyone may receive any number of trees?

Solution: $\binom{r+s-1}{r}$ ▮

(f) In how many ways can all the trees be distributed and planted if the trees are distinct, any family can get any number, and a family must plant its trees in an evenly spaced row along the road?

Solution: $s^{\overline{r}} = (r+s-1)^{\underline{r}}$ ▮

(g) Answer the question in Part 2f assuming that every family must get a tree.

Solution: $r!\binom{r-1}{s-1}$ ▮

(h) Answer the question in Part 2e assuming that each family must get at least one tree.

Solution: $\binom{r-1}{s-1}$ ▮

3. In how many ways can n identical chemistry books, r identical mathematics books, s identical physics books, and t identical astronomy books be arranged on three bookshelves? (Assume there is no limit on the number of books per shelf.)

   Solution: $\frac{(n+r+s+t+2)!}{n!r!s!t!2!}$ ▮

➔4. One formula for the Lah numbers is

$$L(k,n) = \binom{k}{n}(k-1)^{\underline{k-n}}$$

   Find a proof that explains this product.

   Solution: First choose the n elements which will be the first member of the part they lie in. (This, in effect, labels the n parts.) Then assign the remaining $k-n$ elements to their parts by making an ordered function of $n-k$ objects to n recipients in $(n+(k-n)-1)^{k-n} = (k-1)^{k-n}$ ways. ▮

5. What is the number of partitions of n into two parts?

   Solution: $n/2$ if n is even and $(n-1)/2$ if n is odd, equivalently, $\lfloor n/2 \rfloor$. ▮

·6. What is the number of partitions of k into $k-2$ parts?

Solution: A partition of k into $k-2$ parts will have either one part of size 3 and $k-3$ parts of size 1, or two parts of size 2 and $k-4$ parts of size 1. Thus the number of partitions of k into $k-2$ parts is

$$\begin{aligned}\binom{k}{3}+\binom{k}{2}\binom{k-2}{2}/2 &= k(k-1)(n-2)/6+k(k-1)(k-2)(k-3)/8\\ &= k(k-1)(k-2)(1/6+(k-3)/8)\\ &= k(k-1)(k-2)(3k-5)/24.\end{aligned}$$

∎

7. Show that the number of partitions of k into n parts of size at most m equals the number of partitions of $mn-k$ into no more than n parts of size at most $m-1$.

   Solution: If we take the complement of the Young diagram of a partition of k into n parts of size at most m in an rectangle with n rows and m columns, the number we partition will be $mn-k$, and we will have no more than n parts, each of size at most $m-1$. And if we take the complement of a partition of this second kind in the same rectangle, we will get a partition of the first kind. ∎

8. Show that the number of partitions of k into parts of size at most m is equal to the number of partitions of $k+m$ into m parts.

   Solution: Given the first kind of partition, take the conjugate (giving a partition of k into at most m parts), add one to each part, and then add enough parts of size 1 to get a total of m parts. It is straightforward that this process can be reversed. ∎

9. You can say something pretty specific about self-conjugate partitions of k into distinct parts. Figure out what it is and prove it. With that, you should be able to find a relationship between these partitions and partitions whose parts are consecutive integers, starting with 1. What is that relationship?

   Solution: In a self-conjugate partition, the number of parts is the size of the largest part. If these parts are distinct, this means that each number between 1 and the largest part appears once as a part. That is, the parts are a list of consecutive integers, starting with 1. ∎

10. What is $s(k, 1)$?

    Solution: Since s$(k, 1)$ is the coefficient of x^1 in

    $$x^{\underline{k}} = x(x-1)(x-2)\cdot(x-(k-1)),$$

it is $(-1)^{k-1}(k-1)!$. ▮

11. Show that the Stirling numbers of the second kind satisfy the recurrence
$$S(k,n) = \sum_{i=1}^{k} S(k-i, n-1)\binom{k-1}{i-1}.$$
Solution: A partition of $[k]$ into n blocks has a block containing k. If this block has size i, when you remove it, you get a partition of a set of size $k-i$ into $n-1$ blocks. The number of possible sets of size i containing k is $\binom{k-1}{i-1}$, and i can be any number between 1 and k. Each partition of k into n blocks may be constructed exactly once by first choosing the block containing k and then partitioning the remaining elements into $n-1$ blocks. This proves the formula. ▮

➔ 12. Let $c(k,n)$ be the number of ways for k children to hold hands to form n circles, where one child clasping his or her hands together and holding them out to form a circle is considered a circle. (Having Mary hold Sam's right hand is different from having Mary hold Sam's left hand.) Find a recurrence for $c(k,n)$. Is the family of numbers $c(k,n)$ related to any of the other families of numbers we have studied? If so, how?

Solution: The kth child is either in a circle alone, and there are $c(k-1, n-1)$ ways for this to happen, or is in a circle with some other children. In the second case child i can be to the immediate right of any of the other $k-1$ children, so there are $(k-1)c(k-1,n)$ ways for this to happen. Thus $c(k,n) = c(k-1,n-1) + (k-1)c(k-1,n)$. This recurrence is almost the same as the recurrence for $s(k,n)$, except it has a plus sign where the recurrence for the Stirling numbers of the first kind has a minus sign. Further $c(k,1) = (k-1)!$ and $c(k,k) = 1$, which agrees, except for sign, with the Stirling numbers of the first kind. If we experiment with applying the recurrence, we see that whenever we use it to compute $c(k,n)$, we get that $c(k,n) = |s(k,n)|$. It is now straightforward to prove by induction that $c(k,n) = |s(k,n)|$. ▮

➔ 13. How many labeled trees on n vertices have exactly four vertices of degree 1?

Solution: The vertices of degree 1 are the vertices that do not appear in the Prüfer code for the tree. So we first choose four vertices out of n in $\binom{n}{4}$ ways to be our vertices of degree 1, and the Prüfer code may be thought of as a function from the $n-2$ places of the code onto the $n-4$ remaining vertices, so there are $S(n-2, n-4)(n-4)!$ Prüfer codes for each choice of the vertices of degree 1. Thus using Problem 6 from this section of supplementary problems, we have that the number of labeled trees is $\binom{n}{4}(n-2)(n-3)(n-4)(3n-11)(n-4)!/24 = n!(n-2)(n-3)(n-4)(3n-11)/576$. ▮

➔ 14. The *degree sequence* of a graph is a list of the degrees of the vertices in non-increasing order. For example the degree sequence of the first graph in Figure 2.4 is $(4, 3, 2, 2, 1)$. For a graph with vertices labeled 1 through n, the *ordered degree sequence* of the graph is the sequence $d_1, d_2, \ldots d_n$ in which d_i is the degree of vertex i. For example the ordered degree sequence of the first graph in Figure 2.2 is $(1, 2, 3, 3, 1, 1, 2, 1)$.

(a) How many labeled trees are there on n vertices with ordered degree sequence $d_1, d_2, \ldots d_n$?

Solution: We first solve the ordered degree sequence problem in which we assume d_i is the degree of vertex i. The number of times i appears in the Prüfer code of a tree is one less than the degree of i, so vertex i appears $d_i - 1$ times. Thus the sum of the $d_i - 1$ should be $2n-2-n = n-2$. Of the $n-2$ places in the Prüfer code, we want to label $d_1 - 1$ of them with 1, $d_2 - 1$ of them with 2 and in general $d_i - 1$ of them with i. There are

$$\binom{n-2}{d_1 - 1, d_2 - 1, d_3 - 1, \ldots, d_n - 1}$$

ways to do this, so the number of trees in which vertex i has degree d_i is $\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}$. ▮

$*$(b) How many labeled trees are there on n vertices with with the degree sequence in which the degree d appears i_d times?

Solution: Now we modify the solution of the previous part by observing that to count all graphs with a given degree sequence, the actual vertices which have the given degrees is irrelevant, so we must multiply the result of the easier problem by the number of ways to assign the degrees to the vertices. To assign the degrees, we can list the vertices in $n!$ ways, choose the first i_1 of these vertices to have degree 1, then next i_2 to have degree 2, and so on. But the order in which we list the vertices of a given degree is irrelevant. Thus the number of ways to assign the degrees is $\frac{n!}{i_1!i_2!\cdots i_n!}$. Once the degrees are assigned, there are $\frac{(n-2)!}{\prod_{d=1}^{n}(d-1)!^{i_d}}$, by translating our easier result. Thus the total number of trees with the degree sequence in which there are i_d vertices of degree d is

$$\frac{n!(n-2)!}{\prod_{j=1}^{n} i_j!(j-1)!^{i_j}}.$$

∎

Chapter 4

Generating Functions

4.1 The Idea of Generating Functions

4.1.1 Visualizing Counting with Pictures

Suppose you are going to choose three pieces of fruit from among apples, pears and bananas for a snack. We can symbolically represent all your choices as

$$🍎🍎🍎 + 🍐🍐🍐 + 🍌🍌🍌 + 🍎🍎🍐 + 🍎🍎🍌 + 🍎🍐🍐 + 🍐🍐🍌 + 🍎🍌🍌 + 🍐🍌🍌 + 🍎🍐🍌.$$

Here we are using a picture of a piece of fruit to stand for taking a piece of that fruit. Thus 🍎 stands for taking an apple, 🍎🍐 for taking an apple and a pear, and 🍎🍎 for taking two apples. You can think of the plus sign as standing for the "exclusive or," that is, $🍎 + 🍌$ would stand for "I take an apple or a banana but not both." To say "I take both an apple and a banana," we would write 🍎🍌. We can extend the analogy to mathematical notation by condensing our statement that we take three pieces of fruit to

$$🍎^3 + 🍐^3 + 🍌^3 + 🍎^2🍐 + 🍎^2🍌 + 🍎🍐^2 + 🍐^2🍌 + 🍎🍌^2 + 🍐🍌^2 + 🍎🍐🍌.$$

In this notation $🍎^3$ stands for taking a multiset of three apples, while $🍎^2🍌$ stands for taking a multiset of two apples and a banana, and so on. What our notation is really doing is giving us a convenient way to list all three element multisets chosen from the set $\{🍎, 🍐, 🍌\}$.[1]

[1] This approach was inspired by George Pólya's paper "Picture Writing," in the December, 1956 issue of the *American Mathematical Monthly*, page 689. While we are taking a somewhat more formal approach than Pólya, it is still completely in the spirit of his work.

Suppose now that we plan to choose between one and three apples, between one and two pears, and between one and two bananas. In a somewhat clumsy way we could describe our fruit selections as

$$\text{🍎🍐🍌}+\text{🍎}^2\text{🍐🍌}+\cdots+\text{🍎}^2\text{🍐}^2\text{🍌}+\cdots+\text{🍎}^2\text{🍐}^2\text{🍌}^2+\text{🍎}^3\text{🍐🍌}+\cdots+\text{🍎}^3\text{🍐}^2\text{🍌}+\cdots+\text{🍎}^3\text{🍐}^2\text{🍌}^2. \quad (4.1)$$

•178. Using an A in place of the picture of an apple, a P in place of the picture of a pear, and a B in place of the picture of a banana, write out the formula similar to Formula 4.1 without any dots for left out terms. (You may use pictures instead of letters if you prefer, but it gets tedious quite quickly!) Now expand the product $(A+A^2+A^3)(P+P^2)(B+B^2)$ and compare the result with your formula.

Solution: $APB + APB^2 + AP^2B + AP^2B^2 + A^2PB + A^2PB^2 + A^2P^2B + A^2P^2B^2 + A^3PB + A^3PB^2 + A^3P^2B + A^3P^2B^2$

$$\begin{aligned} & (A+A^2+A^3)(P+P^2)(B+B^2) \\ = \;& APB + APB^2 + AP^2B + AP^2B^2 + A^2PB + A^2PB^2 + A^2P^2B \\ + \;& A^2P^2B^2 + A^3PB + A^3PB^2 + A^3P^2B + A^3P^2B^2. \end{aligned}$$

We get the same expression in both cases. ∎

•179. Substitute x for all of A, P and B (or for the corresponding pictures) in the formula you got in Problem 178 and expand the result in powers of x. Give an interpretation of the coefficient of x^n.

Solution: $x^3 + 3x^4 + 4x^5 + 3x^6 + x^7$. There is one way to choose three pieces of fruit, there are three ways to choose four pieces, four ways to chose 5 pieces, three ways to choose 6 pieces , and there is one way to choose 7 pieces of fruit. The coefficient of x^n is the number of ways to choose n pieces of fruit. ∎

If we were to expand the formula

$$(\text{🍎}+\text{🍎}^2+\text{🍎}^3)(\text{🍐}+\text{🍐}^2)(\text{🍌}+\text{🍌}^2), \quad (4.2)$$

we would get Formula 4.1. Thus Formula 4.1 and Formula 4.2 each describe the number of multisets we can choose from the set {🍎, 🍐, 🍌} in which 🍎 appears between one and three times, and 🍐 and 🍌 each appear once or twice. We interpret Formula 4.1 as describing each individual multiset we can choose, and we interpret Formula 4.2 as saying that we first decide how many apples to take, and then decide how many pears to take, and then decide how many bananas to take. At this stage it might seem a bit magical that doing ordinary algebra with the second formula yields the first, but in fact we could define addition and multiplication with these pictures more formally so we could explain in detail why things work out. However, since the pictures are for motivation, and are actually difficult to write out on paper, it doesn't make much sense to work out these details. We will see an explanation in another context later on.

4.1.2 Picture functions

As you've seen, in our descriptions of ways of choosing fruits, we've treated the pictures of the fruit as if they are variables. You've also likely noticed that it is much easier to do algebraic manipulations with letters rather than pictures, simply because it is time consuming to draw the same picture over and over again, while we are used to writing letters quickly. In the theory of generating functions, we associate variables or polynomials or even power series with members of a set. There is no standard language describing how we associate variables with members of a set, so we shall invent[2] some. By a *picture* of a member of a set we will mean a variable, or perhaps a product of powers of variables (or even a sum of products of powers of variables). A function that assigns a picture $P(s)$ to each member s of a set S will be called a *picture function.* The **picture enumerator** for a picture function P defined on a set S will be the sum of the pictures of the elements in S. In symbols we can write this conveniently as.

$$E_P(S) = \sum_{s:s\in S} P(s).$$

We choose this language because the picture enumerator lists, or enumerates, all the elements of S according to their pictures. Thus Formula 4.1 is the picture enumerator of the set of all multisets of fruit with between one and three apples, one and two pears, and one and two bananas.

∘180. How would you write down a polynomial in the variable A that says you should take between zero and three apples?

Solution: $A^0 + A^1 + A^2 + A^3$. ▮

•181. How would you write down a picture enumerator that says we take between zero and three apples, between zero and three pears, and between zero and three bananas?

[2]We are really adapting language introduced by George Pólya.

Solution:
$(A^0 + A^1 + A^2 + A^3)(P^0 + P^1 + P^2 + P^3)(B^0 + B^1 + B^2 + B^3)$. ▮

·182. (Used in Chapter 6.) Notice that when we used A^2 to stand for taking two apples, and P^3 to stand for taking three pears, then we used the product A^2P^3 to stand for taking two apples and three pears. Thus we have chosen the picture of the ordered pair (2 apples, 3 pears) to be the product of the pictures of a multiset of two apples and a multiset of three pears. Show that if S_1 and S_2 are sets with picture functions P_1 and P_2 defined on them, and if we define the picture of an ordered pair $(x_1, x_2) \in S_1 \times S_2$ to be $P((x_1, x_2)) = P_1(x_1)P_2(x_2)$, then the picture enumerator of P on the set $S_1 \times S_2$ is $E_{P_1}(S_1)E_{P_2}(S_2)$. We call this the **product principle for picture enumerators**.

Solution:

$$\begin{aligned}
E_P(S_1 \times S_2) &= \sum_{(x_1,x_2)\in S_1\times S_2} P(x_1)P(x_2) \\
&= \sum_{x_1:x_1\in S_1}\sum_{x_2:x_2\in S_2} P(x_1)P(x_2) \\
&= \sum_{x_1\in S_1} P(x_1) \sum_{x_2\in S_2} P(x_2) \\
&= \sum_{x_1\in S_1} P(x_1)E_{P_2}(S_2) \\
&= \left(\sum_{x_1\in S_1} P(x_1)\right) E_{P_2}(S_2) \\
&= E_{P_1}(S_1)E_{P_2}(S_2)
\end{aligned}$$

▮

4.1.3 Generating functions

•183. Suppose you are going to choose a snack of between zero and three apples, between zero and three pears, and between zero and three bananas. Write down a polynomial in one variable x such that the coefficient of x^n is the number of ways to choose a snack with n pieces of fruit.

Solution: $(1 + x + x^2 + x^3)^3$ ▮

∘184. Suppose an apple costs 20 cents, a banana costs 25 cents, and a pear costs 30 cents. What should you substitute for A, P, and B in Problem

181 in order to get a polynomial in which the coefficient of x^n is the number of ways to choose a selection of fruit that costs n cents?

Solution: Substitute x^{20} for A, x^{25} for B and x^{30} for P. ▮

•185. Suppose an apple has 40 calories, a pear has 60 calories, and a banana has 80 calories. What should you substitute for A, P, and B in Problem 181 in order to get a polynomial in which the coefficient of x^n is the number of ways to choose a selection of fruit with a total of n calories?

Solution: Substitute x^{40} for A, x^{60} for P, and x^{80} for B. ▮

•186. We are going to choose a subset of the set $[n] = \{1, 2, \ldots, n\}$. Suppose we use x_1 to be the picture of choosing 1 to be in our subset. What is the picture enumerator for either choosing 1 or not choosing 1? Suppose that for each i between 1 and n, we use x_i to be the picture of choosing i to be in our subset. What is the picture enumerator for either choosing i or not choosing i to be in our subset? What is the picture enumerator for all possible choices of subsets of $[n]$? What should we substitute for x_i in order to get a polynomial in x such that the coefficient of x^k is the number of ways to choose a k-element subset of n? What theorem have we just reproved (a special case of)?

Solution: The picture enumerator for choosing 1 or not choosing 1 is $x_1 + 1$. The picture enumerator for choosing or not choosing i is $x_i + 1$. The picture enumerator for choosing all possible subsets of $[n]$ is $(x_1 + 1)(x_2 + 1) \cdots (x_n + 1)$. We should substitute x for x_i, thus getting $(1 + x)^n$. Since the number of ways to choose an n-element subset is $\binom{n}{k}$, we have just proved the version of the binomial theorem that says

$$(x + 1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

▮

In Problem 186 we see that we can think of the process of expanding the polynomial $(1+x)^n$ as a way of "generating" the binomial coefficients $\binom{n}{k}$ as the coefficients of x^k in the expansion of $(1+x)^n$. For this reason, we say that $(1+x)^n$ is the "generating function" for the binomial coefficients $\binom{n}{k}$. More generally, the **generating function** for a sequence a_i, defined for i with $0 \le i \le n$ is the expression $\sum_{i=0}^{n} a_i x^i$, and the **generating function** for the sequence a_i with $i \ge 0$ is the expression $\sum_{i=0}^{\infty} a_i x^i$. This last expression is an example of a power series. In calculus it is important to think about whether a power series converges in order to determine whether or not it represents a function. In a nice twist of language, even though we use the phrase generating function as the name of a power series in combinatorics, we don't require the power series to actually represent a function in the usual sense, and so we don't have to worry about convergence.[3] Instead we think of a power series as a convenient way of representing the terms of a sequence of numbers of interest to us. The only justification for saying that such a representation is convenient is because of the way algebraic properties of power series capture some of the important properties of some sequences that are of combinatorial importance. The remainder of this chapter is devoted to giving examples of how the algebra of power series reflects combinatorial ideas.

Because we choose to think of power series as strings of symbols that we manipulate by using the ordinary rules of algebra and we choose to ignore issues of convergence, we have to avoid manipulating power series in a way that would require us to add infinitely many real numbers. For example, we cannot make the substitution of $y+1$ for x in the power series $\sum_{i=0}^{\infty} x^i$, because in order to interpret $\sum_{i=0}^{\infty}(y+1)^i$ as a power series we would have to apply the binomial theorem to each of the $(y+1)^i$ terms, and then collect like terms, giving us infinitely many ones added together as the coefficient of y^0, and in fact infinitely many numbers added together for the coefficient of any y^i. (On the other hand, it would be fine to substitute $y+y^2$ for x. Can you see why?)

4.1.4 Power series

For now, most of our uses of power series will involve just simple algebra. Since we use power series in a different way in combinatorics than we do in calculus, we should review a bit of the algebra of power series.

∘187. In the polynomial $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2 + b_3x^3)$, what is the coefficient of x^2? What is the coefficient of x^4?

Solution: $a_0b^2 + a_1b_1 + a_2b_0$ is the coefficient of x^2. $a_1b_3 + a_2b_2$ is the coefficient of x^4. ▮

[3] In the evolution of our current mathematical terminology, the word function evolved through several meanings, starting with very imprecise meanings and ending with our current rather precise meaning. The terminology "generating function" may be thought of as an example of one of the earlier usages of the term function.

$\circ$188. In Problem 187 why is there a b_0 and a b_1 in your expression for the coefficient of x^2 but there is not a b_0 or a b_1 in your expression for the coefficient of x^4? What is the coefficient of x^4 in

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4)?$$

Express this coefficient in the form

$$\sum_{i=0}^{4} \text{something},$$

where the something is an expression you need to figure out. Now suppose that $a_3 = 0$, $a_4 = 0$, and $b_4 = 0$. To what is your expression equal after you substitute these values? In particular, what does this have to do with Problem 187?

Solution: There is a b_0 in the coefficient of x^2 because b_ox^0 can be paired with a_2x^2 to give the term $a_2b_0x^4$. Similarly there is a b_1 because it can be paired with a_1 for the same purpose. However, there is no a_i that we can pair with b_0 to get a coefficient of x^4 and no a_i that we can pair with b_3 to get a coefficient of x^4.

The coefficient of x^4 in

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4)$$

is $\sum_{i=0}^{4} a_ib_{4-i}$. If we substitute $a_3 = 0$, $a_4 = 0$, and $b_4 = 0$, we get the coefficient of x^4 in $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2 + b_3x^3)$. This exemplifies the idea that we can get a uniform formula for the coefficient of x^i (namely, sum all a_jb_{i-j} from $j = 0$ to i) in a product of two polynomials if we are willing to say that the coefficient of a power of x that does not appear in a polynomial is 0. ▮

$\circ$189. The point of the Problems 187 and 188 is that so long as we are willing to assume $a_i = 0$ for $i > n$ and $b_j = 0$ for $j > m$, then there is a very nice formula for the coefficient of x^k in the product

$$\left(\sum_{i=0}^{n} a_ix^i\right)\left(\sum_{j=0}^{m} b_jx^j\right).$$

Write down this formula explicitly.

Solution: $\sum_{i=0}^{k} a_ib_{k-i}$. ▮

•190. Assuming that the rules you use to do arithmetic with polynomials apply to power series, write down a formula for the coefficient of x^k in the product

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right).$$

Solution: $\sum_{i=0}^{k} a_i b_{k-i}$. ▮

We use the expression you obtained in Problem 190 to *define* the product of power series. That is, we define the product

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)$$

to be the power series $\sum_{k=0}^{\infty} c_k x^k$, where c_k is the expression you found in Problem 190. Since you derived this expression by using the usual rules of algebra for polynomials, it should not be surprising that the product of power series satisfies these rules.[4]

4.1.5 Product principle for generating functions

Each time that we converted a picture function to a generating function by substituting x or some power of x for each picture, the coefficient of x had a meaning that was significant to us. For example, with the picture enumerator for selecting between zero and three each of apples, pears, and bananas, when we substituted x for each of our pictures, the exponent i in the power x^i is the number of pieces of fruit in the fruit selection that led us to x^i. After we simplify our product by collecting together all like powers of x, the coefficient of x^i is the number of fruit selections that use i pieces of fruit. In the same way, if we substitute x^c for a picture, where c is the number of calories in that particular kind of fruit, then the i in an x^i term in our generating function stands for the number of calories in a fruit selection that gave rise to x^i, and the coefficient of x^i in our generating function is the number of fruit selections with i calories. The product principle of picture enumerators translates directly into a product principle for generating functions. However, it is possible to give a proof that does not rely on the product principle for enumerators.

[4]Technically we should explicitly state these rules and prove that they are all valid for power series multiplication, but it seems like overkill at this point to do so!

• 191. Suppose that we have two sets S_1 and S_2. Let v_1 (v stands for value) be a function from S_1 to the nonnegative integers and let v_2 be a function from S_2 to the nonnegative integers. Define a new function v on the set $S_1 \times S_2$ by $v(x_1, x_2) = v_1(x_1) + v_2(x_2)$. Suppose further that $\sum_{i=0}^{\infty} a_i x^i$ is the generating function for the number of elements x_1 of S_1 of value i, that is, with $v_1(x_1) = i$. Suppose also that $\sum_{j=0}^{\infty} b_j x^j$ is the generating function for the number of elements x_2 of S_2 of value j, that is, with $v_2(x_2) = j$. Prove that the coefficient of x^k in

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)$$

is the number of ordered pairs (x_1, x_2) in $S_1 \times S_2$ with total value k, that is, with $v_1(x_1) + v_2(x_2) = k$. This is called the **product principle for generating functions**.

Solution: The generating function for ordered pairs of total value k will have the number of ordered pairs of total value k as the coefficient of x^k. But we get a total value k by taking something of value i in S_1 and something of value $k-i$ in j. And since values cannot be negative, the only values of i available to us are the ones between 0 and k. By the product principle for pairs, the number of ordered pairs (x, y) with $v_1(x) = i$ and $v_2(y) = k - i$ is $a_i b_{k-i}$. To get the number of pairs of total value k, we have to sum over all possible pairs $(i, k-i)$ of values, that is, we have to take the sum $\sum_{i=0}^{k} a_i b_{k-i}$. And this is the coefficient of x^k in the product

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right).$$

This proves the product principle for generating functions. ∎

Problem 191 may be extended by mathematical induction to prove our next theorem.

Theorem 7 (Product Principle for Generating Functions) *If S_1, S_2, …, S_n are sets with a value function v_i from S_i to the nonnegative integers for each i, and $f_i(x)$ is the generating function for the number of elements of S_i of each possible value, then the generating function for the number of n-tuples of each possible total value is $\prod_{i=1}^{n} f_i(x)$.*

4.1.6 The extended binomial theorem and multisets

•192. Suppose once again that i is an integer between 1 and n.

(a) What is the generating function in which the coefficient of x^k is one? This series is an example of what is called an *infinite geometric series*. In the next part of this problem it will be useful to interpret the coefficient one as the number of multisets of size k chosen from the singleton set $\{i\}$. Namely, there is only one way to chose a multiset of size k from $\{i\}$: choose i exactly k times.

Solution: $1 + x + x^2 + \cdots + x^i + \cdots = \sum_{i=0}^{\infty} x^i$. ▮

(b) Express the generating function in which the coefficient of x^k is the number of k-element multisets chosen from $[n]$ as a power of a power series. What does Problem 125 (in which your answer could be expressed as a binomial coefficient) tell you about what this generating function equals?

Solution: The generating function is $\left(\sum_{i=0}^{\infty} x^i\right)^n$. Problem 125 tells us that this equals $\sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$. ▮

∘193. What is the product $(1-x)\sum_{k=0}^{n} x^k$? What is the product

$$(1-x)\sum_{k=0}^{\infty} x^k?$$

Solution:

$$(1-x)\sum_{k=0}^{n} x^k = 1-x+x-x^2+\cdots+x^{n-1}-x^n+x^n-x^{n+1} = 1-x^{n+1}.$$

$$(1-x)\sum_{k=0}^{\infty} x^k = \sum_{i=0}^{\infty} x^i - \sum_{i=0}^{\infty} x^{i+1} = \sum_{i=0}^{\infty} x^i - \sum_{i=1}^{\infty} x^i = 1.$$

▮

➔ •194. Express the generating function for the number of multisets of size k chosen from $[n]$ (where n is fixed but k can be any nonnegative integer) as a 1 over something relatively simple.

Solution: Since $(1-x)\sum_{k=0}^{\infty} x^k = 1$, we have that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Therefore $\left(\sum_{k=0}^{\infty} x^k\right)^n = \frac{1}{(1-x)^n}$ is the generating function for multisets of size k chosen from an n element set. ▮

•195. Find a formula for $(1+x)^{-n}$ as a power series whose coefficients involve binomial coefficients. What does this formula tell you about how we should define $\binom{-n}{k}$ when n is positive?

Solution:

$$(1+x)^{-n} = (1-(-x))^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}(-x)^i = \sum_{i=0}^{\infty}(-1)^i \binom{n+i-1}{i} x^i.$$

We want the coefficient of x^k in $(1+x)^{-n}$ to be $\binom{-n}{k}$, so we want $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$. ▮

➔ 196. If you define $\binom{-n}{k}$ in the way you described in Problem 195, you can write down a version of the binomial theorem for $(x+y)^n$ that is valid for both nonnegative and negative values of n. Do so. This is called the *extended binomial theorem*. Write down a special case with n negative, like $n = -3$, to see an interesting surprise that suggests why we do not use this formula later on.

Solution: $(x+y)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i y^{n-i}$. The proof consists of writing $(x+y) = y(\frac{x}{y}+1)$ and applying Problem 195 when n is negative. When n is positive, we recall that $\binom{n}{k}$ is zero when $k > n$, so replacing the upper limit of n in the standard version of the binomial theorem by ∞ doesn't change the value of the sum. When $n = -3$ we get $(x+y)^n = \sum_{i=0}^{\infty} \binom{-3}{i} x^i y^{-3-i} = \sum_{i=0}^{\infty}(-1)^i\binom{3+i-1}{i} x^i y^{-3-i}$. Expanding our binomial coefficients gives $(x+y)^{-3} = \binom{2}{0}x^0y^{-3} - \binom{3}{1}x^1y^{-4} + \binom{4}{2}x^2y^{-5} - \cdots$. The surprise is that we get an infinite series in positive and negative powers of variables. In order to limit ourselves to infinite series with nonnegative exponents, we do not pursue this idea further. ▮

➔ •197. Write down the generating function for the number of ways to distribute identical pieces of candy to three children so that no child gets more than 4 pieces. Write this generating function as a quotient of polynomials. Using both the extended binomial theorem and the original binomial theorem, find out in how many ways we can pass out exactly ten pieces.

Solution: $(1+x+x^2+x^3+x^4)^3$. We can write

$$(1+x+x^2+x^3+x^4)^3 = \left(\frac{1-x^5}{1-x}\right)^3$$

$$\begin{aligned}
&= (1-x^5)^3(1-x)^{-3}\\
&= (1-3x^5+3x^{10}-x^{15})\sum_{i=0}^{\infty}\binom{3+i-1}{i}x^i\\
&= (1-3x^5+3x^{10}-x^{15})\sum_{i=0}^{\infty}\binom{2+i}{i}x^i
\end{aligned}$$

The coefficient of x^{10} is the number of ways to pass out ten pieces of candy, and is $\binom{12}{10}-3\binom{7}{5}+3\binom{2}{0}$. Thus the number of ways to pass out ten pieces of candy is $66-3\cdot 21+3=6$.)

■

•198. What is the generating function for the number of multisets chosen from an n-element set so that each element appears at least j times and less than m times? Write this generating function as a quotient of polynomials, then as a product of a polynomial and a power series.

Solution:

$$\begin{aligned}
(x^j+x^{j+1}+\cdots+x^{m-1}) &= x^j\left(\sum_{i=0}^{m-j-1}x^i\right)^n\\
&= \left(x^j\frac{1-x^{m-j}}{1-x}\right)^n\\
&= \left(\frac{x^j-x^m}{1-x}\right)^n\\
&= (x^j-x^m)^n\sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i.
\end{aligned}$$

■

➔199. Recall that a tree is determined by its edge set. Suppose you have a tree on n vertices, say with vertex set $[n]$. We can use x_i as the picture of vertex i and x_ix_j as the picture of the edge x_ix_j. Then one possible picture of the tree T is the product $P(T)=\prod_{\{i,j\}:i \text{ and } j\text{are adjacent}} x_ix_j$.

(a) Explain why the picture of a tree is also $\prod_{i=1}^n x_i^{\deg(i)}$.

Solution: Because the number of times x_i appears in an edge is its degree. This number is also the number of times x_i appears in the picture. ■

(b) Write down the picture enumerators for trees on two, three, and four vertices. Factor them as completely as possible.

Solution: x_1x_2, $x_1x_2x_3(x_1+x_2+x_3)$, and $x_1x_2x_3x_4(x_1+x_2+x_3+x_4)^2$. We now explain the third answer. A tree on four vertices either has a vertex of degree three, or it doesn't. If vertex i has degree three, then the picture of the tree is $x_1x_2x_3x_4x_i^2$. If the tree has no vertex of degree 3, it has two vertices of degree 2. If i and j are the vertices of degree 2, then the picture of the tree is $x_1x_1x_3x_4x_ix_j$. Adding up all these pictures gives $x_1x_2x_3x_4(x_1+x_2+x_3+x_4)^2$. ∎

(c) Explain why $x_1x_2\cdots x_n$ is a factor of the picture of a tree on n vertices.

Solution: Every vertex is in at least one edge, because otherwise the tree would not be connected. ∎

(d) Write down the picture of a tree on five vertices with one vertex of degree four, say vertex i. If a tree on five vertices has a vertex of degree three, what are the possible degrees of the other vertices. What can you say about the picture of a tree with a vertex of degree three? If a tree on five vertices has no vertices of degree three or four, how many vertices of degree two does it have? What can you say about its picture? Write down the picture enumerator for trees on five vertices.

Solution: If we have one vertex of degree four, the rest have degree one. So our picture is $x_1x_2x_3x_4x_5x_i^3$. If there is a vertex of degree three, then one vertex must have degree two and the rest must have degree 1, because the sum of the degrees must be 8. Thus the picture must be $x_1x_2x_3x_4x_5x_i^2x_j$, where i is the vertex of degree three and j is the vertex of degree two. If it has no vertices of degree more than 3, then it must have three vertices of degree two in order for the sum of the degrees to be eight. Thus its picture is $x_1x_2x_3x_4x_5x_ix_jx_k$, where i, j, and k are the vertices of degree two. ∎

(e) Find a (relatively) simple polynomial expression for the picture enumerator $\sum_{T:T \text{ is a tree on } [n]} P(T)$. Prove it is correct.

Solution: Based on our examples, the enumerator for trees on $[n]$ must be $x_1x_2\cdots x_n(x_1+x_2+\cdots+x_n)^{n-2}$. When we factor out $x_1x_2\cdots x_n$ from the enumerator of trees, the result is a sum of terms of degree $n-2$. (The degree of $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ is $i_1+i_2+\cdots+i_n$.) We want to show this sum of terms of degree

$n-2$ is $(x_1+x_2+\cdots+x_n)^{n-2}$. So let us analyze this power of a multinomial. It is a sum of terms of the form $x_{i_1}x_{i_2}\cdots x_{i_{n-2}}$ (where x_{i_j} may equal x_{i_k}). The factor x_{i_1} is chosen from the first $(x_1+x_2+\cdots+x_n)$ in the product. The factor x_{i_2} is chosen from the second $(x_1+x_2+\cdots+x_n)$ in the product. The factor $x_{i_{n-2}}$ is chosen from the last $(x_1+x_2+\cdots+x_n)$ in the product. Thus the product is the sum, over all ordered lists $i_i, i_2, \ldots, i_{n-2}$ of $x_{i_1}x_{i_2}\cdots x_{i_{n-2}}$. Thus we want to show that each list determines exactly one tree, and each tree determines exactly one list. But we did exactly this in Problem 112. We do need just a bit more than the bijection of Problem 112, though. The list of numbers we get from a tree determines a monomial in the xs. We need to know that when we multiply this monomial by $x_1x_2\cdots x_n$, we get the picture of the tree we started with. If the tree is a two vertex tree this statement is trivial, and if the tree has three vertices, this is straightforward to prove. Thus we proceed by induction. We assume that $n > 3$ and when applied to a tree with the $(n-1)$-element vertex set $i_1, i_2, \ldots, i_{n-1}$, the bijection of Problem 112 carries the tree to a sequence whose monomial multiplied by $x_{i_1}x_{i_2}\cdots x_{i_{n-1}}$ is the picture of the tree. Then, given a tree on n vertices, our bijection has us remove the lowest numbered vertex of degree one to give a tree on n vertices. Assuming that the edge removed is x_ix_j, the picture of the tree on n vertices is x_ix_j times the picture of the resulting tree. But by our inductive hypothesis, the picture of the $n-1$-vertex tree is the picture provided by the monomial of the list which the bijection gives us and therefore the monomial that the list gives us for the n-vertex tree is also the picture of that tree. Therefore, by the principle of mathematical induction, the monomial given by the Prüfer bijection is the picture of the tree. Therefore, the picture enumerator for trees on n vertices is $x_1x_2\cdots x_n(x_1+x_2+\cdots+x_n)^{n-2}$. ▮

(f) The enumerator for trees by degree sequence is the sum over all trees of $x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$, where d_i is the degree of vertex i. What is the enumerator by degree sequence for trees on the vertex set $[n]$?

Solution: $x_1x_2\cdots x_n(x_1+x_2+\cdots+x_n)^{n-2}$, because the number of times x_i appears in the picture of a tree is the degree of vertex i. ▮

(g) Find the number of trees on n vertices and prove your formula correct.

Solution: n^{n-2}, which we get by substituting 1 for each x_i. This converts the enumerator for trees into the number of trees. ∎

4.2 Generating Functions for Integer Partitions

•200. If we have five identical pennies, five identical nickels, five identical dimes, and five identical quarters, give the picture enumerator for the combinations of coins we can form and convert it to a generating function for the number of ways to make k cents with the coins we have. Do the same thing assuming we have an unlimited supply of pennies, nickels, dimes, and quarters.

Solution: $(1+P+P^2+P^3+P^4+P^5)(1+N+N^2+N^3+N^4+N^5)(1+D+D^2+D^3+D^4+D^5)(1+Q+Q^2+Q^3+Q^4+Q^5)$. Substituting x for P, x^5 for N, x^{10} for D and x^{25} for Q gives us

$$\sum_{i=0}^{5} x^i \sum_{i=0}^{5} x^{5i} \sum_{i=0}^{5} x^{10i} \sum_{i=0}^{5} x^{25i} = \frac{1-x^6}{1-x} \cdot \frac{1-x^{30}}{1-x^5} \cdot \frac{1-x^{60}}{1-x^{10}} \cdot \frac{1-x^{150}}{1-x^{25}}.$$

Although we could write this as a polynomial times a product of four power series, doing so would not significantly increase our understanding, though it would let us make some painful computations of the number of ways to make a certain number of cents. If we actually wanted such numbers we would be better off asking a computer algebra package to expand the product of the polynomials on the left. With unlimited supplies the generating function becomes

$$\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}}.$$

Again, we could write this as a product of power series, and that would allow us to compute painfully the number of ways to create a certain number of cents. If we actually wanted to know the number of ways to make up 200 cents, say, it would be more sensible to ask a computer algebra package to extract the coefficient of x^{200} in the product of the four quotients. ∎

•201. Recall that a partition of an integer k is a multiset of numbers that adds to k. In Problem 200 we found the generating function for the number of partitions of an integer into parts of size 1, 5, 10, and 25.

When working with generating functions for partitions, it is becoming standard to use q rather than x as the variable in the generating function. From now on, write your answers to problems involving generating functions for partitions of an integer in this notation.[5]

(a) Give the generating function for the number of partitions of an integer into parts of size one through ten.

Solution: $\prod_{i=1}^{10} \frac{1}{1-q^i}$ ▮

(b) Give the generating function for the number of partitions of an integer k into parts of size at most m, where m is fixed but k may vary. Notice this is the generating function for partitions whose Young diagram fits into the space between the line $x = 0$ and the line $x = m$ in a coordinate plane. (We assume the boxes in the Young diagram are one unit by one unit.)

Solution: $\prod_{i=1}^{m} \frac{1}{1-q^i}$. ▮

•202. In Problem 201b you gave the generating function for the number of partitions of an integer into parts of size at most m. Explain why this is also the generating function for partitions of an integer into at most m parts. Notice that this is the generating function for the number of partitions whose Young diagram fits into the space between the line $y = 0$ and the line $y = m$.

Solution: Conjugation is a bijection between partitions with largest part at most m and partitions with at most m parts. Thus the coefficient of q^i (the number of partitions of i into parts of size at most m) in the generating function for the number of partitions of integers into parts of size at most m will be the coefficient of q^i (the number of partitions of i with at most m parts) in the generating function for the number of partitions of integers into parts of size at most m. Thus the two generating functions are the same. ▮

•203. When studying partitions of integers, it is inconvenient to restrict ourselves to partitions with at most m parts or partitions with maximum part size m.

[5] The reason for this change in the notation relates to the subject of finite fields in abstract algebra, where q is the standard notation for the size of a finite field. While we will make no use of this connection, it will be easier for you to read more advanced work if you get used to the different notation.

(a) Give the generating function for the number of partitions of an integer into parts of any size. Don't forget to use q rather than x as your variable.

Solution: We start with

$$(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)\cdots(1+q^i+q^{2i})\cdots,$$

which we can write more precisely as

$$\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} q^{ij}.$$

∎

(b) Find the coefficient of q^4 in this generating function.

Solution: From the fifth factor on, there is no way to choose a q^i that has i nonzero and less than five from the factor. Thus we choose a 1 from each of these factors. We can choose a q^4 from the fourth factor and 1 from the rest, a q^3 from the third factor, a q from the first and a 1 from the rest, a q^2 from the second factor, a q^2 from the first and a 1 from the rest, or we can choose a q^4 from the first factor and a 1 from the rest. Therefore the coefficient of q^5 is five. ∎

(c) Find the coefficient of q^5 in this generating function.

Solution: From the sixth factor of the product on, there is no way to choose a q^i that has i nonzero and less than six from the factor, so when we compute the coefficient of q^5, we can only choose the 1 from each of these terms. In the first five factors, we choose any combination of powers that adds to 5. We can choose q^5 from the fifth and 1 from the rest, q from the first, q^4 from the second and 1 from the rest, q^2 from the second, q^3 from the third and 1 from the rest, q^2 from the first and q^3 from the third and 1 from the rest, q^3 from the first, q^2 from the second, and 1 from the rest q^5 from the first and 1 from the rest, and these are the only ways to get a q^5 in the product. Thus the coefficient of q^5 is 7. ∎

(d) This generating function involves an infinite product. Describe the process you would use to expand this product into as many terms of a power series as you choose.

Solution: To get the coefficient of q^k in this product, we look at all ways of choosing one summand from each of the infinite series and multiplying them together to get q^k, and add all these products up. That is, the coefficient of q^k is the number of ways of making these choices of one summand from each series so that the product of our choices is q^k. This lets us write down as many terms of the series as we want, or as we have patience for! ▮

(e) Rewrite any power series that appear in your product as quotients of polynomials or as integers divided by polynomials.

Solution: We can rewrite the infinite product as $\prod_{i=0}^{\infty} \frac{1}{1-q^i}$. ▮

➔ 204. In Problem 203b, we multiplied together infinitely many power series. Here are two notations for infinite products that look rather similar:

$$\prod_{i=1}^{\infty} 1 + q + q^2 + \cdots + q^i \quad \text{and} \quad \prod_{i=1}^{\infty} 1 + q^i + q^{2i} + \cdots + q^{i^2}.$$

However, one makes sense and one doesn't. Figure out which one makes sense and explain why it makes sense and the other one doesn't. If we want to make sense of a product of the form

$$\prod_{i=1}^{\infty} 1 + p_i(q),$$

where each $p_i(q)$ is a nonzero polynomial in q, describe a relatively simple assumption about the polynomials $p_i(q)$ that will make the product make sense. If we assumed the terms $p_i(q)$ were nonzero power series, is there a relatively simple assumption we could make about them in order to make the product make sense? (Describe such a condition or explain why you think there couldn't be one.)

Solution: $\prod_{i=1}^{\infty} 1 + q^i + q^{2i} + \cdots + q^{i^2}$ makes sense because when we look for ways of choosing one summand from each factor so that the summands multiply together to give us q^k, we will find only finitely many ways of making those choices, so the coefficient of q^k can be taken to be the number of such choices. On the other hand, in the expression $\prod_{i=1}^{\infty}(1 + q + q^2 + \cdots + q^i)$, there are infinitely many ways to choose q from one term and 1 from all the rest of the terms so that the product of these summands is q. Thus we can't even specify what the coefficient of q is in the product. On the basis of this analysis, we see that for $\prod_{i=1}^{\infty} 1 + p_i(q)$ to make sense, we need to assume that for each possible positive integer n, there are only a finite number of polynomials p_i whose lowest degree term has degree less than or equal to n. In that way, for each positive integer, there will be only finitely many ways to chose a summand from each factor so that the product of the summands is a multiple of q^k. The same assumption works when the p_i are power series, for the same reason. ∎

•205. What is the generating function (using q for the variable) for the number of partitions of an integer in which each part is even?

Solution: $(1+q^2+q^4+\cdots)(1+q^4+q^8+\cdots)(1+q^6+q^{12}+\cdots)\cdots$, which can be written more concisely as $\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} q^{2ij}$. ∎

•206. What is the generating function (using q as the variable) for the number of partitions of an integer into distinct parts, that is, in which each part is used at most once?

Solution: $(1+q)(1+q^2)(1+q^3)\cdots = \prod_{i=1}^{\infty}(1+q^i)$. ∎

•207. Use generating functions to explain why the number of partitions of an integer in which each part is used an even number of times equals the generating function for the number of partitions of an integer in which each part is even. How does this compare to Problem 166?

Solution: In the generating function $\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} q^{2ij}$, we may interpret the $2ij$ in q^{2ji} the value of using $2i$ as a part j times or as the value of using i as a part $2j$ times. Therefore this is the generating function both for the number of partitions of integers into parts that are even and the number of partitions into parts that are used an even number of times. Therefore the number of partitions of n in which each part is even equals the number of partitions of n in which each part is used an even number of times. In Problem 166 we got the same result bijectively. ∎

➔ •208. Use the fact that

$$\frac{1-q^{2i}}{1-q^i} = 1+q^i$$

and the generating function for the number of partitions of an integer into distinct parts to show how the number of partitions of an integer k into distinct parts is related to the number of partitions of an integer k into odd parts.

Solution: $\displaystyle\prod_{i=1}^{\infty} 1+q^i = \prod_{i=1}^{n} \frac{1-q^{2i}}{1-q^i} = \frac{\prod_{i=1}^{\infty} 1-q^{2i}}{\prod_{j=1}^{\infty} 1-q^j} = \prod_{i=j}^{\infty} \frac{1}{1-q^{2j-1}}$,
because all the terms in the numerator cancel with alternate terms in the denominator leaving only terms with odd powers of q. But

$$\begin{aligned}\prod_{j=1}^{\infty} \frac{1}{1-q^{2j-1}} &= \prod_{j=1}^{\infty}\sum_{i=0}^{\infty}(q^{2j-1})^i \\ &= (1+q+q^{2\cdot 1}+\cdots)(1+q^3+q^{2\cdot 3}+\cdots)(1+q^5+q^{2\cdot 5}+\cdots)\cdots,\end{aligned}$$

which is the generating function for partitions of integers into parts that are odd numbers. ▮

209. Write down the generating function for the number of ways to partition an integer into parts of size no more than m, each used an odd number of times. Write down the generating function for the number of partitions of an integer into parts of size no more than m, each used an even number of times. Use these two generating functions to get a relationship between the two sequences for which you wrote down the generating functions.

Solution:

$$\prod_{i=1}^{m} q^i+q^{3i}+q^{5i}+\cdots = q^{1+2+\cdots m}\prod_{i=1}^{m} 1+q^{2i}+q^{4i}+\cdots = q^{\binom{n+1}{2}}\prod_{i=1}^{m}\frac{1}{1-q^2}$$

is the generating function for the number of ways to partition an integer into parts of size at most m, each used an odd number of times.

$$\prod_{i=1}^{m} 1+q^{2i}+q^{4i}+q^{6i}+\cdots = \prod_{i=1}^{m}\frac{1}{1-q^{2i}}$$

is the generating function for the number of partitions of an integer into parts of size no more than m, each used an even number of times. Therefore the number of partitions of k into parts of size no more than m, each used an even number of times is the number of partitions of $k+\binom{m+1}{2}$, into parts of size no more than m, each used an odd number of times. ▮

➔ 210. In Problem 201b and Problem 202 you gave the generating functions for, respectively, the number of partitions of k into parts the largest of which is at most m and for the number of partitions of k into at most m parts. In this problem we will give the generating function for the number of partitions of k into at most n parts, the largest of which is at most m. That is, we will analyze $\sum_{i=0}^{\infty} a_k q^k$ where a_k is the number of partitions of k into at most n parts, the largest of which is at most m. Geometrically, it is the generating function for partitions whose Young diagram fits into an m by n rectangle, as in Problem 168. This generating function has significant analogs to the binomial coefficient $\binom{m+n}{n}$, and so it is denoted by $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$. It is called a *q-binomial coefficient*.

(a) Compute $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_q$.

Solution: A partition with up to two parts of size up to two can have no parts, one part of size 1, which makes it a partition of 1, one part of size 2 which makes it a partition of 2, two parts of size 1, which makes it a partition of 2, a part of size 2 and a part of size 1, which makes it a partition of 3, or two parts of size 2, which makes it a partition of 4. Thus $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4$. ∎

(b) Find explicit formulas for $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q$.

Solution: Both are $1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^{n-1}}{1-q}$, because they are the generating function for the number of partitions whose Young diagram fits into a rectangle $n-1$ units wide and 1 unit deep or into a rectangle 1 unit wide and $n-1$ units deep respectively. ∎

(c) How are $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ and $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ related? Prove it. (Note this is the same as asking how $\begin{bmatrix} r \\ s \end{bmatrix}_q$ and $\begin{bmatrix} r \\ r-s \end{bmatrix}_q$ are related.)

Solution: By conjugation, $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$. ∎

(d) So far the analogy to $\binom{m+n}{n}$ is rather thin! If we had a recurrence like the Pascal recurrence, that would demonstrate a real analogy. Is $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$?

Solution: No. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4$, but $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = 1+q+q^2$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = 1+q+q^2$. ∎

(e) Recall the two operations we studied in Problem 171.

i. The largest part of a partition counted by $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ is either m or is less than or equal to $m-1$. In the second case, the partition fits into a rectangle that is at most $m-1$ units wide and at most n units deep. What is the generating function for partitions of this type? In the first case, what kind of rectangle does the partition we get by removing the largest part sit in? What is the generating function for partitions that sit in this kind of rectangle? What is the generating function for partitions that sit in this kind of rectangle after we remove a largest part of size m? What recurrence relation does this give you?

Solution: The generating function for partitions that arise in the second case is $\begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$. In the first case, after we delete the largest part, the Young diagram sits in a rectangle of width m and depth $n-1$. The generating function for partitions that arise in the first case *after* we have deleted a part of size m is $\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$. Thus the generating function for partitions that arise in the first case (*before* we delete the part of size m) is $q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$. Thus by the sum principle the generating function for all partitions that fit into a rectangle of width m and depth n is given by

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q .$$

If you don't use the symmetry $\begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$, you get

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q .$$

While this doesn't *quite* look like the Pascal recurrence, it is still a correct answer. ■

ii. What recurrence do you get from the other operation we studied in Problem 171?

Solution: We either have exactly k parts or fewer than k parts. In the first case, removing one from each part gives us a partition whose Young diagram fits into a $m-1$ by n box, while in the second case doing nothing gives us a partition that fits into a m by $n-1$ box. In the first case the partition we get partitions $k-n$, and in the second case it still partitions k. Thus we get

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q.$$

▮

iii. It is quite likely that the two recurrences you got are different. One would expect that they might give different values for $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$. Can you resolve this potential conflict?

Solution: Yes, by conjugation. ▮

(f) Define $[n]_q$ to be $1+q+\cdots+q^{n-1}$ for $n>0$ and $[0]_q=1$. We read this simply as n-sub-q. Define $[n]!_q$ to be $[n]_q[n-1]_q\cdots[3]_q[2]_q[1]_q$. We read this as n cue-torial, and refer to it as a q-ary factorial. Show that

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{[m+n]!_q}{[m]!_q[n]!_q}.$$

Solution: Note that $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$, because only the partition with no parts sits in a rectangle of width or depth 0. But $\frac{[m+0]!_q}{[m]!_q[0]!_q} = 1$ and $\frac{[0+n]!_q}{[0]!_q[n]!_q} = 1$. Thus the formula holds when $m=0$ or $n=0$. But

$$\begin{aligned}
& q^m\frac{[m+n-1]!_q}{[n-1]!_q[m]!_q} + \frac{[m+n-1]!_q}{[n]!_q[m-1]!_q} \\
= \quad & [m+n-1]!_q\left(\frac{q^m}{[n-1]!_q[m]!_q} + \frac{1}{[n]!_q[m-1]!_q}\right) \\
& [m+n-1]!_q\left(\frac{[n]_q q^m}{[n]!_q[m]_q} + \frac{[m]_q}{[n]!_q[m]!_q}\right) \\
= \quad & \frac{[m+n-1]!_q}{[n]!_q[m]!_q}\left((1+q+\cdots+q^{n-1})q^m + 1+q+\cdots+q^{m-1}\right) \\
= \quad & \frac{[m+n-1]!_q[m+n]_q}{[n]!_q[m]!_q} \\
= \quad & \begin{bmatrix} m+n \\ n \end{bmatrix}_q.
\end{aligned}$$

Thus $\frac{[m+n]!_q}{[m]!_q[n]!_q}$ satisfies our recurrence and so by the principle of mathematical induction, $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{[m+n]!_q}{[m]!_q[n]!_q}$. ▮

(g) Now think of q as a variable that we will let approach 1. Find an explicit formula for

i. $\lim_{q\to 1}[n]_q$.

Solution: n ∎

ii. $\lim_{q\to 1}[n]!_q$.

Solution: $n!$ ∎

iii. $\lim_{q\to 1}\begin{bmatrix} m+n \\ n \end{bmatrix}_q$.

Solution: $\binom{m+n}{n}$ ∎

Why is the limit in Part iii equal to the number of partitions (of any number) with at most n parts all of size most m? Can you explain bijectively why this quantity equals the formula you got?

Solution: Since the generating function is a finite sum (we are talking about partitions whose Young diagram fits into a finite rectangle), the limit is obtained by setting $q = 1$, and this sums the number of partitions of each possible number k that have at most n parts all of size at most m. We want a bijection between such partitions and the n element subsets of an $m+n$ element set. Recall that there is a bijection between subsets of an n element set and lattice paths from $(0,0)$ to (m,n) in a coordinate plane. If we draw our rectangle of width m and depth n with its lower left corner at $(0,0)$, then each Young diagram gives us such a lattice path and each such lattice path gives us a Young diagram. ∎

∗(h) What happens to $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ if we let q approach -1?

Solution: $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ is the generating function for the number of partitions whose Young diagram fits into an m by n rectangle. That is,

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{k=0}^{\infty} a_k q^k,$$

where a_k is the number of partitions of k whose Young diagram fits into an m by n rectangle. In particular, $a_k = 0$ if $k > mn$, because the Young diagram of a partition of a number larger than mn certainly cannot fit into an m by n rectangle. Thus

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{k=0}^{mn} a_k q^k.$$

Furthermore, $a_k = a_{mn-k}$, because the complement in an m by n rectangle of a partition of k is a partition of $mn-k$, and complementation in an m by n rectangle is a bijection between partitions of k that fit into the rectangle and partitions of $mn-k$ that fit into the rectangle. Thus $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ is a polynomial of degree mn in which the coefficient of q_k equals the coefficient of q^{mn-k}. For this reason we can compute the limit as q approaches -1 simply by substituting -1 for q in the polynomial; the only trouble is that the formula we know for $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ is a quotient of polynomials that has zero in both the numerator and denominator when we substitute in $q=-1$. Nonetheless, when we substitute -1 for q we get the alternating sum $\sum_{i=0}^{mn}(-1)^i a_i$. Thus if a_i and a_{mn-i} have opposite sign, they will cancel out. If mn is even, i is even exactly when $mn-i$ is even, and so a_i and a_{mn-i} have the same sign. However, when mn is odd, a_i and a_{mn-i} have opposite signs in the sum $\sum_{i=0}^{mn}(-1)^i a_i$, and so the sum is zero. Thus the polynomial $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ is zero at $q=-1$ whenever mn is odd.

Our experience with binomial coefficients might lead us to hope that the alternating sum of the coefficients a_k will always be zero. However, we computed that $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4$, so

$$\lim_{q\to -1} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1-1+2-1+1 = 1.$$

We could compute some more values of the limit by going back to the definition in this way, but it seems unlikely that we could get enough data to make a good conjecture. However, we have the recurrence, and setting $q=-1$ in the recurrence gives us the table

$m+n\backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0
3	1	1	1	1	0	0	0	0	0	0
4	1	0	2	0	1	0	0	0	0	0
5	1	1	2	2	1	1	0	0	0	0
6	1	0	3	0	3	0	1	0	0	0
7	1	1	3	3	3	3	1	1	0	0
8	1	0	4	0	6	0	4	0	1	0
9	1	1	4	4	6	6	4	4	1	1

From the table, it is clear that we get binomial coefficients interspersed with 0s in the even numbered rows of our table and repeated binomial coefficients in the odd numbered rows of our table. In particular when $m+n$ and n are both even, $\begin{bmatrix} m+n \\ n \end{bmatrix}_{-1} = \binom{(m+n)/2}{n/2}$, and if $m+n$ is even and n is odd, $\begin{bmatrix} m+n \\ n \end{bmatrix}_{-1} = 0$. In our table, at least, if $m+n$ is odd, it appears that we get $\begin{bmatrix} m+n \\ n \end{bmatrix}_{-1} = \binom{\lfloor (m+n)/2 \rfloor}{\lfloor n/2 \rfloor}$. In fact, using the recurrence just as we used it to construct the table, we can prove by induction that

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{-1} = \begin{cases} 0 & \text{if } mn \text{ is odd} \\ \binom{\lfloor (m+n)/2 \rfloor}{\lfloor n/2 \rfloor} & \text{otherwise.} \end{cases}$$

(Note that mn is odd if and only if $m+n$ is even and n is odd.) Wouldn't it be fascinating to know what we are counting here? It is the number of lattice paths from $(0,0)$ to $(n-k,k)$ that are symmetric under 180 degree rotation. This and other results are discussed in the paper by John R. Stembridge "Some hidden relations involving the ten symmetry classes of plane partitions." J. Combin. Theory Ser. A 68 (1994), no. 2, 372–409. ∎

4.3 Generating Functions and Recurrence Relations

Recall that a recurrence relation for a sequence a_n expresses a_n in terms of values a_i for $i < n$. For example, the equation $a_i = 3a_{i-1} + 2^i$ is a first order linear constant coefficient recurrence.

4.3.1 How generating functions are relevant

Algebraic manipulations with generating functions can sometimes reveal the solutions to a recurrence relation.

211. Suppose that $a_i = 3a_{i-1} + 3^i$.

•(a) Multiply both sides by x^i and sum both the left hand side and right hand side from $i = 1$ to infinity. In the left-hand side use the fact that

$$\sum_{i=1}^{\infty} a_i x^i = (\sum_{i=0}^{\infty} a_i x^i) - a_0$$

and in the right hand side, use the fact that

$$\sum_{i=1}^{\infty} a_{i-1} x^i = x \sum_{i=1}^{\infty} a_{i-1} x^{i-1} = x \sum_{j=0}^{\infty} a_j x^j = x \sum_{i=0}^{\infty} a_i x^i$$

(where we substituted j for $i-1$ to see explicitly how to change the limits of summation, a surprisingly useful trick) to rewrite the equation in terms of the power series $\sum_{i=0}^{\infty} a_i x^i$. Solve the resulting equation for the power series $\sum_{i=0}^{\infty} a_i x^i$. You can save a lot of writing by using a variable like y to stand for the power series.

Solution:

$$\begin{aligned}
\sum_{i=1}^{\infty} a_i x^i &= 3\sum_{i=1}^{\infty} a_{i-1}x^i + \sum_{i=1}^{i} 3^i x^i \\
\sum_{i=1}^{\infty} a_i x^i &= 3x\sum_{i=1}^{\infty} a_{i-1}x^{i-1} + \sum_{i=0}^{\infty} 3^i x^i - 3^0x^0 \\
\sum_{i=0}^{\infty} a_i x^i - a_0 &= 3x\sum_{i=0}^{\infty} a_i x^i + \frac{1}{1-3x} - 1 \\
(1-3x)\sum_{i=0}^{\infty} a_i x^i &= a_0 + \frac{1}{1-3x} - 1 \\
\sum_{i=0}^{\infty} a_i x^i &= \frac{a_0-1}{1-3x} + \frac{1}{(1-3x)^2}
\end{aligned}$$

▮

• (b) Use the previous part to get a formula for a_i in terms of a_0.

Solution:

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i x^i &= \frac{a_0-1}{1-3x} + \frac{1}{(1-3x)^2} \\
&= (a_0-1)\sum_{i=0}^{\infty} 3^i x^i + \sum_{i=0}^{\infty}\binom{i+1}{i}3^i x^i,
\end{aligned}$$

which gives us $a_i = (a_0-1)3^i + (i+1)3^i = (a_0+i)3^i$. ▮

(c) Now suppose that $a_i = 3a_{i-1} + 2^i$. Repeat the previous two steps for this recurrence relation. (There is a way to do this part using what you already know. Later on we shall introduce yet another way to deal with the kind of generating function that arises here.)

Solution:

$$\sum_{i=1}^{\infty} a_i x^i = 3\sum_{i=1}^{\infty} a_{i-1}x^i + \sum_{i=1}^{\infty} 2^i x^i$$

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i x^i - a_0 &= 3x\sum_{i=1}^{\infty} a_{i-1}x^{i-1} + \sum_{i=0}^{\infty}(2x)^i - 1 \\
\sum_{i=0}^{\infty} a_i x^i - a_0 &= 3x\sum_{i=0}^{\infty} a_i x^i + \frac{1}{1-2x} - 1 \\
(1-3x)\sum_{i=0}^{\infty} a_i x^i &= a_0 + \frac{1}{1-2x} - 1 \\
\sum_{i=0}^{\infty} a_i x^i &= \frac{a_0-1}{1-3x} + \frac{1}{(1-2x)(1-3x)} \\
\sum_{i=0}^{\infty} a_i x^i &= (a_0-1)\sum_{i=0}^{\infty} 3^i x^i + \sum_{i=0}^{\infty} 2^i x^i \sum_{j=0}^{\infty} 3^j x^j.
\end{aligned}$$

But $\displaystyle\sum_{i=0}^{\infty} 2^i x^i \sum_{j=0}^{\infty} 3^j x^j = \sum_{k=0}^{\infty}\sum_{i=0}^{k} 2^i 3^{k-i} x^k = \sum_{k=0}^{\infty} 3^k x^k \sum_{i=0}^{k} \frac{2^i}{3^i} =$ $\displaystyle\sum_{k=0}^{\infty} \frac{1-\left(\frac{2}{3}\right)^{k+1}}{1-\frac{2}{3}} 3^k x^k = \sum_{k=0}^{\infty}(3^{k+1}-2^{k+1})x^k$. Substituting this into the equation for $\sum_{i=0}^{\infty} a_i x^i$ gives us $a_i = (a_0+2)3^i - 2^{i+1}$. ▮

∘212. Suppose we deposit \$5000 in a savings certificate that pays ten percent interest and also participate in a program to add \$1000 to the certificate at the end of each year (from the end of the first year on) that follows (also subject to interest). Assuming we make the \$5000 deposit at the end of year 0, and letting a_i be the amount of money in the account at the end of year i, write a recurrence for the amount of money the certificate is worth at the end of year n. Solve this recurrence. How much money do we have in the account (after our year-end deposit) at the end of ten years? At the end of 20 years?

Solution: $a_n = 1.1a_{n-1} + 1000$, and $a_0 = 5000$.

$$\begin{aligned}
\sum_{i=1}^{\infty} a_i x^i &= 1.1x\sum_{i=1}^{\infty} a_{i-1}x^{i-1} + 1000\sum_{i=1}^{\infty} x^i \\
\sum_{i=0}^{\infty} a_i x^i - a_0 &= 1.1x\sum_{i=0}^{\infty} a_i x^i + 1000(\sum_{i=0}^{\infty} x^i - 1) \\
(1-1.1x)\sum_{i=0}^{\infty} a_i x^i &= a_0 + 1000\sum_{i=0}^{\infty} x^i - 1000
\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{\infty} a_i x^i &= a_0 \sum_{i=0}^{\infty}(1.1)^i x^i + 1000 \sum_{i=0}^{\infty}(1.1)^i x^i \sum_{i=0}^{\infty} x^i - 1000 \sum_{i=0}^{\infty}(1.1)^i x^i \\ \sum_{i=0}^{\infty} a_i x^i &= a_0 \sum_{i=0}^{\infty}(1.1)^i x^i + 1000 \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}(1.1)^i\right) x^k - 1000 \sum_{i=0}^{\infty}(1.1)^i x^i\end{aligned}$$

This gives us that $a_n = 4000(1.1)^n + 1000\frac{1-(1.1)^{n+1}}{-.1}$. This simplifies to $a_n = 4000(1.1)^n + 10000(1.1)^{n+1} - 10000 = 15,000(1.1)^n - 10000$. Courtesy of Maple, after ten years we have \$28,906.14 and after 20 years we have \$90,912.50. ▮

4.3.2 Fibonacci Numbers

The sequence of problems that follows describes a number of hypotheses we might make about a fictional population of rabbits. We use the example of a rabbit population for historic reasons; our goal is a classical sequence of numbers called Fibonacci numbers. When Fibonacci[6] introduced them, he did so with a fictional population of rabbits.

4.3.3 Second order linear recurrence relations

•213. Suppose we start (at the end of month 0) with 10 pairs of baby rabbits, and that after baby rabbits mature for one month they begin to reproduce, with each mature pair producing two new pairs at the end of each month afterwards. Suppose further that over the time we observe the rabbits, none die. Let a_n be the number pairs of rabbits we have at the end of month n. Show that $a_n = a_{n-1} + 2a_{n-2}$. This is an example of a *second order linear* recurrence with constant coefficients. Using a method similar to that of Problem 211, show that

$$\sum_{i=0}^{\infty} a_i x^i = \frac{10}{1 - x - 2x^2}.$$

This gives us the generating function for the sequence a_i giving the population in month i; shortly we shall see a method for converting this to a solution to the recurrence.

Solution:

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n$$

[6]Apparently Leonardo de Pisa was given the name Fibonacci posthumously. It is a shortening of "son of Bonacci" in Italian.

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x &= x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 2x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x &= x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x &= x \left(\sum_{n=0}^{\infty} a_n x^n - a_0 \right) + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\
(1 - x - 2x^2) \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x - a_0 x \\
\sum_{n=0}^{\infty} a_n x^n &= \frac{a_0 + a_1 x - a_0 x}{(1 - x - 2x^2)}
\end{aligned}$$

In this problem $a_0 = a_1 = 10$ because we start with ten pairs of baby rabbits, so they have to mature for a month before they begin reproducing. Thus $\sum_{n=0}^{\infty} a_n x^n = \frac{10}{(1 - x - 2x^2)}$ ▮

•214. In Fibonacci's original problem, each pair of mature rabbits produces one new pair at the end of each month, but otherwise the situation is the same as in Problem 213. Assuming that we start with one pair of baby rabbits (at the end of month 0), find the generating function for the number of pairs of rabbits we have at the end of n months.

Solution: Our recurrence becomes $a_n = a_{n-1} + a_{n-2}$, and following the pattern of Problem 213 we get

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \\
\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x &= x \left(\sum_{n=0}^{\infty} a_n x^n - a_0 \right) + x^2 \sum_{n=0}^{\infty} a_n x^n \\
(1 - x - x^2) \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x - a_0 x \\
\sum_{n=0}^{\infty} a_n x^n &= \frac{a_0 + a_1 x - a_0 x}{(1 - x - x^2)}
\end{aligned}$$

Since now $a_0 = a_1 = 1$, we have $\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - x - x^2}$. ▮

➔215. Find the generating function for the solutions to the recurrence

$$a_i = 5a_{i-1} - 6a_{i-2} + 2^i.$$

Solution:

$$\begin{aligned}
\sum_{i=2}^{\infty} a_i x^i &= \sum_{i=2}^{\infty} 5a_{i-1}x^i - \sum_{i=2}^{\infty} 6a_{i-2}x^i + \sum_{i=2}^{\infty} 2^i x^i \\
\sum_{i=0}^{\infty} a_i x^i - a_0 - a_1 x &= 5x\left(\sum_{i=0}^{\infty} a_i x^i - a_0\right) - 6x^2 \sum_{i=0}^{\infty} a_i x^i + \sum_{n=0}^{\infty} 2^i x^i - 1 - 2x \\
(1 - 5x + 6x^2)\sum_{i=0}^{\infty} a_i x^i &= a_0 - 1 + (a_1 - 5a_0 - 2)x + \sum_{n=0}^{\infty} 2^i x^i \\
\sum_{i=0}^{\infty} a_i x^i &= \frac{a_0 - 1 + (a_1 - 5a_0 - 2)x}{1 - 5x + 6x^2} + \frac{1}{1 - 5x + 6x^2} \cdot \frac{1}{1 - 2x}
\end{aligned}$$

▮

The recurrence relations we have seen in this section are called *second order* because they specify a_i in terms of a_{i-1} and a_{i-2}, they are called *linear* because a_{i-1} and a_{i-2} each appear to the first power, and they are called *constant coefficient recurrences* because the coefficients in front of a_{i-1} and a_{i-2} are constants.

4.3.4 Partial fractions

The generating functions you found in the previous section all can be expressed in terms of the reciprocal of a quadratic polynomial. However, without a power series representation, the generating function doesn't tell us what the sequence is. It turns out that whenever you can factor a polynomial into linear factors (and over the complex numbers such a factorization always exists) you can use that factorization to express the reciprocal in terms of power series.

•216. Express $\frac{1}{x-3} + \frac{2}{x-2}$ as a single fraction.

Solution: $\frac{x-2+2(x-3)}{(x-3)(x-2)} = \frac{3x-8}{x^2-5x+6}$ ▮

◦217. In Problem 216 you see that when we added numerical multiples of the reciprocals of first degree polynomials we got a fraction in which the denominator is a quadratic polynomial. This will always happen unless the two denominators are multiples of each other, because their least common multiple will simply be their product, a quadratic polynomial. This leads us to ask whether a fraction whose denominator is a quadratic polynomial can always be expressed as a sum of fractions whose denominators are first degree polynomials. Find numbers c and d so that

$$\frac{5x+1}{(x-3)(x+5)} = \frac{c}{x-3} + \frac{d}{x+5}.$$

Solution:

$$\begin{aligned} \frac{5x+1}{(x-3)(x+5)} &= \frac{c}{x-3} + \frac{d}{x+5} \\ \frac{5x+1}{(x-3)(x-5)} &= \frac{cx+5c+dx-3d}{(x-3)(x-5)} \end{aligned}$$

gives us

$$\begin{aligned} 5x &= cx+dx \\ 1 &= 5c-3d \end{aligned}$$

$$\begin{aligned} 5 &= c+d \\ 1 &= 5c-3d. \end{aligned}$$

This gives us $16 = 8c$ so that $c = 2$ and then $d = 3$. ▮

•218. In Problem 217 you may have simply guessed at values of c and d, or you may have solved a system of equations in the two unknowns c and d. Given constants a, b, r_1, and r_2 (with $r_1 \neq r_2$), write down a system of equations we can solve for c and d to write

$$\frac{ax+b}{(x-r_1)(x-r_2)} = \frac{c}{x-r_1} + \frac{d}{x-r_2}.$$

Solution: To have

$$\frac{ax+b}{(x-r_1)(x-r_2)} = \frac{c}{x-r_1} + \frac{d}{x-r_2}$$

we must have

$$cx - r_2c + dx - r_1d = ax + b.$$

This gives us the equations $cx + dx = ax$ and $-r_2c - r_1d = b$. Since x can be any value, in particular it can be nonzero, so we can divide by it. This gives us the equations $c + d = a$ and $r_2c + r_1d = -b$. ▮

Writing down the equations in Problem 218 and solving them is called the *method of partial fractions*. This method will let you find power series expansions for generating functions of the type you found in Problems 213 to 215. However, you have to be able to factor the quadratic polynomials that are in the denominators of your generating functions.

•219. Use the method of partial fractions to convert the generating function of Problem 213 into the form

$$\frac{c}{x - r_1} + \frac{d}{x - r_2}.$$

Use this to find a formula for a_n.

Solution: $\frac{10}{(1-x-2x^2)} = \frac{10}{(1-2x)(1+x)} = \frac{c}{1-2x} + \frac{d}{1+x}$. This gives us the equations $c + d = 10$ and $c - 2d = 0$. Thus $3d = 10$, so $d = \frac{10}{3}$, and $c = 2d$ so $c = \frac{20}{3}$. Thus

$$\begin{aligned}\sum_{i=0}^{\infty} a_i x^i &= \frac{10}{1 - x - 2x^2}\\ &= \frac{20/3}{1 - 2x} + \frac{10/3}{1 + x}\\ &= \frac{20}{3}\sum_{i=0}^{\infty}(2x)^i + \frac{10}{3}\sum_{i=0}^{\infty}(-1)^i x^i\end{aligned}$$

Thus $a_i = \frac{20}{3}2^i + \frac{10}{3}(-1)^i$. ▮

•220. Use the quadratic formula to find the solutions to $x^2 + x - 1 = 0$, and use that information to factor $x^2 + x - 1$.

Solution: $r_1 = \frac{-1+\sqrt{5}}{2}$, $r_2 = \frac{-1-\sqrt{5}}{2}$. These roots give us the equation $x^2 + x - 1 = (x - \frac{-1+\sqrt{5}}{2})(x - \frac{-1-\sqrt{5}}{2})$. ▮

•221. Use the factors you found in Problem 220 to write

$$\frac{1}{x^2 + x - 1}$$

in the form

$$\frac{c}{x - r_1} + \frac{d}{x - r_2}.$$

(Hint: You can save yourself a tremendous amount of frustrating algebra if you arbitrarily choose one of the solutions and call it r_1 and call the other solution r_2 and solve the problem using these algebraic symbols in place of the actual roots.[7] Not only will you save yourself some work, but you will get a formula you could use in other problems. When you are done, substitute in the actual values of the solutions and simplify.)

Solution: $\frac{1}{x^2+x-1} = \frac{c}{x-r_1} + \frac{d}{x-r_2}$ gives us $cx - cr_2 + dx - dr_1 = 1$. Thus $c + d = 0$, and $cr_2 + dr_1 = -1$. This gives us $d = -c$ and so $cr_2 - cr_1 = -1$, which yields $c = \frac{1}{r_1-r_2}$, and $d = \frac{1}{r_2-r_1}$. By substitution, $c = 1/\sqrt{5}$ and $d = -1/\sqrt{5}$. This gives us

$$\frac{1}{x^2+x-1} = \frac{1/\sqrt{5}}{x - \frac{-1+\sqrt{5}}{2}} + \frac{-1/\sqrt{5}}{x - \frac{-1-\sqrt{5}}{2}}.$$

∎

•222. (a) Use the partial fractions decomposition you found in Problem 220 to write the generating function you found in Problem 214 in the form

$$\sum_{n=0}^{\infty} a_n x^i$$

and use this to give an explicit formula for a_n. (Hint: once again it will save a lot of tedious algebra if you use the symbols r_1 and r_2 for the solutions as in Problem 221 and substitute the actual values of the solutions once you have a formula for a_n in terms of r_1 and r_2.)

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \frac{1}{1-x-x^2} = -\frac{1}{x^2+x-1} \\
&= \frac{1}{\sqrt{5}} \cdot \frac{1}{r_1 - x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{r_2 - x} \\
&= \frac{1}{r_1\sqrt{5}} \cdot \frac{1}{1 - x/r_1} - \frac{1}{r_2\sqrt{5}} \cdot \frac{1}{1 - x/r_2} \\
&= \frac{1}{r_1\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{x}{r_1}\right)^n - \frac{1}{r_2\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{x}{r_2}\right)^n
\end{aligned}$$

[7] We use the words roots and solutions interchangeably.

This gives us that

$$\begin{aligned}
a_n &= \frac{1}{\sqrt{5}\cdot r_1^{n+1}} + \frac{1}{\sqrt{5}\cdot r_2^{n+1}} \\
&= \frac{2^{n+1}}{\sqrt{5}(-1+\sqrt{5})^{n+1}} + \frac{2^{n+1}}{\sqrt{5}(-1-\sqrt{5})^{n+1}} \\
&= \frac{2^{n+1}(1+\sqrt{5})^{n+1}}{\sqrt{5}\cdot 4^{n+1}} - \frac{2^{n+1}(1-\sqrt{5})^{n+1}}{\sqrt{5}\cdot 4^{n+1}} \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.
\end{aligned}$$

(b) When we have $a_0 = 1$ and $a_1 = 1$, i.e. when we start with one pair of baby rabbits, the numbers a_n are called *Fibonacci Numbers*. Use either the recurrence or your final formula to find a_2 through a_8. Are you amazed that your general formula produces integers, or for that matter produces rational numbers? Why does the recurrence equation tell you that the Fibonacci numbers are all integers?

Solution: Using the recurrence, the Fibonacci numbers from a_0 to a_8 are 1, 1, 2, 3, 5, 8, 13, 21, 34. The recurrence says each term is the sum of the two preceding terms, and since the first two terms are integers, all the sums must be integers. ▮

➔ (c) Explain why there is a real number b such that, for large values of n, the value of the nth Fibonacci number is almost exactly (but not quite) some constant times b^n. (Find b and the constant.)

Solution: Since $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, $\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$ approaches 0 as n becomes large. Therefore if we take b to be $\left(\frac{1+\sqrt{5}}{2}\right)^n$ and we take c to be $\frac{1+\sqrt{5}}{2\sqrt{5}}$ then then nth Fibonacci number is almost exactly cb^n when n is large. In particular, it is the nearest integer to cb^n. ▮

➔ (d) Find an algebraic explanation (not using the recurrence equation) of what happens to make the square roots of five go away in the general formula for the Fibonacci numbers. Explain why there is a real number b such that, for large values of n, the value of the nth Fibonacci number is almost exactly (but not quite) some constant times b^n. (Find b and the constant.)

Solution: From the binomial theorem,

$$
\begin{aligned}
& \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\
= \quad & \frac{1}{2^{n+1}\sqrt{5}}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}\left(\sqrt{5}\right)^i - \sum_{i=0}^{n+1}\binom{n+1}{i}(-1)^i\left(\sqrt{5}\right)^i\right] \\
= \quad & \frac{1}{2^{n+1}\sqrt{5}}\sum_{i:i\in[n+1],\ i \text{ is odd}}\binom{n+1}{i}\left(\left(\sqrt{5}\right)^i - (-1)^i\left(\sqrt{5}\right)^i\right) \\
= \quad & \frac{1}{2^{n+1}\sqrt{5}}\sum_{i:i\in[n+1],\ i \text{ is odd}} 2\binom{n+1}{i}\left(\sqrt{5}\right)^i \\
= \quad & \frac{1}{2^n}\sum_{i:i\in[n+1],\ i \text{ is odd}}\binom{n+1}{i}\left(\sqrt{5}\right)^{i-1} \\
= \quad & \frac{1}{2^n}\sum_{i:i\in[n],\ i \text{ is even}}\binom{n+1}{i+1}5^{i/2} \\
= \quad & \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}5^k,
\end{aligned}
$$

which makes it clear that a_n is at least a rational number. It is not clear from this new formula why the result is always an integer. ▮

➔ ∗(e) As a challenge (which the author has not yet done), see if you can find a way to show algebraically (not using the recurrence relation, but rather the formula you get by removing the square roots of five) that the formula for the Fibonacci numbers yields integers.

Solution: None is yet available. ▮

223. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$.

Solution:

$$
\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= 4\sum_{n=2}^{\infty} a_{n-1}x^n - 4\sum_{n=2}^{\infty} a_{n-2}x^n \\
\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x &= 4x\left(\sum_{n=0}^{\infty} a_n x^n - a_0\right) - 4x^2\sum_{n=0}^{\infty} a_n x^n
\end{aligned}
$$

$$\begin{aligned}
(1-4x+4x^2)\sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x - 4a_0 x \\
\sum_{n=0}^{\infty} a_n x^n &= \frac{a_0 + a_1 x - 4a_0 x}{(1-4x+4x^2)} \\
\sum_{n=0}^{\infty} a_n x^n &= \frac{a_0 + a_1 x - 4a_0 x}{(1-2x)^2} \\
\sum_{n=0}^{\infty} a_n x^n &= (a_0 + a_1 x - 4a_0 x)\sum_{n=0}^{\infty} \binom{n+2-1}{n} 2^n x^n \\
\sum_{n=0}^{\infty} a_n x^n &= (a_0 + a_1 x - 4a_0 x)\sum_{n=0}^{\infty} (n+1)2^n x^n
\end{aligned}$$

Thus $a_n = a_0(n+1)2^n + (a_1 - 4a_0)n2^{n-1} = a_0 2^n + (a_1 - 2a_0)n2^{n-1}$. ▮

4.3.5 Catalan Numbers

➔ 224. (a) Using either lattice paths or diagonal lattice paths, explain why the Catalan Number C_n satisfies the recurrence

$$C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}.$$

Solution: Recall that a Dyck Path is a diagonal lattice path that never goes lower than its starting point and a Catalan Path of length $2n$ is a Dyck path that goes from $(0,0)$ to $(2n,0)$ The Catalan number C_n is the number of Catalan paths of length $2n$. We take $C_0 = 1$. A Catalan path could touch the x-axis several times before it reaches $(2n,0)$. Its first touch can be any point $(2i,0)$ between $(2,0)$ and $(2n,0)$. For the path to touch first at $(2i,0)$, the path must start with an upstep and then proceed as a Dyck path from $(1,1)$ to $(2i-1,1)$. From there it must take a downstep. But the number of Dyck paths from $(1,1)$ to $(2i-1,1)$ is the same as the number of Catalan paths from $(0,0)$ to $(2i-2,0)$. The number of Catalan paths is the number whose first touch of the x-axis is at $x=2$ plus the number whose first touch is at $x=4$,..., through the number whose first touch is at $2n$. After the first touch at $x=2i$, the path then behaves as a Catalan path from $(2i,0)$ to $(2n,0)$. The number of such paths is C_{n-i}. By the product principle, the number of Catalan paths whose first touch is at $x = 2i$ is $C_{i-1}C_{n-i}$. Then by the sum principle, the number of Catalan paths of length 1 or more is

$$C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}.$$

$\blacksquare$

(b) Show that if we use y to stand for the power series $\sum_{i=0}^{\infty} C_n x^n$, then we can find y by solving a quadratic equation. (Hint: does the right hand side of the recurrence remind you of some products you have worked with?) Find y.

Solution: To solve for C_n, write

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n x^n &= 1 + \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_{i-1}C_{n-i}x^n \\
\sum_{n=0}^{\infty} C_n x^n &= 1 + x\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_{i-1}x^{i-1}C_{n-i}x^{n-i} \\
\sum_{n=0}^{\infty} C_n x^n &= 1 + x\sum_{i=1}^{\infty} C_{i-1}x^{i-1}\sum_{j=0}^{\infty} C_j x^j \\
y &= 1 + xy^2
\end{aligned}$$

This gives us $xy^2 - y + 1 = 0$, and solving for y by the quadratic formula gives us $y = \frac{1\pm\sqrt{1-4x}}{2x}$. $\blacksquare$

(c) Taylor's theorem from calculus tells us that the extended binomial theorem

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i$$

holds for any number real number r, where $\binom{r}{i}$ is defined to be

$$\frac{r^{\underline{i}}}{i!} = \frac{r(r-1)\cdots(r-i+1)}{i!}.$$

Use this and your solution for y (note that of the two possible values for y that you get from the quadratic formula, only one gives an actual power series) to get a formula for the Catalan numbers.

Solution: By the extended binomial theorem,

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{i=0}^{\infty} \binom{1/2}{i}(-4x)^i = \sum_{i=0}^{\infty} \frac{(1/2)^{\underline{i}}}{i!}(-1)^i 4^i x^i.$$

The first term of this power series is 1, so to get a power series for y, we must take the negative square root so that the x in the denominator will cancel out. Thus $y = -\frac{1}{2}\sum_{i=1}^{\infty}\frac{(1/2)^{\underline{i}}}{i!}(-1)^i 4^i x^i$. But $(1/2)^{\underline{i}} = (\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})\cdots(\frac{-2i+3}{2})$, so

$$\begin{aligned} y &= -\frac{1}{2}\sum_{i=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2i-3)}{i!}(-1)^{2i-1}2^i x^{i-1} \\ &= \sum_{i=1}^{\infty}\frac{(2i-2)!}{(i-1)!2^i i!}2^i x^{i-1} \\ &= \sum_{i=1}^{\infty}\frac{2i-2!}{i!(i-1)!}x^{i-1} \\ &= \sum_{j=0}^{\infty}\frac{2j!}{(j+1)!j!}x^j \end{aligned}$$

giving us $C_j = \frac{1}{j+1}\binom{2j}{j}$, which is our earlier formula for the Catalan Number C_j. ▮

4.4 Supplementary Problems

1. What is the generating function for the number of ways to pass out k pieces of candy from an unlimited supply of identical candy to n children (where n is fixed) so that each child gets between three and six pieces of candy (inclusive)? Use the fact that

$$(1 + x + x + x^3)(1 - x) = 1 - x^4$$

to find a formula for the number of ways to pass out the candy.

Solution: $(x^3 + x^4 + x^5 + x^6)^n$;

$$\begin{aligned} (x^3 + x^4 + x^5 + x^6)^n &= x^{3n}(1 + x + x^2 + x^3)^n \\ &= x^{3n}\left(\frac{1-x^4}{1-x}\right)^n \\ &= x^{3n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^{4j}\sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i \end{aligned}$$

The number of ways to pass out k pieces of candy is the coefficient of x^k in this expression. Thus the answer is zero if $k < 3n$ because of the x^{3n} in front. Otherwise the answer is the coefficient of x^{k-3n} in $\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^{4j}\sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i$, which is

$$\sum_{i,j:4j+i=k-3n}(-1)^j\binom{n}{j}\binom{n+i-1}{i} = \sum_{j=0}^{\lfloor (k-3n)/4\rfloor}(-1)^j\binom{n}{j}\binom{k-2n-4j-1}{n-1}.$$

▮

○2. (a) In paying off a mortgage loan with initial amount A, annual interest rate $p\%$ (on a monthly basis) with a monthly payment of m, what recurrence describes the amount owed after n months of payments in terms of the amount owed after $n-1$ months? Some technical details: You make the first payment after one month. The amount of interest included in your monthly payment is $.01p/12$. This interest rate is applied to the amount you owed immediately after making your last monthly payment.

Solution: $a_n = (1+\frac{.01p}{12})a_{n-1} - m$. ▮

(b) Find a formula for the amount owed after n months.

Solution: From Problem 98 or by applying generating functions we have

$$\begin{aligned} a_n &= A(1+\frac{.01p}{12})^n - m\frac{1-(1+\frac{.01p}{12})^n}{1-(1+\frac{.01p}{12})} \\ &= \left(A-\frac{1200m}{p}\right)\left(1+\frac{.01p}{12}\right)^n + \frac{1200m}{p} \end{aligned}$$

▮

(c) Find a formula for the number of months needed to bring the amount owed to zero. Another technical point: If you were to make the standard monthly payment m in the last month, you might actually end up owing a negative amount of money. Therefore it is ok if the result of your formula for the number of months needed gives a non-integer number of months. The bank would just round up to the next integer and adjust your payment so your balance comes out to zero.

Solution: $\left(A-\frac{1200m}{p}\right)\left(1+\frac{.01p}{12}\right)^n + \frac{1200m}{p} = 0$ gives us the equation $\left(1+\frac{.01p}{12}\right)^n = \frac{1200m}{1200m-Ap}$. Taking logarithms to any base we choose gives us $n\log(1+\frac{.01p}{12}) = \log 1200m - \log(1200m-Ap)$ and so $n = \frac{\log 1200m - \log(1200m-Ap)}{\log(1+\frac{.01p}{12})}$. ▮

(d) What should the monthly payment be to pay off the loan over a period of 30 years?

Solution: $360 = \frac{\log 1200m - \log(1200m - Ap)}{\log(1+\frac{.01p}{12})}$ is the equation we need to solve for m, the monthly payment. We need to chose some base for the logarithm so we can write its inverse function; suppose we use logs to the base 10. Then

$$\begin{aligned}
360\log(1+\frac{.01p}{12}) &= \log 1200m - \log(1200m - Ap)\\
10^{360\log(1+\frac{.01p}{12})} &= 10^{\log 1200m/(1200m-Ap)}\\
(1+\frac{.01p}{12})^{360} &= 1200m/(1200m - Ap)\\
(1200m - Ap)\,(1+\frac{.01p}{12})^{360} &= 1200m\\
1200m\left((1+\frac{.01p}{12})^{360} - 1\right) &= Ap(1+\frac{.01p}{12})^{360}\\
m &= \frac{Ap(1+\frac{.01p}{12})^{360}}{1200\left((1+\frac{.01p}{12})^{360} - 1\right)}
\end{aligned}$$

is our monthly payment. ▮

➔ 3. We have said that for nonnegative i and positive n we want to define $\binom{-n}{i}$ to be $\binom{n+i-1}{i}$. If we want the Pascal recurrence to be valid, how should we define $\binom{-n}{-i}$ when n and i are both positive?

Solution: The number $\binom{n}{k}$ is the number in row n and column k of the Pascal (right) triangle. We have said we want to have $\binom{-n}{0} = \binom{n+0-1}{0}$, so we want ones everywhere in that column. Now the Pascal recurrence gives us that $\binom{-n}{0} = \binom{-n-1}{-1} + \binom{-n-1}{0}$, so that $\binom{-n}{-1} = 0$, as does $\binom{0}{-1}$. Applying the Pascal recurrence again gives us $\binom{-n}{-1} = \binom{-n-1}{-2} + \binom{-n-1}{-1}$, so we have $\binom{-n-1}{-2} = 0$ as well. Following this pattern, we can prove by induction that $\binom{-n}{-k}$ is zero whenever k and n are positive. ▮

➔ 4. Find a recurrence relation for the number of ways to divide a convex n-gon into triangles by means of non-intersecting diagonals. How do these numbers relate to the Catalan numbers?

Solution: Let d_n be the number of ways to divide an n-gon into triangles by means of nonintersecting diagonals. Take an n-gon and label its vertices cyclically from 1 to n. Any triangulation must have a triangle containing the edge between vertex 1 and vertex n. This edge is denoted by $1n$. The third vertex can be any number between 2 and $n-1$. We consider two cases. First, if the third vertex is 2 or $n-1$, then we have divided our polygon up into a triangle and an $(n-1)$-gon, and any triangulation of that $(n-1)$-gon joins with our original triangle to give us a triangulation of the n-gon. Second, if the third vertex of our original triangle is vertex i with $3 \le i \le n-2$ then we have divided our polygon into the polygon with the i edges 12, 23, ..., $(i-1)i$, $i1$, the polygon with $n-i+1$ edges ni, $i(i+1)$, ..., $(n-1)n$, and the original triangle with edges $n1$, $1i$, in. Triangulations of the first two of these polygons join with the original triangle to give us a triangulation of the original polygon.

The number of triangulations of the original polygon that we get from case 1 is $2d_{n-1}$. The number of triangulations we get from the second case is $\sum_{i=3}^{n-2} d_i d_{n-i+1}$. Thus the total number of triangulations is $2d_{n-1} + d_{n-2}d_3 + d_{n-3}d_4 + \cdots + d_3 d_{n-2}$. If we take $d_2 = 1$, then we may write $d_n = d_{n-1}d_2 + d_{n-2}d_3 + \cdots + d_3 d_{n-2} + d_2 d_{n-1} = \sum\limits_{i=2}^{n-1} d_i d_{n-i+1}$. This is very similar to the recurrence in Problem 224c for the Catalan numbers. We could apply the generating function method we used with the Catalan numbers to find a formula for d_n. We could also experiment with the first few Catalan numbers and the first few "triangulation" numbers to see if they are related. We have $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = C_0C_2 + C_1C_1 + C_2C_0 = 5$, and $C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 14$. We have that $d_2 = 1$, $d_3 = 1$, $d_4 = 2$, $d_5 = d_4d_2 + d_3d_3 + d_2d_4 = 5$, and $d_6 = d_5d_2 + d_4d_3 + d_3d_4 + d_5d_2 = 14$. This makes pretty convincing evidence that $d_i = C_{i-2}$. We have already done a base case (and more) for an inductive proof. So assume inductively that $d_i = C_{i-2}$ for $i < n$. Then

$$\begin{aligned}
d_n &= \sum_{i=2}^{n-1} d_i d_{n-i+1} \\
&= \sum_{i=2}^{n-1} C_{i-2} C_{n-i+1-2} \\
&= \sum_{i=2}^{n-1} C_{i-2} C_{n-i-1} \\
&= \sum_{k=1}^{n-2} C_{k-1} C_{n-(k+1)-1}
\end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{n-2} C_{k-1}C_{(n-2)-k} \\ &= C_{n-2} \end{aligned}$$

Thus by the principle of mathematical induction, $d_n = C_{n-2}$ for all integers $n \geq 2$. ▮

➔ 5. How does $\sum_{k=0}^{n}\binom{n-k}{k}$ relate to the Fibonacci Numbers?

Solution: We begin by computing a few values of $a_n = \sum_{k=0}^{n}\binom{n-k}{k}$. We have $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 1+2 = 3$, $a_4 = 1+3+1 = 5$, $a_5 = 1+4+3 = 8$ and $a_6 = 1+5+6+1 = 13$. So far the sequence agrees with the Fibonacci Numbers. Each term of the sequence is the sum of the two preceding terms, so it makes sense to try to prove that $a_n = a_{n-1} + a_{n-2}$. We may write

$$\begin{aligned} a_{n-1} + a_{n-2} &= \sum_{k=0}^{n-1}\binom{n-1-k}{k} + \sum_{k=0}^{n-2}\binom{n-2-k}{k} \\ &= \sum_{k=0}^{n-1}\binom{n-1-k}{k} + \sum_{j=1}^{n-1}\binom{n-1-j}{j-1} \\ &= 1 + \sum_{k=1}^{n-1}\binom{n-1-k}{k} + \binom{n-1-k}{k-1} \\ &= 1 + \sum_{k=1}^{n-1}\binom{n-k}{k} = \sum_{k=0}^{n-1}\binom{n-k}{k} = a_n. \end{aligned}$$

Thus the sequence satisfies the same recurrence as the Fibonacci numbers and its first two values are the same as the Fibonacci numbers. This lets us prove by induction that a_n is the nth Fibonacci number. More generally, given any second order recurrence, if two sequences satisfy that recurrence and have the same first two values, then they are equal. ▮

6. Let m and n be fixed. Express the generating function for the number of k-element multisets of an n-element set such that no element appears more than m times as a quotient of two polynomials. Use this expression to get a formula for the number of k-element multisets of an n-element set such that no element appears more than m times.

Solution: $(1+x+x^2+\cdots+x^m)^n = \frac{(1-x^{m+1})^n}{(1-x)^n}$. Expanding this gives us $\frac{(1-x^{m+1})^n}{(1-x)^n} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}x^{(m+1)i}\sum_{j=0}^{\infty}\binom{n+j-1}{j}$. The coefficient of

x^k in this product is the number of k-element multisets chosen from an n-element set in which no element appears more than m times. This coefficient is

$$\sum_{i,j:(m+1)i+j=k}(-1)^i\binom{n}{i}\binom{n+j-1}{j} = \sum_{i=1}^{\lfloor\frac{k}{m+1}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-(m+1)i-1}{n-1}. \blacksquare$$

7. One natural but oversimplified model for the growth of a tree is that all new wood grows from the previous year's growth and is proportional to it in amount. To be more precise, assume that the (total) length of the new growth in a given year is the constant c times the (total) length of new growth in the previous year. Write down a recurrence for the total length a_n of all the branches of the tree at the end of growing season n. Find the general solution to your recurrence relation. Assume that we begin with a one meter cutting of new wood (from the previous year) which branches out and grows a total of two meters of new wood in the first year. What will the total length of all the branches of the tree be at the end of n years?

   Solution: $a_n = a_{n-1} + c(a_{n-1} - a_{n-2}) = (1+c)a_{n-1} - ca_{n-2}$.

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty}(1+c)a_{n-1}x^n - c\sum_{n=2}^{\infty} a_{n-2}x^n \\
(1-(1+c)x+cx^2)\sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x - a_0(1+c)x \\
\sum_{n=0}^{\infty} a_n x^n &= \frac{a_0 + (a_1 - a_0(1+c))x}{1-(1+c)x+cx^2} \\
&= \frac{a_0 + (a_1 - a_0(1+c))x}{(1-x)(1-cx)}
\end{aligned}$$

   Assuming $c \neq 1$ and using the method of partial fractions gives us

$$\begin{aligned}
&\frac{a_0 + (a_1 - a_0(1+c))x}{(1-x)(1-cx)} \\
= \quad & (a_0 + (a_1 - a_0(1+c))x)\left[\frac{1/(1-c)}{(1-x)} - \frac{c/(1-c)}{1-cx}\right] \\
= \quad & (a_0 + (a_1 - a_0(1+c))x)\left[\frac{1}{1-c}\sum_{i=0}^{\infty} x^i - \frac{c}{1-c}\sum_{i=0}^{\infty} c^i x^i\right] \\
= \quad & \frac{(a_0 + (a_1 - a_0(1+c))x)}{1-c}\sum_{i=0}^{\infty}(1-c^{i+1})x^i.
\end{aligned}$$

From this we get that

$$a_i = \frac{a_0}{1-c}(1-c^{i+1}) + \frac{a_1 - a_0(1+c)}{1-c}(1-c^i).$$

Assuming that we begin with one meter of new wood means $a_0 = 1$, and assuming we have a total of two meters of new wood at the end of the first year means $c = 2$ and $a_1 = 3$. Substituting these into our formula for a_i gives us $a_i = 2^{i+1} - 1$. ▮

➔8. (Relevant to Appendix C) We have some chairs which we are going to paint with red, white, blue, green, yellow and purple paint. Suppose that we may paint any number of chairs red or white, that we may paint at most one chair blue, at most three chairs green, only an even number of chairs yellow, and only a multiple of four chairs purple. In how many ways may we paint n chairs?

Solution: The generating function for the number of ways to paint n chairs is

$$\begin{aligned}&(1+x+x^2+\cdots)^2(1+x)(1+x+x^2+x^3)(1+x^2+x^4+\cdots)(1+x^4+x^8+\cdots)\\ &= \frac{(1+x)(1+x+x^2+x^3)}{(1-x)^2(1-x^2)(1-x^4)} = \frac{1}{(1-x)^4}\end{aligned}$$

Thus the number of ways to paint n chairs is $\binom{n+4-1}{n} = \binom{n+3}{n}$. ▮

9. What is the generating function for the number of partitions of an integer in which each part is used at most m times? Why is this also the generating function for partitions in which consecutive parts (in a decreasing list representation) differ by at most m and the smallest part is also at most m?

Solution:

$$(1+q+\cdots+q^m)(1+q^2+\cdots+q^{2m})\cdots = \prod_{i=1}^{\infty}\sum_{j=0}^{m} q^{ij} = \prod_{i=1}^{\infty}\frac{1-q^{i(m+1)}}{1-q^i}$$

This is also the generating function for the number of partitions of an integer in which consecutive parts differ by at most m, because when we conjugate a partition in which each part is used at most m times, we get a partition in which each distinct column of the Young diagram occurs at most m times, which means that the difference between two consecutive parts (in the decreasing list representation) is at most m, and that the last (smallest) part is at most m. ▮

Chapter 5

The Principle of Inclusion and Exclusion

5.1 The Size of a Union of Sets

One of our very first counting principles was the sum principle which says that the size of a union of disjoint sets is the sum of their sizes. Computing the size of overlapping sets requires, quite naturally, information about how they overlap. Taking such information into account will allow us to develop a powerful extension of the sum principle known as the "principle of inclusion and exclusion."

5.1.1 Unions of two or three sets

$\circ$225. In a biology lab study of the effects of basic fertilizer ingredients on plants, 16 plants are treated with potash, 16 plants are treated with phosphate, and among these plants, eight are treated with both phosphate and potash. No other treatments are used. How many plants receive at least one treatment? If 32 plants are studied, how many receive no treatment?

Solution: The number of plants receiving treatment was $16 + 16 - 8 = 24$. The number of plants receiving no treatment was eight. ∎

$+$226. Give a formula for the size of the union $A \cup B$ of two sets A and B in terms of the sizes $|A|$ of A, $|B|$ of B, and $|A \cap B|$ of $A \cap B$. If A and B are subsets of some "universal" set U, express the size of the complement $U - (A \cup B)$ in terms of the sizes $|U|$ of U, $|A|$ of A, $|B|$ of B, and $|A \cap B|$ of $A \cap B$.

Solution: $|A \cup B| = |A| + |B| - |A \cap B|$.
$|U - (A \cup B)| = |U| - |A| - |B| + |A \cap B|$. ▮

∘227. In Problem 225, there were just two fertilizers used to treat the sample plants. Now suppose there are three fertilizer treatments, and 15 plants are treated with nitrates, 16 with potash, 16 with phosphate, 7 with nitrate and potash, 9 with nitrate and phosphate, 8 with potash and phosphate and 4 with all three. Now how many plants have been treated? If 32 plants were studied, how many received no treatment at all?

Solution: $15 + 16 + 16 - 7 - 9 - 8 + 4 = 27$ plants were treated and five received no treatment. ▮

•228. Give a formula for the size of $A \cup B \cup A_3$ in terms of the sizes of A, B, C and the intersections of these sets.

Solution:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. ▮

5.1.2 Unions of an arbitrary number of sets

•229. Conjecture a formula for the size of a union of sets

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

in terms of the sizes of the sets A_i and their intersections.

Solution:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i=1}^{n} \sum_{j=i+1}^{n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

▮

The difficulty of generalizing Problem 228 to Problem 229 is not likely to be one of being able to see what the right conjecture is, but of finding a good notation to express your conjecture. In fact, it would be easier for some people to express the conjecture in words than to express it in a notation. We will describe some notation that will make your task easier. It is similar to the notation

$$E_P(S) = \sum_{s: s \in S} P(s).$$

that we used to stand for the sum of the pictures of the elements of a set S when we introduced picture enumerators.

Let us define

$$\bigcap_{i:i\in I} A_i$$

to mean the intersection over all elements i in the set I of A_i. Thus

$$\bigcap_{i:i\in\{1,3,4,6\}} A_i = A_1 \cap A_3 \cap A_4 \cap A_6. \tag{5.1}$$

This kind of notation, consisting of an operator with a description underneath of the values of a dummy variable of interest to us, can be extended in many ways. For example

$$\begin{aligned}\sum_{I:I\subseteq\{1,2,3,4\},\ |I|=2} |\cap_{i\in I} A_i| \quad &= \quad |A_1\cap A_2| + |A_1\cap A_3| + |A_1\cap A_4| \\ &+ \quad |A_2\cap A_3| + |A_2\cap A_4| + |A_3\cap A_4|.\end{aligned} \tag{5.2}$$

•230. Use notation something like that of Equation 5.1 and Equation 5.2 to express the answer to Problem 229. Note there are many different correct ways to do this problem. Try to write down more than one and choose the nicest one you can. Say why you chose it (because your view of what makes a formula nice may be different from somebody else's). The nicest formula won't necessarily involve all the elements of Equations 5.1 and 5.2. (The author's version doesn't use all those elements.)

Solution:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{S:S\subseteq[n],S\neq\emptyset} (-1)^{|S|-1}\left|\bigcap_{i:i\in S} A_i\right|$$

I chose this way of writing the formula partly because it is efficient with symbols; for example, it uses only one sum sign. But more importantly I chose it because it captures what I would want to say in words: "You sum, over all ways of choosing an intersection of the sets A_i, the size of the intersection times a sign factor that is -1 if you are intersecting an even number of sets and 1 if you are intersecting an odd number." If I were writing my solution out in words, I would probably assume that nobody would think about the possibility of an intersection of the empty set of the A_is, but I had to put the $S \neq \emptyset$ in my formula because otherwise the formula would have had us consider the possibility that S was empty. ▮

•231. A group of n students goes to a restaurant carrying backpacks. The manager invites everyone to check their backpack at the check desk and everyone does. While they are eating, a child playing in the check room randomly moves around the claim check stubs on the backpacks. We will try to compute the probability that, at the end of the meal, at least one student receives his or her own backpack. This probability is the fraction of the total number of ways to return the backpacks in which at least one student gets his or her own backpack back.

(a) What is the total number of ways to pass back the backpacks?

Solution: $n!$, because there are n students and n backpacks and a distribution of backpacks to students will be a bijection. ∎

(b) In how many of the distributions of backpacks to students does at least one student get his or her own backpack? (Hint: For each student, how big is the set of backpack distributions in which that student gets the correct backpack? It might be a good idea to first consider cases with $n = 3$, 4, and 5.)

Solution: If we let A_i be the set of backpack distributions in which student i gets the correct backpack, then the number we want to compute is the size of the union of the sets A_i. For this purpose we need to know, for every nonempty subset $S \subseteq [n]$ the size of $\cap_{i \in S} A_i$. That is, we need to know the number of ways to pass out the backpacks so that student i gets the correct one for each i in S. It won't matter whether or not other students get the correct backpacks, so we can just assume that for each $i \in S$, student i gets the correct backpack and then hand out the remaining $n - |S|$ backpacks to the remaining $n - |S|$ students in $(n - |S|)!$ ways. Thus $(n - |S|)!$ is $|\cap_{i:i \in S} A_i|$. Using our formula from Problem 230 we get

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{S:S\subseteq[n],S\neq\emptyset} (-1)^{|S|-1} \left|\bigcap_{i:i\in S} A_i\right| \\
&= \sum_{S:S\subseteq[n],S\neq\emptyset} (-1)^{|S|-1}(n-|S|)! \\
&= \sum_{s=1}^{n} \binom{n}{s} (-1)^{s-1}(n-s)! \\
&= \sum_{s=1}^{n} (-1)^{s-1}\frac{n!}{s!}
\end{aligned}$$

▮

(c) What is the probability that at least one student gets the correct backpack?

Solution: Dividing the answer in the last part by $n!$, the total number of ways to pass back the backpacks, we get $\sum_{s=1}^{n} \frac{(-1)^{s-1}}{s!}$ for the probability that at least one student gets the correct backpack. ▮

(d) What is the probability that no student gets his or her own backpack?

Solution: Subtracting from 1 to get the probability that no student gets the correct backpack gives us

$$1 - \sum_{s=1}^{n} \frac{(-1)^{s-1}}{s!} = \sum_{s=0}^{n} \frac{(-1)^{s}}{s!}.$$

▮

➔ (e) As the number of students becomes large, what does the probability that no student gets the correct backpack approach?

Solution: From calculus, we know that $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$. Substituting $x = -1$ gives us $e^{-1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}$, which is the limit as n becomes infinite of the probability in the solution to Problem 231. Thus the probability approaches $1/e$. ▮

Problem 231 is "classically" called the *hatcheck problem*; the name comes from substituting hats for backpacks. If is also sometimes called the *derangement problem*. A *derangement* of an n-element set is a permutation of that set (thought of as a bijection) that maps no element of the set to itself. One can think of a way of handing back the backpacks as a permutation f of the students: $f(i)$ is the owner of the backpack that student i receives. Then a derangement is a way to pass back the backpacks so that no student gets his or her own.

5.1.3 The Principle of Inclusion and Exclusion

The formula you have given in Problem 230 is often called **the principle of inclusion and exclusion** for unions of sets. The reason is the pattern in which the formula first adds (includes) all the sizes of the sets, then subtracts (excludes) all the sizes of the intersections of two sets, then adds (includes) all the sizes of the intersections of three sets, and so on. Notice that we haven't yet proved the principle. There are a variety of proofs. Perhaps one of the most straightforward (though not the most elegant) is an inductive proof that relies on the fact that

$$A_1 \cup A_2 \cup \cdots \cup A_n = (A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cup A_n$$

and the formula for the size of a union of two sets.

232. Give a proof of your formula for the principle of inclusion and exclusion.

Solution: The principle of inclusion and exclusion for one set says $|A_1| = |A_1|$. The principle of inclusion and exclusion for two sets says $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, and was proved in our solution to Problem 226. Now suppose the formula is true for a union of $n-1$ or fewer sets and $n > 2$. Since

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = (A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cup A_n,$$

we can apply the formula of Problem 226 to get

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= |\left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n| \\
&= |\bigcup_{i=1}^{n-1} A_i| + |A_n| - |\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n| \\
&= \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\bigcup_{i=1}^{n-1} A_i \cap A_n\right| \qquad (*)
\end{aligned}$$

By the inductive hypothesis, we may apply the principle of inclusion and exclusion to the first and last term in the line of the equation marked $(*)$. We can rewrite the line $(*)$ as

$$\begin{aligned}
&\sum_{S:S\subseteq[n-1],S\neq\emptyset} (-1)^{|S|-1} \left|\bigcap_{i:i\in S} A_i\right| + |A_n| - \sum_{S:S\subseteq[n-1],S\neq\emptyset} (-1)^{|S|-1} \left|\bigcap_{i:i\in S} A_i \cap A_n\right| \\
&= \sum_{S:S\subseteq[n],S\neq\emptyset} (-1)^{|S|-1} \left|\bigcap_{i:i\in S} A_i\right|,
\end{aligned}$$

where the last line follows because every nonempty subset of $[n]$ is either (1) a nonempty subset of $[n-1]$, (2) a nonempty subset of

$[n-1]$ with n added in, or (3) the set $\{n\}$. Thus by the principle of mathematical induction, the formula for the principle of inclusion and exclusion holds for all nonnegative integers n. ▮

➔ 233. We get a more elegant proof if we ask for a picture enumerator for $A_1 \cup A_2 \cup \cdots \cup A_n$. So let us assume A is a set with a picture function P defined on it and that each set A_i is a subset of A.

(a) By thinking about how we got the formula for the size of a union, write down instead a conjecture for the picture enumerator of a union. You could use a notation like $E_P(\bigcap_{i:i\in S} A_i)$ for the picture enumerator of the intersection of the sets A_i for i in a subset S of $[n]$.

Solution: $E_P(\bigcup_{i=1}^{n} A_i) = \sum_{S:S\subseteq[n],S\neq\emptyset} (-1)^{|S|-1} E_P(\bigcap_{i:i\in S} A_i)$. ▮

(b) If $x \in \bigcup_{i=1}^{n} A_i$, what is the coefficient of $P(x)$ in (the inclusion-exclusion side of) your formula for $E_P(\bigcup_{i=1}^{n} A_i)$?

Solution: Let T be the set of all i such that $x \in A_i$. Then $x \in A_i$ for exactly those i with $i \in T$. Note that $|T| > 0$ because x is at least one A_i. Then the coefficient of $P(x)$ is $\sum_{S:S\subseteq T,S\neq\emptyset} (-1)^{|S|-1}$.

But

$$\sum_{S:S\subseteq T,S\neq\emptyset} (-1)^{|S|-1} = -\sum_{i=1}^{|T|} \binom{|T|}{i} (-1)^i = -[(1-1)^{|T|}-1] = 0+1 = 1.$$

Thus if $x \in E_P(\bigcup_{i=1}^{n} A_i)$, then the coefficient of $P(x)$ in $E_P(\bigcup_{i=1}^{n} A_i)$ is 1. ▮

(c) If $x \notin \bigcup_{i=1}^{n} A_i$, what is the coefficient of $P(x)$ in (the inclusion-exclusion side of) your formula for $E_P(\bigcup_{i=1}^{n} A_i)$?

Solution: Since $P(x)$ is not in any of the enumerators $E_P(\bigcap_{i:i\in S} A_i)$, its coefficient is 0. ▮

(d) How have you proved your conjecture for the picture enumerator of the union of the sets A_i?

Solution: We have shown that $P(x)$ appears in the right hand side of our formula with coefficient one if it is in the union and with coefficient 0 otherwise. ▮

(e) How can you get the formula for the principle of inclusion and exclusion from your formula for the picture enumerator of the union?

Solution: Substitute 1 for the picture of each element x. Then $E_P(\bigcap_{i:i\in S} A_i)$ becomes $|\bigcap_{i:i\in S} A_i|$, and our formula follows. ▮

234. Frequently when we apply the principle of inclusion and exclusion, we will have a situation like that of Problem 231d. That is, we will have a set A and subsets $A_1, A_2, \ldots, A_n$ and we will want the size or the probability of the set of elements in A that are *not* in the union. This set is known as the *complement* of the union of the A_is in A, and is denoted by $A - \bigcup_{i=1}^n A_i$, or if A is clear from context, by $\overline{\bigcup_{i=1}^n A_i}$. Give the formula for $\overline{\bigcup_{i=1}^n A_i}$. The principle of inclusion and exclusion generally refers to both this formula and the one for the union.

Solution: Since all the A_is are subsets of A, one way to write this size is as $|A| - \sum_{S:\ S\subseteq[n], S\neq\emptyset}(-1)^{|S|-1}|\bigcap_{i:i\in S} A_i|$. Letting $|A| = |\bigcap_{i:i\in\emptyset} A_i|$, we may write $|\overline{\bigcup_{i=1}^n A_i}| = \sum_{S:\ S\subseteq[n]} (-1)^{|S|}|\bigcap_{i:i\in S} A_i|$. ▮

We can find a very elegant way of writing the formula in Problem 234 if we let $\bigcap_{i:i\in\emptyset} A_i = A$. For this reason, if we have a family of subsets A_i of a set A, we define[1] $\bigcap_{i:i\in\emptyset} A_i = A$.

[1]For those interested in logic and set theory, given a family of subsets A_i of a set A, we define $\bigcap_{i:i\in S} A_i$ to be the set of all members x of A that are in A_i for all $i \in S$. (Note that this allows x to be in some other A_js as well.) Then if $S = \emptyset$, our intersection consists of all members x of A that satisfy the statement "if $i \in \emptyset$, then $x \in A_i$." But since the hypothesis of the 'if-then' statement is false, the statement itself is true for all $x \in A$. Therefore $\bigcap_{i:i\in\emptyset} A_i = A$.

5.2 Applications of Inclusion and Exclusion

5.2.1 Multisets with restricted numbers of elements

235. In how many ways may we distribute k identical apples to n children so that no child gets more than four apples? Compare your result with your result in Problem 197.

Solution: Let S be the set of all distributions of k identical apples to the n children. Let A_i be the set of distributions in which child i gets five or more apples. Then we are asking for the number of distributions of apples that lie in none of the sets, so we are asking for $|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}|$. From the formula you gave in Problem 234 we see that to find this number we need to know $|\bigcap_{i \in S} A_i|$ for every subset S of $[n]$. But if S has size s, then S determines a distribution such that all the children in a particular set of size s will get five or more apples. By Problem 128, we can pass out the apples so that the children in a particular set $\hat{S}$ of children get at least five apples as follows: we give everyone in $\hat{S}$ five apples, and then pass out the remaining $k-5s$ apples to the children in $\binom{k-5s+n-1}{k-5s} = \binom{k-5s+n-1}{n-1}$ ways. This counts the number of ways to give at least five apples to every child in $\hat{S}$, and maybe give five apples to some other children as well. Thus $|\bigcap_{i \in S} A_i| = \binom{k-5|S|+n-1}{n-1}$. In particular, if $S = \emptyset$, we get $\binom{k+n-1}{n-1}$, which is the total number of ways to pass out k identical apples to n children. Applying the formula from Problem 234 gives us

$$\begin{aligned}
|\overline{\bigcup_{i=1}^{n} A_i}| &= \sum_{S:S\subseteq[n]} (-1)^{|S|} |\bigcap_{i:i\in S} A_i| \\
&= \sum_{s=0}^{n} \binom{n}{s} (-1)^s \binom{k-5s+n-1}{n-1} \\
&= \sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{k-5s+n-1}{n-1}
\end{aligned}$$

∎

5.2.2 The Ménage Problem

➔ 236. A group of n married couples comes to a group discussion session where they all sit around a round table. In how many ways can they sit so that no person is next to his or her spouse? (Note that two people of the same sex can sit next to each other.)

Solution: Let A be the set of all seating arrangements for $2n$ people around a round table. Let A_i be the set of arrangements in which couple i sits together. We are interested in $|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}|$. Thus for a set $S \subseteq [n]$, we need to compute $|\bigcap_{i:i\in S} A_i|$. If we let each couple described by S sit together, we will seat $|S|$ couples and $2n - 2|S|$ individuals around the table. We can do this in $2^{|S|}(|S|+2n-2|S|-1)!$ ways, because once we choose a place for a couple (i.e. two adjacent seats) there are two ways the couple can sit down. Thus as long as S is nonempty we have the right formula for $|\bigcap_{i:i\in S} A_i|$. Notice that in the case where $S = \emptyset$ the formula gives us $(2n-1)!$ seating arrangements, which is exactly the number of way to seat $2n$ people around a round table. This is the size of our set A. Therefore

$$|\bigcap_{i:i\in S} A_i| = 2^{|S|}(2n - |S| - 1)!.$$

Substituting this into the formula from Problem 234 gives us

$$\begin{aligned} |\overline{\bigcup_{i=1}^{n} A_i}| &= \sum_{S:S\subseteq[n]} (-1)^{|S|} |\bigcap_{i:i\in S} A_i| \\ &= \sum_{s=0}^{n} (-1)^s \binom{n}{s} 2^s (2n-s-1)! \end{aligned}$$

∎

➔ ∗237. A group of n married couples comes to a group discussion session where they all sit around a round table. In how many ways can they sit so that no person is next to his or her spouse or a person of the same sex? This problem is called the *ménage problem.* (Hint: Reason somewhat as you did in Problem 236, noting that if the set of couples who do sit side-by-side is nonempty, then the sex of the person at each place at the table is determined once we seat one couple in that set, or, for that matter, once we seat one person.)

Solution: We are going to consider arrangements of the couples alternating sex around the table. This will be our set A. The set A_i is the set of arrangements in which couple i sits together. We are interested in the number of arrangements that are in none of these sets. Thus for each subset S of $[n]$, we consider the number of arrangements in $\bigcap_{i:\ i\in S} A_i$. We distinguish the case that S is empty from the others. The number of arrangements with S empty is just the number of ways to seat $2n$ couples around the table, alternating sex, but with no other restrictions. We can arrange one of the sexes in a circle in $(n-1)!$ ways and then assign the members of the opposite sex to the places between them in $n!$ ways, so $\bigcap_{i:\ i\in\emptyset} A_i = (n-1)!n!$. (Another way to get this result is to let one person sit down. This determines the sex of the person at each place of the table, so there are $(n-1)!$ ways to assign the people of the same sex of the first person, and $n!$ ways to assign the people of the opposite sex. It appears that there are $2n$ choices for where the first person sits, but we can break the seating charts up into blocks of $2n$ seating charts, each of which gives the same circular arrangement. Thus there are $(n-1)!n!$ inequivalent seating arrangements.)

Now if S is nonempty and has s members, we seat one of the couples that must sit together (say the first in alphabetical order), and this determines the sex of the person that must sit at each other place. There are $2n$ pairs of adjacent seats where we can seat that couple and two ways they can sit in the pair of adjacent seats that we choose. Then we have $s-1$ couples, $n-s$ men and $n-s$ women to seat in the remaining places. First we arrange the $s-1$ couples and $2n-2s$ identical empty chairs in places at the table in $(2n-2s+s-1)!/(2n-2s)! = (2n-s-1)!/(2n-2s)!$ ways. Each couple can sit in only one way in the places they have chosen, because the sex of the person in a given place has been determined by how the first couple sits. The sex of the person in each of the remaining chairs has been determined, so we assign the men to their seats in $(n-s)!$ ways and we assign the women to their seats in $(n-s)!$ ways. Thus we have $2\cdot 2n(2n-s-1)!(n-s)!^2/(2n-2s)!$ ways to place the people. But we can partition the placements into blocks of $2n$ equivalent placements, because shifting everyone the same number of places to the right or left gives an equivalent placement. Thus the number of inequivalent seating arrangements is

$$\begin{aligned}\frac{2(2n-s-1)!(n-s)!^2}{(2n-2s)!} &= \frac{2(2n-s-1)!(n-s)!^2}{2(n-s)(2n-2s-1)!}\\ &= \frac{(2n-s-1)!(n-s)!(n-s-1)!}{(2n-2s-1)!}.\end{aligned}$$

Notice that if we take $s=0$, this formula reduces to $(n-1)!n!$. Thus for all sets S

$$\left|\bigcap_{i:i\in S} A_i\right| = \frac{(2n-s-1)!(n-s)!(n-s-1)!}{(2n-2s-1)!}.$$

Then from Problem 234

$$\begin{aligned}
|\overline{\bigcup_{i=1}^{n} A_i}| &= \sum_{S:S\subseteq[n]} (-1)^{|S|}\frac{(2n-|S|-1)!(n-|S|)!(n-|S|-1)!}{(2n-2|S|-1)!} \\
&= \sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{(2n-s-1)!(n-s)!(n-s-1)!}{(2n-2s-1)!} \\
&= \sum_{s=0}^{n}(-1)^s\frac{n!}{s!(n-s)!}\frac{(2n-s-1)!(n-s)!(n-s-1)!}{(2n-2s-1)!} \\
&= \sum_{s=0}^{n}(-1)^s\frac{n!(2n-s-1)!(n-s-1)!}{s!(2n-2s-1)!} \\
&= \sum_{s=0}^{n}(-1)^s\binom{2n-s-1}{s}n!(n-s-1)!
\end{aligned}$$

is the number of ways to seat the people, alternating sex, so that no couple sits together. ▮

5.2.3 Counting onto functions

•238. Given a function f from the k-element set K to the n-element set $[n]$, we say f is in the set A_i if $f(x) \neq i$ for every x in K. How many of these sets does an onto function belong to? What is the number of functions from a k-element set onto an n-element set?

Solution: An onto function is in none of these sets. Since we want the number of functions that are in none of these sets, we let our set A be the set of all functions from K to $[n]$. Then the number of onto functions is $|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}|$. For a nonempty subset S of $[n]$, the set $\bigcap_{i:\ i\in\emptyset} A_i$ is the set of functions from K to $[n]-S$. The size of this set is $(n-|S|)^k$. When $S=\emptyset$ this gives the size of A. Thus by Problem 234

$$\begin{aligned}
|\overline{\bigcup_{i=1}^{n} A_i}| &= \sum_{S:S\subseteq[n]} (-1)^{|S|}(n-|S|)^k \\
&= \sum_{s=0}^{n}(-1)^s\binom{n}{s}(n-s)^k
\end{aligned}$$

is the number of functions from K onto $[n]$. ▮

➔ 239. Find a formula for the Stirling number (of the second kind) $S(k,n)$.

Solution: Since the number of functions from $[k]$ onto $[n]$ is $S(k,n)n!$, we get from the solution to Problem 238

$$S(k,n) = \frac{1}{n!}\sum_{s=0}^{n}(-1)^s\binom{n}{s}(n-s)^k.$$

▮

240. If we roll a die eight times, we get a sequence of 8 numbers, the number of dots on top on the first roll, the number on the second roll, and so on.

(a) What is the number of ways of rolling the die eight times so that each of the numbers one through six appears at least once in our sequence? To get a numerical answer, you will likely need a computer algebra package.

Solution: By the formula for the number of onto functions, we have $\sum_{s=0}^{6}(-1)^s\binom{6}{s}(6-s)^8$ sequences in which each number between one and six appears. Courtesy of Maple, this number is 191,520. ▮

(b) What is the probability that we get a sequence in which all six numbers between one and six appear? To get a numerical answer, you will likely need a computer algebra package, programmable calculator, or spreadsheet.

Solution: $191520/6^8 = 665/5832$, which is about .1140260631, courtesy of Maple. ▮

➔ (c) How many times do we have to roll the die to have probability at least one half that all six numbers appear in our sequence. To answer this question, you will likely need a computer algebra package, programmable calculator, or spreadsheet.

Solution: Some experimenting with Maple shows that if we roll our die 13 times, we get probability approximately .5138581940 of having all six numbers appear, but with 12 rolls the probability is approximately .4378156806. Thus, 13 rolls are required. ▮

5.2.4 The chromatic polynomial of a graph

We defined a graph to consist of set V of elements called vertices and a set E of elements called edges such that each edge joins two vertices. A *coloring* of a graph by the elements of a set C (of colors) is an assignment of an element of C to each vertex of the graph; that is, a function from the vertex set V of the graph to C. A coloring is called *proper* if for each edge joining two distinct vertices[2], the two vertices it joins have different colors. You may have heard of the famous four color theorem of graph theory that says if a graph may be drawn in the plane so that no two edges cross (though they may touch at a vertex), then the graph has a proper coloring with four colors. Here we are interested in a different, though related, problem: namely, in how many ways may we properly color a graph (regardless of whether it can be drawn in the plane or not) using k or fewer colors? When we studied trees, we restricted ourselves to connected graphs. (Recall that a graph is connected if, for each pair of vertices, there is a walk between them.) Here, disconnected graphs will also be important to us. Given a graph which might or might not be connected, we partition its vertices into blocks called *connected components* as follows. For each vertex v we put all vertices connected to it by a walk into a block together. Clearly each vertex is in at least one block, because vertex v is connected to vertex v by the trivial walk consisting of the single vertex v and no edges. To have a partition, each vertex must be in one and only one block. To prove that we have defined a partition, suppose that vertex v is in the blocks B_1 and B_2. Then B_1 is the set of all vertices connected by walks to some vertex v_1 and B_2 is the set of all vertices connected by walks to some vertex v_2.

·241. (Relevant in Appendix C as well as this section.) Show that $B_1 = B_2$.

Solution: Since v is in B_1, there is a walk from v_1 to v. Since there is a walk from every vertex in B_1 to v_1, there is a walk from every vertex in in B_1 to v. But there is a walk from v to v_2 since $v \in B_2$. Thus there is a walk from every vertex in B_1 to v_2. Then by our description of B_2 just before the problem, every vertex in B_1 is also in B_2. A similar argument shows that every vertex in B_2 is also in B_1. Thus $B_1 = B_2$. ∎

Since $B_1 = B_2$, these two sets are the same block, and thus all blocks containing v are identical, so v is in only one block. Thus we have a partition of

[2]If a graph had a loop connecting a vertex to itself, that loop would connect a vertex to a vertex of the same color. It is because of this that we only consider edges with two distinct vertices in our definition.

the vertex set, and the blocks of the partition are the connected components of the graph. Notice that the connected components depend on the edge set of the graph. If we have a graph on the vertex set V with edge set E and another graph on the vertex set V with edge set E', then these two graphs could have different connected components. It is traditional to use the Greek letter γ (gamma)[3] to stand for the number of connected components of a graph; in particular, $\gamma(V, E)$ stands for the number of connected components of the graph with vertex set V and edge set E. We are going to show how the principle of inclusion and exclusion may be used to compute the number of ways to color a graph properly using colors from a set C of c colors.

·242. Suppose we have a graph G with vertex set V and edge set $E = \{e_1, e_2, \ldots e_{|E|}\}$. Suppose F is a subset of E. Suppose we have a set C of c colors with which to color the vertices.

(a) In terms of $\gamma(V, F)$, in how many ways may we color the vertices of G so that each edge in F connects two vertices of the same color?

Solution: For each edge in F to connect two vertices of the same color, we must have all the vertices in a connected component of the graph with vertex set V and edge set F colored the same color. Thus the number of such colorings is $c^{\gamma(V,F)}$. ▮

(b) Given a coloring of G, for each edge e_i in E, let us consider the set A_i of colorings that the endpoints of e are colored the same color. In which sets A_i does a proper coloring lie?

Solution: A proper coloring is in none of those sets. ▮

(c) Find a formula (which may involve summing over all subsets F of the edge set of the graph and using the number $\gamma(V, F)$ of connected components of the graph with vertex set V and edge set F) for the number of proper colorings of G using colors in the set C.

Solution: $|\overline{\bigcup_{i=1}^{|E|} A_i}| = \sum_{F:F\subseteq E}(-1)^{|F|}c^{\gamma(V,F)}$. ▮

The formula you found in Problem 242c is a formula that involves powers of c, and so it is a polynomial function of c. Thus it is called the "chromatic polynomial of the graph G." Since we like to think about polynomials as having a variable x and we like to think of c as standing for some constant,

[3] The Greek letter gamma is pronounced gam-uh, where gam rhymes with ham.

people often use x as the notation for the number of colors we are using to color G. Frequently people will use $\chi_G(x)$ to stand for the number of ways to color G with x colors, and call $\chi_G(x)$ the *chromatic polynomial* of G.

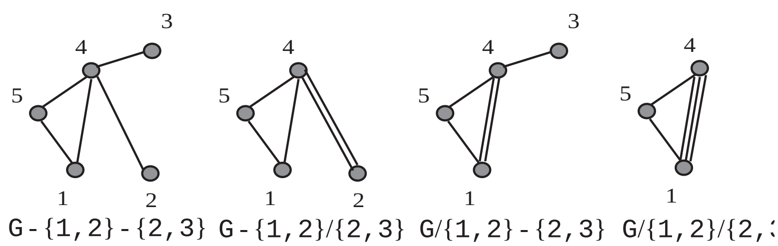
5.3 Deletion-Contraction and the Chromatic Polynomial

➔ 243. In Chapter 2 we introduced the deletion-contraction recurrence for counting spanning trees of a graph. Figure out how the chromatic polynomial of a graph is related to those resulting from deletion of an edge e and from contraction of that same edge e. Try to find a recurrence like the one for counting spanning trees that expresses the chromatic polynomial of a graph in terms of the chromatic polynomials of $G-e$ and G/e for an arbitrary edge e. Use this recurrence to give another proof that the number of ways to color a graph with x colors is a polynomial function of x.

Solution: The number of colorings of $G-e$ is equal to the number of proper colorings of G plus the number of colorings of G that are proper except for giving both ends of e the same color. But the number of colorings of G that are proper except for giving both ends of e the same color is the number of proper colorings of G/e. Therefore $\chi_{G-e}(x) = \chi_G(x) - \chi_{G/e}(x)$. This gives us $\chi_G(x) = \chi_{G-e}(x) - \chi_{G/e}(x)$. We can use this to prove inductively that $\chi_G(x)$ is a polynomial in x. If G has one vertex, then the number of ways to color G properly with x colors is x. This is a polynomial in x. Now suppose inductively that G has more than one vertex and whenever a graph H has fewer vertices than G, the function $\chi_H(x)$ is a polynomial function in x. Then $\chi_G(x) = \chi_{G-e}(x) - \chi_{G/e}(x)$, is a difference of of two polynomial functions in x, so it is a polynomial function in x. Therefore by the principle of mathematical induction, for all graphs G on a finite vertex set, the number of ways to properly color G in x colors is a polynomial in x. ▮

244. Use the deletion-contraction recurrence to reduce the computation of the chromatic polynomial of the graph in Figure 5.1 to computation of chromatic polynomials that you can easily compute. (You can simplify your computations by thinking about the effect on the chromatic polynomial of deleting an edge that is a loop, or deleting one of several edges between the same two vertices.)

Solution: If a graph has a loop it has no proper colorings. The graph in Figure 5.1 has no loops and no multiple edges between two vertices. The only way we could get a loop is by contracting one of several multiple edges between two vertices, and the resulting graph would have no contribution to the chromatic polynomial of the original graph. Thus whenever a contraction gives us a graph with multiple edges between two vertices, we can replace the multiple edges by one edge and go on with our computation from there. The graphs we get when we delete and contract the edges $\{1,2\}$ and $\{2,3\}$ are $(G-\{1,2\})-\{2,3\}$, $(G-\{1,2\})/\{2,3\}$, $(G-\{2,3\})/\{1,2\}$, and $(G/\{2,3\})/\{1,2\}$. These are shown in the following picture.

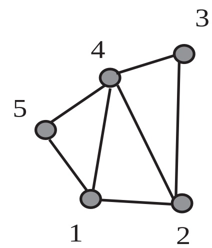


The chromatic polynomial of a triangle is $x(x-1)(x-2)$ because for one vertex we have x colors, for a second we have $x-1$, and for the third vertex, because it is adjacent to both of the other vertices, we have $x-2$ choices of colors. For a vertex of degree 1 there are $x-1$ choices of colors, those colors not used on the one vertex to which it is adjacent. As we mentioned, the extra edges do not change the chromatic polynomial, so we have that the chromatic polynomial of $(G-\{1,2\})-\{2,3\}$ is $x(x-1)^3(x-2)$, the chromatic polynomial of $(G-\{1,2\})/\{2,3\}$ is $x(x-1)^2(x-2)$, as is that of $(G-\{2,3\})/\{1,2\}$, and the chromatic polynomial of $(G/\{2,3\})/\{1,2\}$ is $(x-1)(x-2)(x-3)$. Using the deletion-contraction recurrence, we get that

$$\begin{aligned}
\chi_G(x) &= \chi_{G-\{1,2\}}(x) - \chi_{G/\{1,2\}}(x) \\
&= \chi_{G-\{1,2\}-\{2,3\}}(x) - \chi_{(G-\{1,2\})/\{2,3\}}(x) - \chi_{(G/\{1,2\})-\{2,3\}}(x) \\
&+ \chi_{G/\{1,2\}/\{2,3\}}(x) \\
&= x(x-1)^3(x-2) - 2x(x-1)^2(x-2) + x(x-1)(x-2) \\
&= x(x-1)(x-2)(x^2-2x+1+2x-2+1) \\
&= x^3(x-1)(x-2)
\end{aligned}$$

for the chromatic polynomial of G. ▮

Figure 5.1: A graph.



➔245. (a) In how many ways may you properly color the vertices of a path on n vertices with x colors? Describe any dependence of the chromatic polynomial of a path on the number of vertices.

Solution: To color the vertices of a path, start at one end. There are x colors for that vertex, and $x-1$ colors for each of the next $n-1$, since each of them must be different from the preceding one. Thus the chromatic polynomial of a path on n vertices is $x(x-1)^{n-1}$. The dependence on the number of vertices appears in the exponent on $x-1$. ▮

$*$(b) (Not tremendously hard.) In how many ways may you properly color the vertices of a cycle on n vertices with x colors? Describe any dependence of the chromatic polynomial of a cycle on the number of vertices.

Solution: If we use C_n to stand for a path on n vertices and P_n to stand for a path on n vertices, then by the deletion-contraction recurrence, we may write

$$\begin{aligned}
\chi_{C_n}(x) &= \chi_{P_n}(x) - \chi_{C_{n-1}}(x) \\
&= \chi_{P_n}(x) - \chi_{P_{n-1}}(x) + \chi_{C_{n-2}}(x) \\
&= \chi_{P_n}(x) - \chi_{P_{n-1}}(x) + \chi_{P_{n-3}}(x) - \cdots + (-1)^{n-3}(\chi_{P_3}(x) - \chi_{C_2}(x)) \\
&= x(x-1)^{n-1} - x(x-1)^{n-2} + x(x-1)^{n-3} \cdots \\
&+ (-1)^{n-3}[x(x-1)^2 - x(x-1)] \\
&= x(x-1)\sum_{i=0}^{n-2}(x-1)^i(-1)^{n-2-i} \\
&= x(x-1)(-1)^{n-2}\sum_{i=0}^{n-2}(1-x)^i
\end{aligned}$$

$$\begin{aligned}
&= x(x-1)(-1)^{n-2}\frac{1-(1-x)^{n-1}}{1-(1-x)} \\
&= (x-1)[(x-1)^{n-1}+(-1)^n].
\end{aligned}$$

Here the dependence on n is interesting; effectively, we are taking $(x-1)$ times the result of dropping the constant term from $(x-1)^{n-1}$. ▮

246. In how many ways may you properly color the vertices of a tree on n vertices with x colors?

Solution: Color an arbitrary vertex; you have x choices for the color of that vertex. No two vertices adjacent to it are adjacent (otherwise we'd have a cycle), so for for each of them you have $x-1$ choices of colors. No two vertices adjacent to colored vertices are adjacent to each other, nor is one of them adjacent to two colored vertices (in either case you'd have a cycle), so for each of them you'd have $x-1$ colors. You can continue this argument until all vertices are colored, so you have $x(x-1)^{n-1}$ ways to color the vertices.

You can also prove by induction that the chromatic polynomial of a tree is $x(x-1)^{n-1}$. This is clearly true if there is one vertex. Otherwise, choose a vertex of degree 1 in an n-vertex tree and remove it. You may inductively assume that the chromatic polynomial of the remaining tree is $x(x-1)^{n-2}$. Now there are $x-1$ choices for the color of the vertex you removed since it has degree 1, and so the chromatic polynomial of the tree is $x(x-1)^{n-1}$. There is also an inductive argument in which you delete and contract an arbitrary edge. ▮

➔ 247. What do you observe about the signs of the coefficients of the chromatic polynomial of the graph in Figure 5.1? What about the signs of the coefficients of the chromatic polynomial of a path? Of a cycle? Of a tree? Make a conjecture about the signs of the coefficients of a chromatic polynomial and prove it.

Solution: Not all powers of x appear, but the signs alternate as the power of x increases; that is, the sign of x^i is opposite that of x^{i+1}. More precisely, if c_i is the coefficient of x^i, then $(-1)^{n-i}c_i \geq 0$. To prove this, note it is trivially true for a graph with no edges. Choose an edge e of G. Then $\chi_G(x) = \chi_{G-e}(x) - \chi_{G/e}(x)$. In $G-e$, we may assume inductively that $(-1)^{n-i}c_i' \geq 0$ and in G/e we can assume inductively that $c_i''(-1)^{n-1-i} \geq 0$, where we use c_i' and c_i''

as the coefficient of x^i in $\chi_{G-e}(x)$ and $\chi_{G/e}(x)$, respectively. Then $c_i = c'_i - c''_i$, and

$$c_i(-1)^{n-i} = c'_i(-1)^{n-i} - c''_i(-1)^{n-i} = c'_i(-1)^{n-i} + c''_i(-1)^{n-1-i} \geq 0.$$

Therefore by the principle of mathematical induction, $c_i(-1)^i \geq 0$ for all finite graphs. ∎

5.4 Supplementary Problems

1. Each person attending a party has been asked to bring a prize. The person planning the party has arranged to give out exactly as many prizes as there are guests, but any person may win any number of prizes. If there are n guests, in how many ways may the prizes be given out so that nobody gets the prize that he or she brought?

   Solution: We use inclusion and exclusion. Let A be the set of all ways to distribute the prizes. Let A_i be the the set of distributions in which person i gets the prize he or she brought. We are interested in $|\overline{\bigcup_{i=1}^n A_i}|$. We need to compute $|\bigcap_{i:\ i\in S} A_i|$ for every subset S of $[n]$. But $|\bigcap_{i:\ i\in S} A_i|$ is the number of functions from the prizes to the people that assign the prize that person i brought to person i for each i in the set S. Think in terms of distributing those prizes first. Then there are $n-|S|$ other prizes that we may pass out to the n people as we please, so we may do that in $n^{n-|S|}$ ways. Thus $|\bigcap_{i:\ i\in S} A_i| = n^{n-|S|}$. When S is empty, this gives A. Applying Equation 234, we get

   $$|\overline{\bigcup_{i=1}^{n} A_i}| = \sum_{S:s\subseteq P} (-1)^{|S|} n^{n-|S|} = \sum_{s=0}^{n} (-1)^{|S|} \binom{n}{s} n^{n-s}.$$

   ∎

2. There are m students attending a seminar in a room with n seats. The seminar is a long one, and in the middle the group takes a break. In how many ways may the students return to the room and sit down so that nobody is in the same seat as before?

   Solution: We use inclusion and exclusion. We let A be the set of all seating arrangements. We let A_i be the set of seating arrangements such that student i sits in the same seat as before. We are interested in $|\overline{\bigcup_{i=1}^n A_i}|$. For this purpose, for each subset S of the set $[n]$, we need to compute $|\bigcap_{i:\ i\in S} A_i|$, the number of ways for the students to return so that every student represented by an i in S sits in his or her previous seat. This leaves us with $n - |S|$ seats to be filled in a one-to-one fashion by $m - |S|$ students. There are $(n-|S|)^{\underline{m-|S|}}$ such seating arrangements, so $|\bigcap_{i:\ i\in S} A_i| = (n-|S|)^{\underline{m-|S|}}$. Thus we have

$$|\overline{\bigcup_{i=1}^{n} A_i}| = \sum_{S:S\subseteq[n]} (-1)^{|S|}(n-|S|)^{\underline{m-|S|}} = \sum_{s=0}^{m}(-1)^s\binom{m}{s}(n-s)^{\underline{m-s}}$$

ways for the students to return so that nobody sits in his or her previous place. ▮

3. What is the number of ways to pass out k pieces of candy from an unlimited supply of identical candy to n children (where n is fixed) so that each child gets between three and six pieces of candy (inclusive)? If you have done Supplementary Problem 1 in Chapter 4 compare your answer in that problem with your answer in this one.

   Solution: We could do the problem as a generating functions problem. But, as an inclusion-exclusion problem, we would let A_i be the set of i such that child i gets more than six pieces of candy. We would then observe that the number of ways to pass out the candy so that the children determined by a subset S of $[i]$ all get more than six pieces, and everyone else gets at least 3, is the number of ways to pass out the remaining candy after giving 7 pieces to each child identified by S and 3 pieces to each of the other children. This number is $\binom{k-7|S|-3(n-|S|)-1}{n-1} = \binom{k-2n-4|S|-1}{n-1}$. From here we would substitute into formula of Problem 234, make any simplifications we could, and we would be done. This will give the same answer as Problem 1 in Chapter 4. ▮

➔ 4. In how many ways may k distinct books be arranged on n shelves so that no shelf gets more than m books?

   Solution: We use inclusion and exclusion. Let A be the set of all arrangements of the books on the shelves. Let A_i be the set of arrangements in which shelf i gets more than m books. Then the number of arrangements of books in which the shelves determined by a subset S of $[n]$ get more than m books is $|\bigcap_{i:\ i\in S} A_i| = k^{\underline{(m+1)|S|}} n^{\underline{k-(m+1)|S|}}$, because in order to get an arrangement in $\bigcup_{i:\ i\in S} A_i$ we may choose $(m+1)|S|$ books and arrange $m+1$ of them on each of the shelves represented by the elements of S, after which we arrange the remainder of the books. Thus

$$\begin{aligned}
|\overline{\bigcup_{i=1}^{n} A_i}| &= \sum_{S:S\subseteq[n]} (-1)^{|S|} k^{\underline{(m+1)|S|}} n^{\underline{k-(m+1)|S|}} \\
&= \sum_{s=0}^{n} (-1)^s \binom{n}{s} k^{\underline{(m+1)s}} n^{\underline{k-(m+1)s}}
\end{aligned}$$

is the number of ways to arrange the books so that no shelf gets more than m. ▮

➔ 5. Suppose that n children join hands in a circle for a game at nursery school. The game involves everyone falling down (and letting go). In how many ways may they join hands in a circle again so that nobody has the same person immediately to the right both times the group joins hands?

Solution: We use inclusion and exclusion, with A being the set of all circular arrangements of the children (where rotation of an arrangement gives the same arrangement, but flipping gives a different arrangement). The set A_i is the set of arrangements such that child i has the same child to the immediate right the both times they join hands. Given a set $S \subseteq [n]$, we can think of arranging units consisting of individual children and strings of children holding hands in a circle. We have $n-s$ of these units because s children are to the immediate right of someone in units of size more than one (and everyone else is lefmost in a unit or not in a string of length 2 or more). Each string of children can be arranged in only one way, because our set specifies who has to have the same child on the right. Thus $|\bigcap_{i:\ i\in S} A_i| = (n-s-1)!$. This gives us

$$\begin{aligned}
|\overline{\bigcup_{i=1}^{n} A_i}| &= \sum_{S:S\subseteq P} (-1)^{|S|} (n-s-1)! \\
&= \sum_{s=0}^{n} (-1)^s \binom{n}{s} (n-s-1)! \\
&= \sum_{s=0}^{n} (-1)^s \frac{n!}{s!(n-s)}
\end{aligned}$$

ways for the children to join hands the second time so that none of them has the same child to the right. ▮

➔ $*6$. Suppose that n people link arms in a folk-dance and dance in a circle. Later on they let go and dance some more, after which they link arms in a circle again. In how many ways can they link arms the second time so that no one links with a person with whom he or she linked arms before?

Solution: We use the principle of inclusion and exclusion. The set A will be the set of arrangements of people in a circle where two arrangements are the same if we get one from the other by rotating or flipping the second. Set A_i will be the set of arrangements in which person i links arms with someone previously to his or her immediate right. (Saying it is the person to the right gives us more control over our formulas.) Given a subset S of $[n]$, the number of ways for the people determined by that set to link arms with the people previously on their right is the number of ways to arrange $n-|S|$ strings of people around a circle with strings of length more than 1 having two ways to arrange themselves. (Once we have two or more people linked, another person can be added to this string only at one end or not at all, because this person must have been to the right of one of the people on an end of the string. However, a string of length two or more can unlink and then link in the opposite order, and each person will still be linked to exactly the same people.) Thus $|\bigcap_{i:\ i\in S} A_i| = (n-|S|-1)!2^{m(S)}$, where $m(S)$ is the number of strings of length more than one determined by S. The number $m(S)$ can be any number from 1 to S, so long as S is not too big; namely so long as $|S| \leq \lfloor n/2 \rfloor$. (This is because if $m(S) = |S|$, then each person determined by an integer in S must be adjacent to a person not determined by an integer in S.) In particular, $|\bigcap_{i:\ i\in S} A_i|$ is not completely determined by the size of S, as in all our other inclusion-exclusion problems. How do we compute $m(S)$? Let us call a subset R of S a run if

(a) the people determined by R are linked together in a row in both linkings, and

(b) no other person in S is in a row with these people in both linkings.

Some runs might determine just one person, but a run could also equal all of S. Each run will have one more person not in S who was originally to the right of the person in the run who was rightmost in the first linking, and so this person will have to sit in a row with the people in R in the second linking as well. Thus the number r of runs in S is the number of strings $m(S)$ that may be linked in two ways, and there are $n - |S| - r$ people who do not have to be linked with runs. Thus $|\bigcap_{i:\ i\in S} A_i| = (n - |S| - 1)!2^r$, because the total number of strings of people (including strings of just one person) we need to arrange is $n - |S| - r$, and there are $(k-1)!$ ways to arrange k objects in a circle. If we try to use the information we have so far to compute $|\overline{\bigcup_{i=1}^{n} A_i}|$, we get

$$|\overline{\bigcup_{i=1}^{n} A_i}| = \sum_{S:S\subseteq P} (-1)^{|S|}(n - |S|_1)!2^r = \sum_{s=0}^{n}\sum_{r=1}^{|S|} N(s,r)(n-s-1)!2^r,$$

in which $N(s,r)$ stands for the number of sets S with size s and r runs.

Picking out runs in a circular arrangement adds a layer of difficulty, so to compute $N(s,r)$, we first compute how many subsets of $[n]$ we have with r runs and then adjust for putting 1 through n around a circle in order. Imagine writing 1 through n in a straight line, each integer occupying one unit of distance along the line. We now place r sticks whose lengths add to s (each stick has positive integer length) along that line. Each stick picks out a set of consecutive integers, as many as its length, so the sticks together pick out s integers. In order to be sure the sticks correspond to runs, we need to make sure they do not touch each other, so we place $n-s$ identical stones along the line too, making sure there is at least one stone between any two sticks. The stones thus pick out the integers not in S. The sticks are not quite identical, though the sticks of a given length are. In other words, which lengths of sticks are in which places is what matters. So the sticks give us a composition of s, a list of distinct positive integers that add to s. We know there are $\binom{s-1}{r-1}$ such compositions. Once we have chosen an ordering for the sticks, we need to distribute the stones among the sticks so that no two sticks are adjacent. Since the stones are identical, we can do this by putting one stone between each pair of sticks in our composition, and then distribute the remaining $n-s-r+1$ stones in any way we want among the $r-1$ places between the sticks and the two places to the left and right of all the sticks.. We can do this in

$$\binom{r+1+(n-s-r+1)-1}{n-s-r+1} = \binom{n-s+1}{n-s-r+1} = \binom{n-s+1}{r}$$

ways. Thus there are $\binom{s-1}{r-1}\binom{n-s+1}{r}$ ways to choose a subset S of $[n]$ that has r runs.

Now we have to deal with the fact that our n people (who we have replaced with the integers 1 through n in order) were arranged around a circle. That means that a run is now a set of consecutive integers on the circle, where n and 1 are considered consecutive. Recall that the set S is picked out by the sticks. If we arrange 1 through n around a circle in order, the set S that originally had r runs will have $r-1$ runs if sticks covered both the first and last integer (1 and n), but otherwise it will still have n runs. Thus the number of subsets of $[n]$ that have n runs when 1 through n are arranged in a circle is the number of subsets of $[n]$ with $r+1$ runs that have both 1 and n in S plus the number of subsets of $[n]$ with r runs that do not have both 1 and n in S. To compute the number of subsets S that *do* contain both 1 and n, we compute the number of arrangements of r sticks and $n-s$ stones that do start and end with a stick; that means that after we choose our composition into r parts to get our arrangement of sticks and place one stone between each pair of previously adjacent sticks, we now place the remaining $n-s-r+1$ stones in the $r-1$ places between previously adjacent sticks in

$$\binom{r-1+(n-s-r+1)-1}{n-s-r+1} = \binom{n-s-1}{n-s-r+1} = \binom{n-s-1}{r-2}$$

ways. For the sticks and stones to determine a subset we must assign lengths to the sticks; the number of ways to do this is, as above, $\binom{s-1}{r-1}$, the number of compositions of s with r parts. Thus there are $\binom{s-1}{r-1}\binom{n-s-1}{r-2}$ subsets of $[n]$ that have r runs and include both 1 and n. For our computation we will also want the number of subsets of $[n]$ that have $r+1$ runs and contain both 1 and n; this is $\binom{s-1}{r}\binom{n-s-1}{r-1}$.

On the other hand, the number of subsets of $[n]$ that have r runs and do not contain both 1 and n is the total number of subsets with r runs minus the number that do contain both 1 and n; this is

$$\binom{s-1}{r-1}\left(\binom{n-s+1}{r} - \binom{n-s-1}{r-2}\right).$$

This gives us

$$N(s,r) = \binom{s-1}{r}\binom{n-s-1}{r-1} + \binom{s-1}{r-1}\left(\binom{n-s+1}{r} - \binom{n-s-1}{r-2}\right)$$

ways to choose an s-element subset of $[n]$ that has r runs when $[n]$ is arranged around a circle. Thus there are

$$\sum_{s=0}^{n}\sum_{r=1}^{s}(-1)^s\left[\binom{s-1}{r}\binom{n-s-1}{r-1}+\binom{s-1}{r-1}\left(\binom{n-s+1}{r}-\binom{n-s-1}{r-2}\right)\right](n-s-1)!2^r$$

ways for people to arrange themselves in the second circle so that no one is adjacent to anyone he or she was previously adjacent to. ∎

➔ $*$7. (A challenge; the author has not tried to solve this one!) Redo Problem 6 in the case that there are n men and n women and when people arrange themselves in a circle they do so alternating gender.

➔ 8. Suppose we take two graphs G_1 and G_2 with disjoint vertex sets, we choose one vertex on each graph, and connect these two vertices by an edge e to get a graph G_{12}. How does the chromatic polynomial of G_{12} relate to those of G_1 and G_2?

Solution: By the deletion-contraction recurrence,

$$\chi_{G_{12}}(x) = \chi_{G_{12}-e}(x) - \chi_{G_{12}/e}(x).$$

Now $\chi_{G_{12}-e}(x) = \chi_{G_1}(x)\cdot\chi_{G_2}(x)$ because each ordered pair of proper colorings of G_1 and G_2 is a proper coloring of $G_{12}-e$. G_{12}/e is the graph we get by identifying the endpoint of e in G_1 with the endpoint of e in G_2. Notice that if you fix one vertex of a graph G, fix one color, and ask how many proper colorings with x colors G has in which the fixed vertex is the fixed color, you get $\chi_G(x)/x$ (by the quotient principle). Thus $\chi_{G_2}(x)/x$ is the number of ways to extend a proper coloring of G_1 to a proper coloring of G_{12}/e. Then, by the product principle, the number of proper colorings of G_{12}/e with x colors is $\chi_{G_1}(x)\chi_{G_2}(x)/x$. Therefore by the deletion-contraction recurrence, $\chi_{G_{12}}(x) = \chi_{G_1}(x)\chi_{G_2}(x)(1-\frac{1}{x})$. ∎

Chapter 6

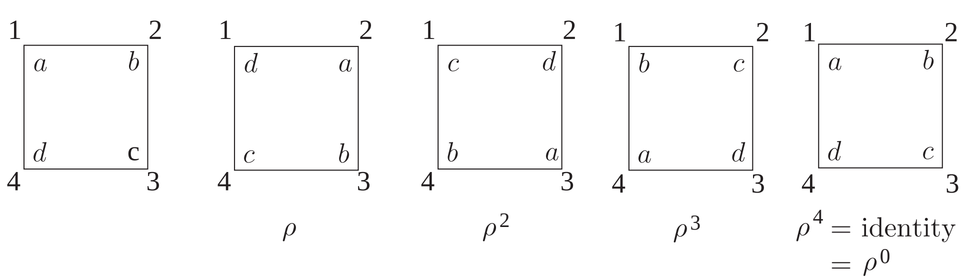
Groups Acting on Sets

6.1 Permutation Groups

Until now we have thought of permutations mostly as ways of listing the elements of a set. In this chapter we will find it very useful to think of permutations as functions. This will help us in using permutations to solve enumeration problems that cannot be solved by the quotient principle because they involve counting the blocks of a partition in which the blocks don't have the same size. We begin by studying the kinds of permutations that arise in situations where we have used the quotient principle in the past.

6.1.1 The rotations of a square

Figure 6.1: The four possible results of rotating a square but maintaining its location.



In Figure 6.1 we show a square with its four vertices labeled a, b, c, and d. We have also labeled the spots in the plane where each of these vertices falls with the label 1, 2, 3, or 4. Then we have shown the effect of rotating the square clockwise through 90, 180, 270, and 360 degrees (which is the same as rotating through 0 degrees). Underneath each of the rotated squares we have named the function that carries out the rotation. We use ρ, the Greek letter pronounced "row," to stand for a 90 degree clockwise rotation. We use ρ^2 to stand for two 90 degree rotations, and so on. We can think of the function ρ as a function on the four element set[1] $\{1, 2, 3, 4\}$. In particular, for any function φ (the Greek letter phi, usually pronounced "fee," but sometimes "fie") from the plane back to itself that may move the square around but otherwise leaves it in the same location, we let $\varphi(i)$ be the label of the place where vertex previously in position i is now. Thus $\rho(1) = 2$, $\rho(2) = 3$, $\rho(3) = 4$ and $\rho(4) = 1$. Notice that ρ is a permutation on the set $\{1, 2, 3, 4\}$.

$\circ$248. The composition $f \circ g$ of two functions f and g is defined by $f \circ g(x) = f(g(x))$. Is ρ^3 the composition of ρ and ρ^2? Does the answer depend on the order in which we write ρ and ρ^2? How is ρ^2 related to ρ?

Solution: Yes, ρ^3 is the composition of ρ and ρ^2, and also of ρ^2 and ρ, so it doesn't matter in which order we write them. $\rho^2 = \rho \circ \rho$. ▮

$\circ$249. Is the composition of two permutations always a permutation?

Solution: Yes, because the composition of one-to-one functions is one-to-one and the composition of onto functions is onto. ▮

In Problem 248 you see that we can think of $\rho^2 \circ \rho$ as the result of first rotating by 90 degrees and then by another 180 degrees. In other words, the composition of two rotations is the same thing as first doing one and then doing the other. Of course there is nothing special about 90 degrees and 180 degrees. As long as we first do one rotation through a multiple of 90 degrees and then another rotation through a multiple of 90 degrees, the composition of these rotations is a rotation through a multiple of 90 degrees.

If we first rotate by 90 degrees and then by 270 degrees then we have rotated by 360 degrees, which does nothing visible to the square. Thus we say that ρ^4 is the "identity function." In general the **identity function** on a set S, denoted by ι (the Greek letter iota, pronounced eye-oh-ta) is the function that takes each element of the set to itself. In symbols, $\iota(x) = x$

[1] What we are doing is restricting the rotation ρ to the set $\{1, 2, 3, 4\}$.

for every x in S. Of course the identity function on a set is a permutation of that set.

6.1.2 Groups of permutations

◦250. For any function φ from a set S to itself, we define φ^n (for nonnegative integers n) inductively by $\varphi^0 = \iota$ and $\varphi^n = \varphi^{n-1} \circ \varphi$ for every positive integer n. If φ is a permutation, is φ^n a permutation? Based on your experience with previous inductive proofs, what do you expect $\varphi^n \circ \varphi^m$ to be? What do you expect $(\varphi^m)^n$ to be? There is no need to prove these last two answers are correct, for you have, in effect, already done so in Chapter 2.

Solution: It is a permutation because the composition of permutations is a permutation. (You could be more precise and use the inductive definition of φ^n to prove inductively that φ^n is a permutation.) We expect $\varphi^m \circ \varphi^n = \varphi^{m+n}$, and we expect $(\varphi^m)n = \varphi^{mn}$, and we would prove this by induction. ▮

◦251. If we perform the composition $\iota \circ \varphi$ for any function φ from S to S, what function do we get? What if we perform the composition $\varphi \circ \iota$?

Solution: $\iota \circ \varphi = \varphi \circ \iota = \varphi$ ▮

What you have observed about iota in Problem 251 is called the *identity property* of iota. In the context of permutations, people usually call the function ι "the identity" rather than calling it "iota."

Since rotating first by 90 degrees and then by 270 degrees has the same effect as doing nothing, we can think of the 270 degree rotation as undoing what the 90 degree rotation does. For this reason we say that in the rotations of the square, ρ^3 is the "inverse" of ρ. In general, a function $\varphi : T \to S$ is called an **inverse** of a function $\sigma : S \to T$ (σ is the lower case Greek letter sigma) if $\varphi \circ \sigma = \sigma \circ \varphi = \iota$. For a slower introduction to inverses and practice with them, see Section A.1.3 in Appendix A. Since a permutation is a bijection, it has a unique inverse, as in Section A.1.3 of Appendix A. And since the inverse of a bijection is a bijection (again, as in the Appendix), the inverse of a permutation is a permutation.

We use φ^{-1} to denote the inverse of the permutation φ. We've seen that the rotations of the square are functions that return the square to its original location but may move the vertices to different places. In this way we create permutations of the vertices of the square. We've observed three important properties of these permutations.

- (Identity Property) These permutations include the identity permutation.
- (Inverse Property) Whenever these permutations include φ, they also include φ^{-1}.
- (Closure Property) Whenever these permutations include φ and σ, they also include $\varphi \circ \sigma$.

A set of permutations with these three properties is called a **permutation group**[2] or a group of permutations. We call the group of permutations corresponding to rotations of the square the *rotation group* of the square. There is a similar rotation group with n elements for any regular n-gon.

252. If $f : S \to T$, $g : T \to X$, and $h : X \to Y$, is

$$h \circ (g \circ f) = (h \circ g) \circ f?$$

What does this say about the status of the *associative law*

$$\rho \circ (\sigma \circ \varphi) = (\rho \circ \sigma) \circ \varphi$$

in a group of permutations?

Solution: Since $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$ and $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$, we have that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

This says that the associative law holds for the composition operation in a group of permutations. ▮

253. •(a) How should we define φ^{-n} for an element φ of a permutation group?

Solution: We define $\varphi^{-n} = \left(\varphi^{-1}\right)^n$. ▮

•(b) Will the two standard rules for exponents

$$a^m a^n = a^{m+n} \text{ and } (a^m)^n = a^{mn}$$

still hold in a group if one or more of the exponents may be negative? (No proof required yet.)

Solution: Yes. ▮

[2] The concept of a permutation group is a special case of the concept of a *group* that one studies in abstract algebra. When we refer to a group in what follows, if you know what groups are in the more abstract sense, you may use the word in this way. If you do not know about groups in this more abstract sense, then you may assume we mean permutation group when we say group.

(c) Proving that $(\varphi^{-m})^n = \varphi^{-mn}$ when m and n are nonnegative is different from proving that $(\varphi^m)^{-n} = \varphi^{-mn}$ when m and n are nonnegative. Make a list of all such formulas we would need to prove in order to prove that the rules of exponents of Part 253b do hold for all nonnegative and negative m and n.

Solution: To prove this we need to show that for nonnegative m and n we have that $\varphi^m \circ \varphi^{-n} = \varphi^{m-n}$, that $\varphi^{-m} \circ \varphi^n = \varphi^{n-m}$, that $\varphi^{-m}\varphi^{-n} = \varphi^{-m-n}$, that $(\varphi^m)^{-n} = \varphi^{-mn}$, that $(\varphi^{-m})^n = \varphi^{-mn}$ and that $(\varphi^{-m})^{-n} = \varphi^{mn}$. ▮

(d) If the rules hold, give enough of the proof to show that you know how to do it; otherwise give a counterexample.

Solution: We prove the first and last formula. We will induct on n in the first proof. If $n = 0$ the formula automatically holds. Assume inductively that $\varphi^m \circ \varphi^{-(n-1)} = \varphi^{m-(n-1)}$. Compose both sides of this equation on the right by σ^{-1} and use the associative law for composition of functions to get, in the case that $m - n$ is nonnegative

$$\begin{aligned}
\varphi^m \circ (\varphi^{-(n-1)} \circ \varphi^{-1}) &= \varphi^{m-(n-1)} \circ \varphi^{-1} \\
\varphi^m \circ (\varphi^{-1})^{n-1}\varphi^{-1} &= (\varphi^{m-n} \circ \varphi) \circ \varphi^{-1} \\
\varphi^m \circ (\varphi^{-1})^n &= \varphi^{m-n} \\
\varphi^m\varphi^{-n} &= \varphi^{m-n}.
\end{aligned}$$

We do the same in the case that $m - n$ is negative, getting

$$\begin{aligned}
\varphi^m \circ (\varphi^{-(n-1)} \circ \varphi^{-1}) &= \varphi^{m-(n-1)} \circ \varphi^{-1} \\
\varphi^m \circ (\varphi^{-1})^{n-1}\varphi^{-1} &= (\varphi^{-1})^{n-m-1} \circ \varphi^{-1} \\
\varphi^m \circ (\varphi^{-1})^n &= (\varphi^{-1})^{n-m} \\
\varphi^m \circ \varphi^{-n} &= \varphi^{m-n}.
\end{aligned}$$

To prove the last statement we need to prove, we write the following, in which the line marked $(*)$ follows from $(\varphi^m)^{-n} = \varphi^{-mn}$ and the line marked $(**)$ follows from the fact that $(\varphi^{-1})^{-1} = \varphi$.

$$\begin{aligned}
(\varphi^{-m})^{-n} &= ([\varphi^{-1})^m]^{-1})^n \\
&= [(\varphi^{-1})^{-m}]^n \qquad (*) \\
&= ([(\varphi^{-1})^{-1}]^m)^n \\
&= (\varphi^m)^n \qquad (**) \\
&= \varphi^{mn}
\end{aligned}$$

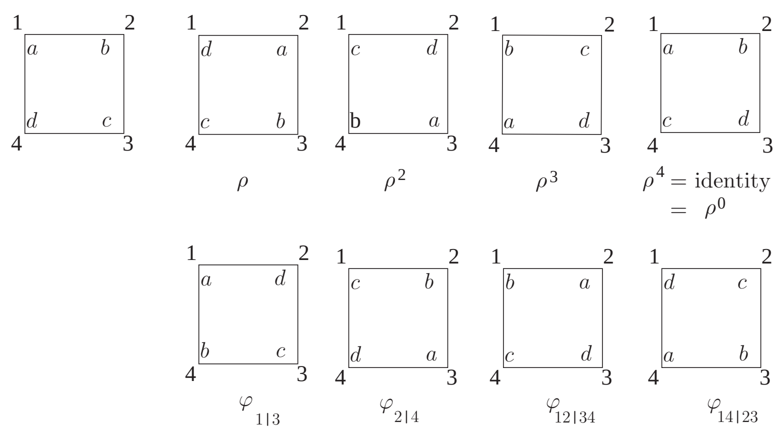
This completes our proof. ▮

•254. If a finite set of permutations satisfies the closure property is it a permutation group?

Solution: Yes, because the permutations σ, σ^2, $\ldots$, σ^n, $\ldots$ must eventually start repeating. But if $\sigma^i = \sigma^j$, and $i < j$, then $\sigma^{j-i} = \iota$, so σ^{j-i-1} is the inverse of σ. This means that every member of the set of permutations has an inverse in that set. But then since $\sigma \circ \sigma^{-1} = \iota$, we have that the identity is in the set as well. Thus it is a permutation group. ▮

•255. There are three-dimensional geometric motions of the square that return it to its original location but move some of the vertices to other positions. For example, if we flip the square around a diagonal, most of it moves out of the plane during the flip, but the square ends up in the same location. Draw a figure like Figure 6.1 that shows all the possible results of such motions, including the ones shown in Figure 6.1. Do the corresponding permutations form a group?

Solution:



The set of all permutations corresponding to motions forms a group because it is closed under composition. ▮

➔256. Let σ and φ be permutations.

(a) Why must $\sigma \circ \varphi$ have an inverse?

Solution: One could either say that the composition of one-to-one and onto functions is one-to-one and onto, or note that

$(\sigma \circ \varphi) \circ (\varphi^{-1} \circ \sigma^{-1}) = \iota$ by the associative law. Note that this says that $(\sigma \circ \varphi)^{-1} = \varphi^{-1} \circ \sigma^{-1}$. ▮

(b) Is $(\sigma \circ \varphi)^{-1} = \sigma^{-1}\varphi^{-1}$? (Prove or give a counter-example.)

Solution: The set of all permutations of $\{1, 2, 3\}$ is a permutation group. Let σ be given by $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. Let φ be given by $\varphi(1) = 1$, $\varphi(2) = 3$, and $\varphi(3) = 2$. Then $\sigma \circ \varphi(1) = 2$, $\sigma \circ \varphi(2) = 1$, and $\sigma \circ \varphi(3) = 3$. But $\sigma^{-1} \circ \varphi^{-1}(2) = \sigma^{-1}(3) = 2$. Thus $(\sigma \circ \varphi)^{-1} \neq \sigma^{-1}\varphi^{-1}$. ▮

(c) Is $(\sigma \circ \varphi)^{-1} = \varphi^{-1}\sigma^{-1}$? (Prove or give a counter-example.)

Solution: See the second solution for Part (a). ▮

∘257. Explain why the set of all permutations of four elements is a permutation group. How many elements does this group have? This group is called the *symmetric group on four letters* and is denoted by S_4.

Solution: It is a finite set of permutations that satisfies the closure property. It has $4! = 24$ elements. ▮

6.1.3 The symmetric group

In general, the set of all permutations of an n-element set is a group. It is called the **symmetric group on n letters**. We don't have nice geometric descriptions (like rotations) for all its elements, and it would be inconvenient to have to write down something like "Let $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 4$, and $\sigma(4) = 1$" each time we need to introduce a new permutation. We introduce a new notation for permutations that allows us to denote them *reasonably* compactly and compose them reasonably quickly. If σ is the permutation of $\{1, 2, 3, 4\}$ given by $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 4$ and $\sigma(4) = 2$, we write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

We call this notation the *two row notation* for permutations. In the two row notation for a permutation of $\{a_1, a_2, \ldots, a_n\}$, we write the numbers a_1 through a_n in one row and we write $\sigma(a_1)$ through $\sigma(a_n)$ in a row right below, enclosing both rows in parentheses. Notice that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix},$$

although the second ordering of the columns is rarely used.

If φ is given by

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$

then, by applying the definition of composition of functions, we may compute $\sigma \circ \varphi$ as shown in Figure 6.2.

Figure 6.2: How to multiply permutations in two row notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

We don't normally put the circle between two permutations in two row notation when we are composing them, and refer to the operation as multiplying the permutations, or as the product of the permutations. To see how Figure 6.2 illustrates composition, notice that the arrow starting at 1 in φ goes to 4. Then from the 4 in φ it goes to the 4 in σ and then to 2. This illustrates that $\varphi(1) = 4$ and $\sigma(4) = 2$, so that $\sigma(\varphi(1)) = 2$.

258. For practice, compute $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$.

Solution: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ ▮

6.1.4 The dihedral group

We found four permutations that correspond to rotations of the square. In Problem 255 you found four permutations that correspond to flips of the square in space. One flip fixes the vertices in the places labeled 1 and 3 and interchanges the vertices in the places labeled 2 and 4. Let us denote it by $\varphi_{1|3}$. One flip fixes the vertices in the positions labeled 2 and 4 and interchanges those in the positions labeled 1 and 3. Let us denote it by $\varphi_{2|4}$. One flip interchanges the vertices in the places labeled 1 and 2 and also interchanges those in the places labeled 3 and 4. Let us denote it by $\varphi_{12|34}$. The fourth flip interchanges the vertices in the places labeled 1 and 4 and interchanges those in the places labeled 2 and 3. Let us denote it by $\varphi_{14|23}$. Notice that $\varphi_{1|3}$ is a permutation that takes the vertex in place 1 to the vertex in place 1 and the vertex in place 3 to the vertex in place 3, while $\varphi_{12|34}$ is a permutation that takes the edge between places 1 and 2 to the edge between places 2 and 1 (which is the same edge) and takes the edge between places 3 and 4 to the edge between places 4 and 3 (which is the same edge). This should help to explain the similarity in the notation for the two different kinds of flips.

•259. Write down the two row notation for ρ^3, $\varphi_{2|4}$, $\varphi_{12|34}$ and $\varphi_{2|4} \circ \varphi_{12|34}$.

Solution:
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$. ▮

260. (You may have already done this problem in Problem 255, in which case you need not do it again!) In Problem 255, if a rigid motion in three-dimensional space returns the square to its original location, in how many places can vertex number one land? Once the location of vertex number one is decided, how many possible locations are there for vertex two? Once the locations of vertex one and vertex two are decided, how many locations are there for vertex three? Answer the same question for vertex four. What does this say about the relationship between the four rotations and four flips described just before Problem 259 and the permutations you described in Problem 255?

Solution: Vertex number one can go in four places. After the place for vertex 1 is chosen, there are two choices for vertex 2. After vertices 1 and 2 are placed, one side of the square has been placed, and there is only one way the rest of the square can fill in the same location where the square used to be, so there is only one choice for where vertex 3 goes and one choice for where vertex 4 goes. Thus the permutations described in Problem 255 are exactly the rotations and flips. ▮

The four rotations and four flips of the square described before Problem 259 form a group called the dihedral group of the square. Sometimes the group is denoted D_8 because it has eight elements, and sometimes the group is denoted by D_4 because it deals with four vertices! Let us agree to use the notation D_4 for the dihedral group of the square. There is a similar **dihedral group**, denoted by D_n, of all the rigid motions of three-dimensional space that return a regular n-gon to its original location (but might put the vertices in different places).

261. Another view of the dihedral group of the square is that it is the group of all distance preserving functions, also called *isometries*, from

a square to itself. Notice that an isometry must be a bijection. Any rigid motion of the square preserves the distances between all points of the square. However, it is conceivable that there might be some isometries that do not arise from rigid motions. (We will see some later on in the case of a cube.) Show that there are exactly eight isometries (distance preserving functions) from a square to itself.

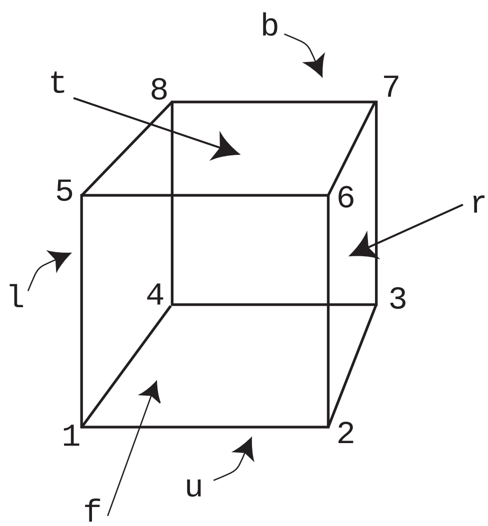
Solution: We can use three of the vertices of a square to set up a rectangular coordinate system, and every point is determined by its signed distances from the two coordinate lines. But the distance of a point from a line is simply the height of a triangle given by that point and two points on the line. All unsigned distances of points in the square from the coordinate lines are determined by the distances of points from the three points we use to construct our coordinate system. We can set up our coordinate system so all distances of points in the square from the coordinate lines are positive. Therefore each point in the square is determined by its distances from the three points we have chosen. But a distance preserving map must take corners of the square to corners of the square (it has to preserve diagonal distances). Therefore any isometry is determined by what it does to the corners of the square. But two adjacent corners must go to two adjacent corners for distances to be preserved and so a corner and the two edges adjacent to it must be mapped to a corner and the two edges adjacent to it. There are four choices for the corner that we map to and two choices of which edges go to which once the corner is chosen. Thus there are 8 choices for an isometry of the square. Each element of the dihedral group of the square is a permutation of the corners determined by a rigid motion, and thus an isometry. Thus the dihedral group can also be thought of as the permutations of the vertices (corners) of a square induced by isometries. ▮

➔ 262. How many elements does the group D_n have? Prove that you are correct.

Solution: D_n has $2n$ elements, because once you have chosen one of the n places for vertex one, there are two choices for vertex two, and the remaining vertices can go in only one place each. ▮

• 263. In Figure 6.3 we show a cube with the positions of its vertices and faces labeled. As with motions of the square, we let we let $\varphi(x)$ be the label of the place where vertex previously in position x is now.

Figure 6.3: A cube with the positions of its vertices and faces labeled. The curved arrows point to the faces that are blocked by the cube.



(a) Write in two row notation the permutation ρ of the vertices that corresponds to rotating the cube 90 degrees around a vertical axis through the faces t (for top) and u (for underneath). (Rotate in a right-handed fashion around this axis, meaning that vertex 6 goes to the back and vertex 8 comes to the front.)

Solution: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \end{pmatrix}$ ∎

(b) Write in two row notation the permutation φ that rotates the cube 120 degrees around the diagonal from vertex 1 to vertex 7 and carries vertex 8 to vertex 6.

Solution: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 8 & 5 & 2 & 3 & 7 & 6 \end{pmatrix}$ ∎

(c) Compute the two row notation for $\rho \circ \varphi$.

Solution:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 8 & 5 & 2 & 3 & 7 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 5 & 6 & 3 & 4 & 8 & 7 \end{pmatrix}.$$

∎

(d) Is the permutation $\rho \circ \varphi$ a rotation of the cube around some axis? If so, say what the axis is and how many degrees we rotate around the axis. If $\rho \circ \varphi$ is not a rotation, give a geometric description of it.

Solution: It is a rotation of 180 degrees around the axis between the center of the edge 1,2 and the center of the edge 7,8. ▮

➔ $\cdot$264. How many permutations are in the group R? R is sometimes called the "rotation group" of the cube. Can you justify this?

Solution: There are eight places where vertex one can go. Once vertex 1 is placed, there are three ways the three faces including vertex 1 can be placed so that the cube returns to its original location, and then the location of all the vertices is determined. Thus there are 24 elements of the group. Of them, we have the identity, three rotations about each of the three axes through the midpoints of the faces, one additional rotation about each of the six axes joining opposite faces, and $2 \cdot 4 = 8$ additional rotations about each of the four axes joining a vertex to the diagonally (in three dimensions) opposite vertex. This gives us a total of 10+6+8=24 rotations, so every element of the group is a rotation. ▮

➔ 265. As with a two-dimensional figure, it is possible to talk about isometries of a three-dimensional figure. These are distance preserving functions from the figure to itself. The function that reflects the cube in Figure 6.3 through a plane halfway between the bottom face and top face exchanges the vertices 1 and 5, 2 and 6, 3 and 7, and 4 and 8 of the cube. This function preserves distances between points in the cube. However, it cannot be achieved by a rigid motion of the cube because a rigid motion that takes vertex 1 to vertex 5, vertex 2 to vertex 6, vertex 3 to vertex 7, and vertex 4 to vertex 8 would not return the cube to its original location; rather it would put the bottom of the cube where its top previously was and would put the rest of the cube above that square rather than below it.

(a) How many elements are there in the group of permutations of [8] that correspond to isometries of the cube?

Solution: There are 48, because there are 8 places for vertex one to go, and once it is placed, there are six ways to place the three vertices adjacent to vertex 1, and each of them corresponds to an isometry. ▮

(b) Is every permutation of [8] that corresponds to an isometry either a rotation or a reflection?

Solution: It suffices by symmetry to show that half the permutations that take vertex 1 to itself are rotations and half are reflections. There are six permutations corresponding to isometries that take vertex 1 to vertex 1. We already know that three of them are rotations of the cube around an axis through vertex 1 and vertex 7. One of them reflects through the plane bisecting and perpendicular to the edges $1, 5$ and $3, 7$. One of them reflects through the plane bisecting and perpendicular to the edges $1, 4$ and $6, 7$. One of them reflects through the plane bisecting and perpendicular to the edges $1, 2$ and $7, 8$. Thus half of the isometries that take vertex 1 to vertex 1 are reflections, and by symmetry, half of the isometries (or in other words, 24 of them) are reflections. The other half (again, 24 isometries) are rotations. Thus every isometry is either a rotation or a reflection. ∎

6.1.5 Group tables (Optional)

We can always figure out the composition of two permutations of the same set by using the definition of composition, but if we are going to work with a given permutation group again and again, it is worth making the computations once and recording them in a table. For example, the group of rotations of the square may be represented as in Table 6.1. We list the elements of our group, with the identity first, across the top of the table and down the left side of the table, using the same order both times. Then in the row labeled by the group element σ and the column labeled by the group element φ, we write the composition $\sigma \circ \varphi$, expressed in terms of the elements we have listed on the top and on the left side. Since a group of permutations is closed under composition, the result $\sigma \circ \varphi$ will always be expressible as one of these elements.

266. In Table 6.1, all the entries in a row (not including the first entry, the one to the left of the line) are different. Will this be true in any group table for a permutation group? Why or why not? Also in Table 6.1 all the entries in a column (not including the first entry, the one above the line) are different. Will this be true in any group table for a permutation group? Why or why not?

Solution: It will always be the case that all the entries of a row or column below and to the left of the lines in a group table are different.

Table 6.1: The group table for the rotations of a square.

$\circ$	ι	ρ	ρ^2	ρ^3
ι	ι	ρ	ρ^2	ρ^3
ρ	ρ	ρ^2	ρ^3	ι
ρ^2	ρ^2	ρ^3	ι	ρ
ρ^3	ρ^3	ι	ρ	ρ^2

If two entries of the row labeled by σ were equal, that would mean $\sigma \circ \varphi_1 = \sigma \circ \varphi_2$ for two different elements φ_1 and φ_2. But if we multiply both sides of the equation by σ^{-1} we get $\sigma^{-1}(\sigma \circ \varphi_1) = \sigma^{-1}(\sigma \circ \varphi_2)$, and by using the associative law and the identity property, we get $\varphi_1 = \varphi_2$, a contradiction. The same sort of argument (with σ on the right) works for columns. ▮

267. In Table 6.1, every element of the group appears in every row (even if you don't include the first element, the one before the line). Will this be true in any group table for a permutation group? Why or why not? Also in Table 6.1 every element of the group appears in every column (even if you don't include the first entry, the one before the line). Will this be true in any group table for a permutation group? Why or why not?

Solution: Every element appears in each row and every element appears in each column (after and below the lines). This is because the number of entries in a row or column is the number of elements of the group. By the pigeonhole principle, if not all the elements appear in a row, then two are the same, so $\tau \circ \sigma_1 = \tau \circ \sigma_2$ for some σ_1, σ_2, and τ (τ is the Greek letter tau that rhymes with cow) in the group with $\sigma_1 \neq \sigma_2$. Composing by τ^{-1} gives us $\sigma_1 = \sigma_2$, a contradiction. The same kind of argument works on columns, though you now have $\sigma_1 \circ \tau = \sigma_2 \circ \tau$ for some σ_1, σ_2, and τ in the group with $\sigma_1 \neq \sigma_2$. ▮

·268. Write down the group table for the dihedral group D_4. Use the φ notation described earlier to denote the flips. (Hints: Part of the table has already been written down. Will you need to think hard to write down the last row? Will you need to think hard to write down the last column?)

Solution:

$\circ$	ι	ρ	ρ^2	ρ^3	$\varphi_{1\|3}$	$\varphi_{2\|4}$	$\varphi_{12\|34}$	$\varphi_{14\|23}$
ι	ι	ρ	ρ^2	ρ^3	$\varphi_{1\|3}$	$\varphi_{2\|4}$	$\varphi_{12\|34}$	$\varphi_{14\|23}$
ρ	ρ	ρ^2	ρ^3	ι	$\varphi_{12\|34}$	$\varphi_{23\|14}$	$\varphi_{2\|4}$	$\varphi_{1\|3}$
ρ^2	ρ^2	ρ^3	ι	ρ	$\varphi_{2\|4}$	$\varphi_{1\|3}$	$\varphi_{14\|23}$	$\varphi_{12\|34}$
ρ^3	ρ^3	ι	ρ	ρ^2	$\varphi_{23\|14}$	$\varphi_{12\|34}$	$\varphi_{1\|3}$	$\varphi_{2\|4}$
$\varphi_{1\|3}$	$\varphi_{1\|3}$	$\varphi_{14\|23}$	$\varphi_{2\|4}$	$\varphi_{12\|34}$	ι	ρ^2	ρ	ρ^3
$\varphi_{2\|4}$	$\varphi_{2\|4}$	$\varphi_{12\|34}$	$\varphi_{1\|3}$	$\varphi_{14\|23}$	ρ^2	ι	ρ^3	ρ
$\varphi_{12\|34}$	$\varphi_{13\|24}$	$\varphi_{1\|3}$	$\varphi_{14\|23}$	$\varphi_{2\|4}$	ρ	ρ^3	ι	ρ^2
$\varphi_{14\|23}$	$\varphi_{14\|23}$	$\varphi_{2\|4}$	$\varphi_{12\|34}$	$\varphi_{1\|3}$	ρ^3	ρ	ρ^2	ι

∎

You may notice that the associative law, the identity property, and the inverse property are three of the most important rules that we use in regrouping parentheses in algebraic expressions when solving equations. There is one property we have not yet mentioned, the *commutative law*, which would say that $\sigma \circ \varphi = \varphi \circ \sigma$. It is easy to see from the group table of the rotation group of a square that it satisfies the commutative law.

269. Does the commutative law hold in all permutation groups?

Solution: No. In the group D_3 or D_4, for example, a nontrivial rotation (other than ρ^2 in D_4) does not commute with a flip. ∎

6.1.6 Subgroups

We have seen that the dihedral group D_4 contains a copy of the group of rotations of the square. When one group G of permutations of a set S is a subset of another group G' of permutations of S, we say that G is a **subgroup** of G'.

∘270. Find all subgroups of the group D_4 and explain why your list is complete.

Solution: $\{\rho^2,\iota\}$, $\{\varphi_{1|3},\iota\}$, $\{\varphi_{2|4},\iota\}$, $\{\varphi_{12|34},\iota\}$, $\{\varphi_{14|23},\iota\}$, $\{\iota,\rho,\rho^2,\rho^3\}$, $\{\iota,\varphi_{1|3},\rho^2,\varphi_{2|4}\}$, $\{\iota,\varphi_{12|34},\rho^2,\varphi_{12|34}\}$, $\{\iota\}$, and all of D_4. Notice that once a subgroup has ρ or ρ^3 in it, it has all powers of ρ, and if it has all powers of ρ and just one flip, it has all the flips in it. But the product of two "different kinds" of flips is ρ or ρ^3, and this shows that our list of subgroups is complete. ∎

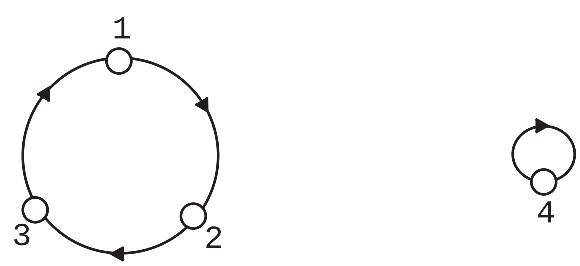
271. Can you find subgroups of the symmetric group S_4 with two elements? Three elements? Four elements? Six elements? (For each positive answer, describe a subgroup. For each negative answer, explain why not.)

Solution: Yes to all of the above. For two elements, $\{\iota, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}\}$, for three elements, $\{\iota, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}\}$ (which is the rotations of a triangle on $\{1, 2, 3\}$), for four elements, the rotations of a square, and for six elements, the rotations and flips of a triangle on $\{1, 2, 3\}$. (There is no subgroup with five elements, but at this point, that is quite hard to verify.) ▮

6.1.7 The cycle decomposition of a permutation

The digraph of a permutation gives us a nice way to think about it. Notice that the product in Figure 6.2 is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$. We have drawn the directed graph of this permutation in Figure 6.4. You see that the graph

Figure 6.4: The directed graph of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$.



has two directed cycles, the rather trivial one with vertex 4 pointing to itself, and the nontrivial one with vertex 1 pointing to vertex 2 pointing to vertex 3 which points back to vertex 1. A permutation is called a **cycle** if its digraph consists of exactly one cycle. Thus $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is a cycle but $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ is not a cycle by our definition. We write (1 2 3) or (2 3 1) or (3 1 2) to stand for the cycle $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

We can describe cycles in another way as well. A **cycle** of the permutation σ is a list $(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i))$ that does not have repeated elements while the list $(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i)\ \sigma^{n+1}(i))$ does have repeated elements.

272. If the list $(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i))$ does not have repeated elements but the list $(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i)\ \sigma^{n+1}(i))$ does have repeated elements, then what is $\sigma^{n+1}(i)$?

    Solution: Only one element can be repeated, and that element must be $\sigma^{n+1}(i)$. Further, we have that $\sigma^{n+1}(i)$ must be i, because if we had

    $$\sigma^j(i) = \sigma^{n+1}(i) \qquad (*)$$

    and $0 < j < n+1$, applying σ^{-j} to both sides of Equation $(*)$ would give us an earlier repeat. ▮

We say $\sigma^j(i)$ is an *element* of the cycle $(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i))$. Notice that the case $j = 0$ means i is an element of the cycle. Notice also that if $j > n$, $\sigma^j(i) = \sigma^{j-n-1}(i)$, so the distinct elements of the cycle are i, $\sigma(i)$, $\sigma^2(i)$, through $\sigma^n(i)$. We think of the cycle $(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i))$ as representing the permutation σ restricted to the set of elements of the cycle. We say that the cycles

$$(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i))$$

and

$$(\sigma^j(i)\ \sigma^{j+1}(i)\ \ldots\ \sigma^n(i)\ i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^{j-1}(i))$$

are *equivalent.* Equivalent cycles represent the same permutation on the set of elements of the cycle. For this reason, we consider equivalent cycles to be equal in the same way we consider $\frac{1}{2}$ and $\frac{2}{4}$ to be equal. In particular, this means that $(i_1\ i_2\ \ldots i_n) = (i_j\ \ i_{j+1}\ \ldots\ i_n\ \ i_1\ \ i_2\ \ldots\ i_{j-1})$.

•273. Find the cycles of the permutations ρ, $\varphi_{1|3}$ and $\varphi_{12|34}$ in the group D_4.

    Solution: The permutation ρ has one cycle, $(1\ 2\ 3\ 4)$, $\varphi_{1|3}$ has the cycles $(2\ 4)$ (1), and (3), $\varphi_{12|34}$ has two cycles, namely $(1\ 2)$ and $(3\ 4)$. ▮

◦274. Find the cycles of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 6 & 2 & 9 & 7 & 1 & 5 & 8 \end{pmatrix}$.

    Solution: $(1\quad 3\quad 6\quad 7)$, $(2\quad 4)$, and $(5\quad 9\quad 8)$. ▮

275. If two cycles of σ have an element in common, what can we say about them?

Solution: They are the same cycle, since if the cycles are

$$(i\ \sigma(i)\ \sigma^2(i)\ \ldots\ \sigma^n(i))$$

and

$$(j\ \sigma(j)\ \sigma^2(j)\ \ldots\ \sigma^m(j))$$

and $\sigma^p(i) = \sigma^q(j)$ is a common element, then, $j = \sigma^{p-q}(i)$. Thus if $p \geq q$, j is in the first cycle. Otherwise $\sigma^{p-q}(i) = \sigma^{n+1=p-q}(i) = j$, and so j is in the first cycle. Thus if we start the first cycle with $\sigma^k(i)$ rather than i, we will get an equivalent cycle. But we will get the second cycle, so the two cycles are equal. ∎

Problem 275 leads almost immediately to the following theorem.

Theorem 8 *For each permutation σ of a set S, there is a unique partition of S each of whose blocks is the set of elements of a cycle of σ.*

More informally, we may say that every permutation partitions its domain into disjoint cycles. We call the set of cycles of a permutation the *cycle decomposition* of the permutation. Since the cycles of a permutation σ tell us $\sigma(x)$ for every x in the domain of σ, the cycle decomposition of a permutation completely determines the permutation. Using our informal language, we can express this idea in the following corollary to Theorem 8.

Corollary 2 *Every partition of a set S into cycles determines a unique permutation of S.*

276. Prove Theorem 8.

Solution: Suppose the (inequivalent) cycles of σ are $\gamma_1, \gamma_2, \ldots, \gamma_k$. Then let B_j be the set of elements of γ_j. Since each element i is in the cycle $(i\ \sigma(i)\ \ldots\ )$, every element is in a set B_j. Since no two (inequivalent) cycles have an element in common, every element of S is in exactly one set B_j. Then the sets B_j are the blocks of a partition, and it is the only partition each of whose blocks is the set of elements of a cycle of σ. This completes the proof. ∎

In Problems 273 and 274 you found the cycle decompositions of typical elements of the group D_4 and of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 6 & 2 & 9 & 7 & 1 & 5 & 8 \end{pmatrix}.$$

The group of all rotations of the square is simply the set of the four powers of the cycle $\rho = (1\ 2\ 3\ 4)$. For this reason, it is called a *cyclic group*[3] and often denoted by C_4. Similarly, the rotation group of an n-gon is usually denoted by C_n.

➔ 277. Write a recurrence for the number $c(k, n)$ of permutations of $[k]$ that have exactly n cycles, including 1-cycles. Use it to write a table of $c(k, n)$ for k between 1 and 7 inclusive. Can you find a relationship between $c(k, n)$ and any of the other families of special numbers such as binomial coefficients, Stirling numbers, Lah numbers, etc. we have studied? If you find such a relationship, prove you are right.

Solution: The element k is either in a cycle by itself or it isn't. The number of permutations in which it is in a cycle by itself is $c(k-1, n-1)$. If it is in a cycle with something else, it can come after any of the $k-1$ elements, and each choice of which one it comes after gives a different permutation. (You can always shift a cycle around so that k doesn't come first.) Thus $c(k, n) = c(k-1, n-1) + (k-1)c(k-1, n)$. The table is:

k/n	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	2	3	1	0	0	0	0
4	0	6	11	6	1	0	0	0
5	0	24	50	35	10	1	0	0
6	0	120	274	225	85	15	1	0
7	0	720	1764	1624	735	175	21	1

We see that the table is identical with our earlier table of the Stirling numbers of the first kind, except that in that table there was an alternating pattern of minus signs, offset by one place in successive rows. Thus it must be the case that $c(k, n) = |s(k, n)|$. To prove this, note that it is the case when $k = 0$, and also when $n = 1$. Now assume inductively it is true when $k = m - 1$ and notice that if $n > 0$, $s(m, n) = s(m-1, n-1) - (m-1)s(m-1, n)$, that $s(m-1, n-1)$ and $s(m-1, n)$ have opposite signs, so

$$|s(m-1, n-1) - (m-1)s(m-1, n)|$$

[3]The phrase cyclic group applies in a more general (but similar) situation. Namely the set of all powers of any member of a group is called a cyclic group.

$$
\begin{aligned}
&= |s(m-1,n-1)| + (m-1)|s(m-1,n)| \\
&= c(m-1,n-1) + (m-1)c(m-1,n) \\
&= c(m,n).
\end{aligned}
$$

Thus by the principle of mathematical induction $c(k,n) = |s(k,n)|$ for all nonnegative numbers k (and all n between 0 and k). ▮

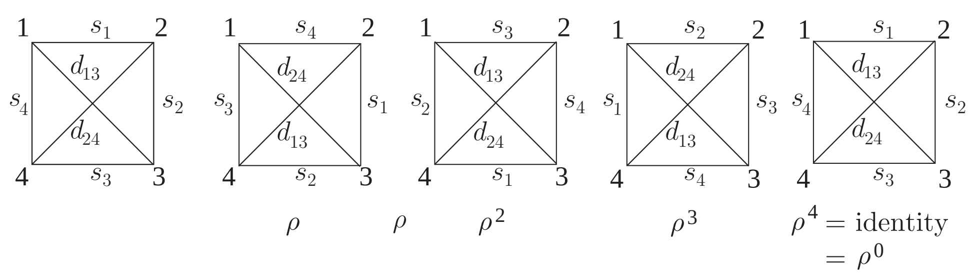
➔ · 278. (Relevant to Appendix C.) A permutation σ is called an **involution** if $\sigma^2 = \iota$. When you write down the cycle decomposition of an involution, what is special about the cycles?

Solution: They are all 2-cycles or 1-cycles. ▮

6.2 Groups Acting on Sets

We defined the rotation group C_4 and the dihedral group D_4 as groups of permutations of the vertices of a square. These permutations represent rigid motions of the square in the plane and in three-dimensional space respectively. The square has geometric features of interest other than its vertices; for example, its diagonals, or its edges. Any geometric motion of the square that returns it to its original location takes each diagonal to a possibly different diagonal, and takes each edge to a possibly different edge. In Figure 6.5 we show the results on the sides and diagonals of the rotations of a square. The rotation group permutes the sides of the square

Figure 6.5: The results on the sides and diagonals of rotating the square



and permutes the diagonals of the square as it rotates the square. Thus we say the rotation group "acts" on the sides and diagonals of the square.

∘279. (a) Write down the two-line notation for the permutation $\overline{\rho}$ that a 90 degree rotation does to the sides of the square.

Solution: $\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_3 & s_3 & s_4 & s_1 \end{pmatrix}$. ▮

(b) Write down the two-line notation for the permutation $\overline{\rho^2}$ that a 180 degree rotation does to the sides of the square.

Solution: $\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_3 & s_4 & s_1 & s_2 \end{pmatrix}$. ▮

(c) Is $\overline{\rho^2} = \overline{\rho} \circ \overline{\rho}$? Why or why not?

Solution: Yes, because

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_1 \end{pmatrix}\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_1 \end{pmatrix} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_3 & s_4 & s_1 & s_2 \end{pmatrix}.$$

▮

(d) Write down the two-line notation for the permutation $\widehat{\rho}$ that a 90 degree rotation does to the diagonals d_{13}, and d_{24} of the square.

Solution: $\begin{pmatrix} d_{13} & d_{24} \\ d_{24} & d_{13} \end{pmatrix}$. ▮

(e) Write down the two-line notation for the permutation $\widehat{\rho^2}$ that a 180 degree rotation does to the diagonals of the square.

Solution: $\begin{pmatrix} d_{13} & d_{24} \\ d_{13} & d_{24} \end{pmatrix}$. ▮

(f) Is $\widehat{\rho^2} = \widehat{\rho} \circ \widehat{\rho}$? Why or why not? What familiar permutation is $\widehat{\rho^2}$ in this case?

Solution: Yes, because

$$\begin{pmatrix} d_{13} & d_{24} \\ d_{24} & d_{13} \end{pmatrix}\begin{pmatrix} d_{13} & d_{24} \\ d_{24} & d_{13} \end{pmatrix} = \begin{pmatrix} d_{13} & d_{24} \\ d_{13} & d_{24} \end{pmatrix}.$$

Interestingly, $\widehat{\rho^2} = \iota$. ▮

We have seen that the fact that we have defined a permutation group as the permutations of some specific set doesn't preclude us from thinking of the elements of that group as permuting the elements of some other set as well. In order to keep track of which permutations of which set we are using to define our group and which other set is being permuted as well, we introduce some new language and notation. We are going to say that the group D_4 "acts" on the edges and diagonals of a square and the group R of permutations of the vertices of a cube that arise from rigid motions of the cube "acts" on the edges, faces, diagonals, etc. of the cube.

•280. In Figure 6.3 we show a cube with the positions of its vertices and faces labeled. As with motions of the square, we let we let $\varphi(x)$ be the label of the place where vertex previously in position x is now.

·(a) In Problem 263 we wrote in two row notation the permutation ρ of the vertices that corresponds to rotating the cube 90 degrees around a vertical axis through the faces t (for top) and u (for underneath). (We rotated in a right-handed fashion around this axis, meaning that vertex 6 goes to the back and vertex 8 comes to the front.) Write in two row notation the permutation $\overline{\rho}$ of the faces that corresponds to this member ρ of R.

Solution: $\begin{pmatrix} t & f & r & b & l & u \\ t & r & b & l & f & u \end{pmatrix}$ ∎

(b) In Problem 263 we wrote in two row notation the permutation φ that rotates the cube 120 degrees around the diagonal from vertex 1 to vertex 7 and carries vertex 8 to vertex 6. Write in two row notation the permutation $\overline{\varphi}$ of the faces that corresponds to this member of R.

Solution: $\begin{pmatrix} t & f & r & b & l & u \\ r & u & b & t & f & l \end{pmatrix}$ ∎

(c) In Problem 263 we computed the two row notation for $\rho \circ \varphi$. Now compute the two row notation for $\overline{\rho} \circ \overline{\varphi}$ ($\overline{\rho}$ was defined in Part 280a), and write in two row notation the permutation $\overline{\rho \circ \varphi}$ of the faces that corresponds to the action of the permutation $\rho \circ \varphi$ on the faces of the cube (for this question it helps to think geometrically about what motion of the cube is carried out by $\rho \circ \varphi$). What do you observe about $\overline{\rho \circ \varphi}$ and $\overline{\rho} \circ \overline{\varphi}$?

Solution:

$$\begin{pmatrix} t & f & r & b & l & u \\ t & r & b & l & f & u \end{pmatrix}\begin{pmatrix} t & f & r & b & l & u \\ r & u & b & t & f & l \end{pmatrix} = \begin{pmatrix} t & f & r & b & l & u \\ b & u & l & t & r & f \end{pmatrix}$$

$$\overline{\rho \circ \varphi} = \begin{pmatrix} t & f & r & b & l & u \\ b & u & l & t & r & f \end{pmatrix} = \overline{\rho} \circ \overline{\varphi}$$

∎

We say that a permutation group G **acts** on a set S if, for each member σ of G there is a permutation $\overline{\sigma}$ of S such that

$$\overline{\sigma \circ \varphi} = \overline{\sigma} \circ \overline{\varphi}$$

for every member σ and φ of G. In Problem 280c you saw one example of this condition. If we think intuitively of ρ and φ as motions in space, then following the action of φ by the action of ρ does give us the action of $\rho \circ \varphi$. We can also reason directly with the permutations in the group R of rigid motions (rotations) of the cube to show that R acts on the faces of the cube.

$\circ$281. Show that a group G of permutations of a set S acts on S with $\overline{\varphi} = \varphi$ for all φ in G.

Solution: Clearly $\overline{\varphi}$ is a permutation of S. Further, $\overline{\sigma} \circ \overline{\varphi} = \sigma \circ \varphi = \overline{\sigma \circ \varphi}$. ▮

$\circ$282. The group D_4 is a group of permutations of $\{1, 2, 3, 4\}$ as in Problem 255. We are going to show in this problem how this group acts on the two-element subsets of $\{1, 2, 3, 4\}$. In Problem 287 we will see a natural geometric interpretation of this action. In particular, for each two-element subset $\{i, j\}$ of $\{1, 2, 3, 4\}$ and each member σ of D_4 we define $\overline{\sigma}(\{i, j\}) = \{\sigma(i), \sigma(j)\}$. Show that with this definition of $\overline{\sigma}$, the group D_4 acts on the two-element subsets of $\{1, 2, 3, 4\}$.

Solution: The action has been defined for us, so all we need to show is that it is indeed an action. We must show that for each permutation σ in D_4, $\overline{\sigma}$ is a permutation of the two-element sets of $[4]$ and in addition we must show that $\overline{\sigma} \circ \overline{\tau} = \overline{\sigma \circ \tau}$. If $\overline{\sigma}(\{i, j\}) = \overline{\sigma}(\{h, k\})$, then either $\sigma(i) = \sigma(h)$ and $\sigma(j) = \sigma(k)$ or else $\sigma(i) = \sigma(k)$ and $\sigma(j) = \sigma(h)$. Since σ is a permutation, in the first case we get $i = h$ and $j = k$, so that $\{i, j\} = \{h, k\}$, and in the second case we get that $i = k$ and $j = h$ so that $\{i, j\} = \{k, h\} = \{h, k\}$. Thus in either case $\{i, j\} = \{h, k\}$, so that $\overline{\sigma}$ a one-to-one function from the finite set of two element subsets of $\{1, 2, 3, 4\}$ to itself, and so it is a permutation.

To show that $\overline{\sigma} \circ \overline{\tau} = \overline{\sigma \circ \tau}$, we note that

$$\begin{aligned}
(\overline{\sigma} \circ \overline{\tau})(\{i, j\}) &= \overline{\sigma}(\overline{\tau}(\{i, j\})) = \overline{\sigma}(\{\tau(i), \tau(j)\}) = \\
\sigma(\tau(i), \tau(j)) &= \{(\sigma \circ \tau)(i), (\sigma \circ \tau)(j)\} = \\
& \overline{\sigma \circ \tau}(\{i, j\})
\end{aligned}$$

▮

$\cdot$283. Suppose that σ and φ are permutations in the group R of rigid motions of the cube. We have argued already that each rigid motion sends a face to a face. Thus σ and φ both send the vertices on one face to the vertices on another face. Let $\{h, i, j, k\}$ be the set of labels next to the vertices on a face F.

(a) What are the labels next to the vertices of the face F' that F is sent to by φ? (The function φ may appear in your answer.)
Solution: $\varphi(h)$, $\varphi(i)$, $\varphi(j)$, $\varphi(k)$. ▮

(b) What are the next to the vertices of the face F'' that F' is sent to by σ?
Solution: $\sigma(\varphi(h))$, $\sigma(\varphi(i))$, $\sigma(\varphi(j))$, $\sigma(\varphi(k))$. ▮

(c) What are the labels next to the vertices of the face F''' that F is sent to by $\sigma \circ \varphi$?
Solution: $(\sigma \circ \varphi)(h) = \sigma(\varphi(h))$, $\sigma(\varphi(i))$, $\sigma(\varphi(j))$, $\sigma(\varphi(k))$. ▮

(d) How have you just shown that the group R acts on the faces?
Solution: We have just shown that for each face F, $\overline{\sigma \circ \varphi}(F) = \overline{\sigma} \circ \overline{\varphi}(F)$, so that $\overline{\sigma \circ \varphi} = \overline{\sigma} \circ \overline{\varphi}$. ▮

6.2.1 Groups acting on colorings of sets

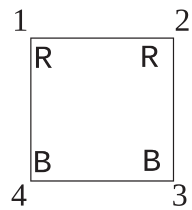
Recall that when you were asked in Problem 45 to find the number of ways to place two red beads and two blue beads at the corners of a square free to move in three-dimensional space, you were not able to apply the quotient principle to answer the question. Instead you had to see that you could divide the set of six lists of two Rs and two Bs into two sets, one of size two in which the Rs and Bs alternated and one of size four in which the two reds (and therefore the two blues) would be side-by-side on the square. Saying that the square is free to move in space is equivalent to saying that two arrangements of beads on the square are equivalent if a member of the dihedral group carries one arrangement to the other. Thus an important ingredient in the analysis of such problems will be how a group can act on colorings of a set of vertices. We can describe the coloring of the square in Figure 6.6 as the function f with

$$f(1) = R,\ f(2) = R,\ f(3) = B,\ \text{and}\ f(4) = B,$$

but it is more compact and turns out to be more suggestive to represent the coloring in Figure 6.6 as the set of ordered pairs

$$(1, R), (2, R), (3, B), (4, B). \tag{6.1}$$

Figure 6.6: The colored square with coloring $\{(1,R),(2,R),(3,B),(4,B)\}$



This gives us an explicit list of which colors are assigned to which vertex.[4] Then if we rotate the square through 90 degrees, we see that the set of ordered pairs becomes

$$\{(\rho(1),R),(\rho(2),R),(\rho(3),B),(\rho(4),B)\} \tag{6.2}$$

which is the same as

$$\{(2,R),(3,R),(4,B),(1,B)\}.$$

or, in a more natural order,

$$\{(1,B),(2,R),(3,R),(4,B)\}. \tag{6.3}$$

The reordering we did in 6.3 suggests yet another simplification of notation. So long as we know we that the first elements of our pairs are labeled by the members of $[n]$ for some integer n and we are listing our pairs in increasing order by the first component, we can denote the coloring

$$\{(1,B),(2,R),(3,R),(4,B)\}$$

by $BRRB$. In the case where we have numbered the elements of the set S we are coloring, we will call this list of colors of the elements of S in order the *standard notation* for the coloring. We will call the ordering used in 6.3 the *standard ordering* of the coloring.

Thus we have three natural ways to represent a coloring of a set as a function, as a set of ordered pairs, and as a list. Different representations are useful for different things. For example, the representation by ordered

[4] The reader who has studied Appendix A will recognize that this set of ordered pairs is the relation of the function f, but we won't need to make any specific references to the idea of a relation in what follows.

pairs will provide a natural way to define the action of a group on colorings of a set. Given a coloring as a function f, we denote the set of ordered pairs

$$\{(x, f(x))|x \in S\},$$

suggestively as (S, f) for short. We use $f(1)f(2)\cdots f(n)$ to stand for a particular coloring (S, f) in the standard notation.

∘284. Suppose now that instead of coloring the vertices of a square, we color its edges. We will use the shorthand 12, 23, 34, and 41 to stand for the edges of the cube between vertex 1 and vertex 2, vertex 2 and vertex 3, and so on. Then a coloring of the edges with 12 red, 23 blue, 34 red and 41 blue can be represented as

$$\{(12, R), (23, B), (34, R), (41, B)\}. \tag{6.4}$$

If ρ is the rotation through 90 degrees, then we have a permutation $\overline{\rho}$ acting on the edges. This permutation acts on the colorings to give us a permutation $\overline{\overline{\rho}}$ of the set of colorings.

(a) What is $\overline{\overline{\rho}}$ of the coloring in 6.4?
Solution: $\{(12, B), (23, R), (34, B), (41, R)\}$. ▮

(b) What is $\overline{\overline{\rho^2}}$ of the coloring in 6.4?
Solution: $\{(12, R), (23, B), (34, R), (41, B)\}$. ▮

If G is a group that acts the set S, we define the **action of G on the colorings** (S, f) by

$$\overline{\overline{\sigma}}((S, f)) = \overline{\overline{\sigma}}(\{(x, f(x))|x \in S\}) = \{(\overline{\sigma}(x), f(x))|x \in S\}. \tag{6.5}$$

We have the two bars over σ, because σ is a permutation of one set that gives us a permutation $\overline{\sigma}$ of a second set, and then $\overline{\sigma}$ acts to give a permutation $\overline{\overline{\sigma}}$ of a third set, the set of colorings. For example, suppose we want to anlayze colorings of the faces of a cube under the action of the rotation group of the cube as we have defined it on the vertices. Each vertex-permutation σ in the group gives a permutation $\overline{\sigma}$ of the faces of the cube. Then each permutation $\overline{\sigma}$ of the faces gives us a permutation $\overline{\overline{\sigma}}$ of the colorings of the faces.

In the special case that G is a group of permutations of S rather than a group acting on S, Equation 6.5 becomes

$$\overline{\sigma}((S, f)) = \overline{\sigma}(\{(x, f(x))|x \in S\}) = \{(\sigma(x), f(x))|x \in S\}.$$

In the case where G is the rotation group of the square acting on the vertices of the square, the example of acting on a coloring by ρ that we saw in 6.3 is an example of this kind of action. In the standard notation, when we act on a coloring by σ, the color in position i moves to position $\sigma(i)$.

285. Why does the action we have defined on colorings in Equation 6.5 take a coloring to a coloring?

Solution: G acts on S. Since $\overline{\sigma}$ is a permutation of S when $\sigma \in G$, we get a set of pairs in which each element of S is listed once as a first element and in which each second element is a color. This is a coloring. ▮

286. Show that if G is a group of permutations of a set S, and f is a coloring function on S, then the equation

$$\overline{\overline{\sigma}}(\{(x, f(x))|x \in S\}) = \{(\overline{\sigma}(x), f(x))|x \in S\}$$

defines an action of G on the colorings (S, f) of S.

Solution: By Problem 285, $\overline{\overline{\sigma}}$ takes a coloring to a coloring. We will delay showing that $\overline{\overline{\sigma}}$ is a permutation of the colorings of S. For the second condition for a group action,

$$\begin{aligned}
\overline{\overline{\sigma}} \circ \overline{\overline{\varphi}}(\{(x, f(x))|x \in S\}) &= \overline{\overline{\sigma}}(\overline{\overline{\varphi}}(\{(x, f(x))|x \in S\})) \\
&= \overline{\overline{\sigma}}(\{\overline{\varphi}(x), f(x)|x \in S\}) \\
&= \{\overline{\sigma}(\overline{\varphi}(x)), f(x))|x \in S\} \\
&= \{(\overline{\sigma \circ \varphi}(x), f(x))|x \in S\} \\
&= \overline{\overline{\sigma \circ \varphi}}(S, f),
\end{aligned}$$

so that $\overline{\overline{\sigma}} \circ \overline{\overline{\varphi}} = \overline{\overline{\sigma \circ \varphi}}$.

By the condition we just proved $\overline{\overline{\sigma}} \circ \overline{\overline{\sigma^{-1}}} = \overline{\overline{\sigma^{-1} \circ \sigma}} = \overline{\overline{\iota}}$. Therefore, $\overline{\overline{\sigma}}$ has an inverse (because $\overline{\overline{\iota}}$ is the identity), so it is a permutation. ▮

6.2.2 Orbits

•287. In Problem 282

(a) What is the set of two element subsets that you get by computing $\overline{\sigma}(\{1, 2\})$ for all σ in D_4?

Solution: From $\{1, 2\}$ we get $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, and $\{1, 4\}$. ▮

(b) What is the multiset of two-element subsets that you get by computing $\overline{\sigma}(\{1,2\})$ for all σ in D_4?

Solution:
$\{\{1,2\},\{1,2\},\{2,3\},\{2,3\},\{3,4\},\{3,4\},\{1,4\},\{1,4\}\}$. ▮

(c) What is the set of two-element subsets you get by computing $\overline{\sigma}(\{1,3\})$ for all σ in D_4?

Solution: From $\{1,3\}$ we get $\{1,3\}$ and $\{2,4\}$. ▮

(d) What is the multiset of two-element subsets that you get by computing $\overline{\sigma}(\{1,3\})$ for all σ in D_4?

Solution:
$\{\{1,3\},\{1,3\},\{1,3\},\{1,3\},\{2,4\},\{2,4\},\{2,4\},\{2,4\}\}$. ▮

(e) Describe the two sets in parts (a) and (c) geometrically in terms of the square.

Solution: We get the set of edges and the set of diagonals of the square. ▮

∘288. This problem uses the notation for permutations in the dihedral group of the square introduced before Problem 259. What is the effect of a 180 degree rotation ρ^2 on the diagonals of a square? What is the effect of the flip $\varphi_{1|3}$ on the diagonals of a square? How many elements of D_4 send each diagonal to itself? How many elements of D_4 interchange the diagonals of a square?

Solution: The 180 degree rotation sends the diagonals to themselves, i.e., it fixes the diagonals. $\varphi_{1|3}$ fixes the diagonals. So does $\varphi_{2|4}$ and the identity. However, ρ, ρ^3, $\varphi_{12|34}$ and $\varphi_{14|23}$ all exchange the diagonals. Thus four elements fix each diagonal and four elements interchange them. ▮

In Problem 287 you saw that the action of the dihedral group D_4 on two element subsets of $\{1,2,3,4\}$ seems to split them into two blocks, one with two elements and one with 4. We call these two blocks the "orbits" of D_4 acting on the two element subsets of $\{1,2,3,4\}$. More generally, given a group G acting on a set S, the **orbit** of G determined by an element x of S is the set

$$\{\overline{\sigma}(x)|\sigma \in G\},$$

and is denoted by Gx. In Problem 287 it was possible to have $Gx = Gy$. In fact in that problem, $Gx = Gy$ for every y in Gx.

289. Suppose a group acts on a set S. Could an element of S be in two different orbits? (Say why or why not.)

   Solution: No, because if $z = \overline{\sigma}(x)$ and $z = \overline{\varphi}(y)$, then $Gz = Gx = Gy$, where the last two equalities follow from the fact that as τ ranges over G, so does $\tau\sigma$ and $\tau\varphi$. ▮

Problem 289 almost completes the proof of the following theorem.

Theorem 9 *Suppose a group acts on a set S. The orbits of G form a partition of S.*

It is probably worth pointing out that this theorem tells us that the orbit Gx is also the orbit Gy for any element y of Gx.

290. Complete the proof of Theorem 9.

   Solution: Every element x of S lies in the orbit Gx. In Problem 289 we showed that the element x can lie in only one orbit. Therefore each x in S lies in one and only one orbit, so the orbits partition S. ▮

Notice that thinking in terms of orbits actually hides some information about the action of our group. When we computed the multiset of all results of acting on $\{1, 2\}$ with the elements of D_4, we got an eight-element multiset containing each side twice. When we computed the multiset of all results of acting on $\{1, 3\}$ with the elements of D_4, we got an eight-element multiset containing each diagonal of the square four times. These multisets remind us that we are acting on our two-element sets with an eight-element group. The **multiorbit** of G determined by an element x of S is the multiset

$$\{\overline{\sigma}(x)|\sigma \in G\},$$

and is denoted by Gx_{multi}.

When we used the quotient principle to count circular seating arrangements or necklaces, we partitioned up a set of lists of people or beads into blocks of equivalent lists. In the case of seating n people around a round table, what made two lists equivalent was, in retrospect, the action of the rotation group C_n. In the case of stringing n beads on a string to make a necklace, what made two lists equivalent was the action of the dihedral group. Thus the blocks of our partitions were orbits of the rotation group or the dihedral group, and we were counting the number of orbits of the group action. In Problem 45, we were not able to apply the quotient principle because we had blocks of different sizes. However, these blocks were still orbits of the action of the group D_4. And, even though the orbits have different sizes, we expect that each orbit corresponds naturally to a multiorbit and that the multiorbits all have the same size. Thus if we had a version of the quotient rule for a union of multisets, we could hope to use it to count the number of multiorbits.

∘291. (a) Find the orbit and multiorbit of D_4 acting on the coloring

$$\{(1, R), (2, R), (3, B), (4, B)\},$$

or, in standard notation, $RRBB$, of the vertices of a square.

Solution: The orbit is, in the standard notation, $\{RRBB, BRRB, BBRR, RBBR.\}$
The multiorbit, in the standard notation, is
$\{RRBB, RRBB, BRRB, BRRB, BBRR, BBRR, RBBR, RBBR\}$. ▮

(b) How many group elements map the coloring RRBB to itself? What is the multiplicity of $RRBB$ in its multiorbit?

Solution: 2, 2. ▮

(c) Find the orbit and multiorbit of D_4 acting on the coloring

$$\{(1, R), (2, B), (3, R), (4, B)\}.$$

Solution: The orbit is, in the standard notation, $\{RBRB, BRBR\}$.
The multiorbit, in the standard notation, is
$\{RBRB, RBRB, RBRB, RBRB, BRBR, BRBR, BRBR, BRBR\}$. ▮

(d) How many elements of the group send the coloring $RBRB$ to itself? What is the multiplicity of $RBRB$ in its orbit?

Solution: 4, 4. ▮

292. (a) If G is a group, how is the set $\{\tau\sigma|\tau \in G\}$ related to G?

Solution: It is G, because composition on the right by σ gives a bijection from G to G. ▮

(b) Use this to show that y is in the multiorbit Gx_{multi} if and only if $Gx_{\text{multi}} = Gy_{\text{multi}}$.

Solution: If $y = \sigma x$, Part (a) tells us that $Gy_{\text{multi}} = G\sigma(x)_{\text{multi}} = \{\tau\sigma(x)|\tau \in G\}_{\text{multi}} = \{\sigma(x)|\tau \in G\}_{\text{multi}} = Gx_{\text{multi}}$. ▮

Problem 292b tells us that, when G acts on S, each element x of S is in one and only one multiorbit. Since each orbit is a subset of a multiorbit and each element x of S is in one and only one orbit, this also tells us there is a bijection between the orbits of G and the multiorbits of G, so that we have the same number of orbits as multiorbits.

When a group acts on a set, a group element is said to **fix** an element of $x \in S$ if $\overline{\sigma}(x) = x$. The set of all elements fixing an element x is denoted by $\mathrm{Fix}(x)$.

293. Suppose a group G acts on a set S. What is special about the subset $\mathrm{Fix}(x)$ for an element x of S?

   Solution: $\mathrm{Fix}(x)$ is a group; in fact a subgroup of G. ▮

•294. Suppose a group G acts on a set S. What is the relationship of the multiplicity of $x \in S$ in its multiorbit and the size of $\mathrm{Fix}(x)$?

   Solution: The multiplicity of x is the size of $\mathrm{Fix}(x)$, because in Gx_{multi} the multiplicity of x will be exactly the number of elements that send x to itself. ▮

295. What can you say about relationships between the multiplicity of an element y in the multiorbit Gx_{multi} and the multiplicites of other elements? Try to use this to get a relationship between the size of an orbit of G and the size of G.

   Solution: Every element y of a multiorbit has the same multiplicity. This is because if $\sigma(x) = y$, then the permutations that send x to y are the permutations $\sigma\tau$ where τ fixes x. Thus the size of the multiorbit is $|\mathrm{Fix}(x)| \cdot |Gx|$, so the size of the orbit divides the size of the multiorbit, which is the size of G. In particular, the size of an orbit is $|G|/|\mathrm{Fix}(x)|$ for any x in the orbit. ▮

We suggested earlier that a quotient principle for multisets might prove useful. The quotient principle came from the sum principle, and we do not have a sum principle for multisets. Such a principle would say that the size of a union of disjoint multisets is the sum of their sizes. We have not yet defined the union of multisets or disjoint multisets, because we haven't needed the ideas until now. We define the *union* of two multisets S and T to be the multiset in which the multiplicity of an element x is the maximum[5] of the multiplicity of x in S and its multiplicity in T. Similarly, the union of a family of multisets is defined by defining the multiplicity of an element x to be the maximum of its multiplicities in the members of the family. Two

[5]We choose the maximum rather than the sum so that the union of sets is a special case of the union of multisets.

multisets are said to be *disjoint* if no element is a member of both, that is, if no element has multiplicity one or more in both. Since the size of a multiset is the sum of the multiplicities of its members, we immediately get the *sum principle for multisets*.

> The size of a union of disjoint multisets is the sum of their sizes.

Taking the multisets all to have the same size, we get the *product principle for multisets*.

> The union of a set of m disjoint multisets, each of size n has size mn.

The *quotient principle for multisets* then follows immediately.

> If a p-element multiset is a union of q disjoint multisets, each of size r, then $q = p/r$.

•296. How does the size of the union of the set of multiorbits of a group G acting on a set S relate to the number of multiorbits and the size of G?

Solution: It is simply the product of the number of multiorbits and the size of G. ▮

•297. How does the size of the union of the set of multiorbits of a group G acting on a set S relate to the numbers $|\mathrm{Fix}(x)|$?

Solution: Since the size of the union is the sum of the multiplicities of the elements of S in the union, it is the sum of $|\mathrm{Fix}(x)|$ over all x in S. ▮

•298. In Problems 296 and 297 you computed the size of the union of the set of multiorbits of a group G acting on a set S in two different ways, getting two different expressions which must be equal. Write the equation that says they are equal and solve for the number of multiorbits, and therefore the number of orbits.

Solution: Using m for the number of multiorbits, we get

$$m|G| = \sum_{x:x\in S} |\mathrm{Fix}(x)|.$$

Therefore,

$$m = \frac{1}{|G|}\sum_{x:x\in S} |\mathrm{Fix}(x)|.$$

▮

6.2.3 The Cauchy-Frobenius-Burnside Theorem

•299. In Problem 298 you stated and proved a theorem that expresses the number of orbits in terms of the number of group elements fixing each element of S. It is often easier to find the number of elements fixed by a given group element than to find the number of group elements fixing an element of S.

(a) For this purpose, how does the sum $\sum_{x:x\in S} |\mathrm{Fix}(x)|$ relate to the number of ordered pairs (σ, x) (with $\sigma \in G$ and $x \in S$) such that σ fixes x?

Solution: They are equal, because $\mathrm{Fix}(x)$ computes the number of ordered pairs using that particular x. ▮

(b) Let $\chi(\sigma)$ denote the number of elements of S fixed by σ. How can the number of ordered pairs (σ, x) (with $\sigma \in G$ and $x \in S$) such that σ fixes x be computed from $\chi(\sigma)$? (It is ok to have a summation in your answer.)

Solution: The number of ordered pairs is $\sum_{\sigma:\ \sigma\in G} \chi(\sigma)$. ▮

(c) What does this tell you about the number of orbits?

Solution: $m = \frac{1}{|G|}\sum_{\sigma:\ \sigma\in G} \chi(\sigma)$. ▮

300. A second computation of the result of problem 299 can be done as follows.

(a) Let $\hat{\chi}(\sigma, x) = 1$ if $\sigma(x) = x$ and let $\hat{\chi}(\sigma, x) = 0$ otherwise. Notice that $\hat{\chi}$ is different from the χ in the previous problem, because it is a function of two variables. Use $\hat{\chi}$ to convert the single summation in your answer to Problem 298 into a double summation over elements x of S and elements σ of G.

Solution:

$$m = \frac{1}{|G|} \sum_{x:x\ \in S} \sum_{\sigma:\sigma\in G} \hat{\chi}(\sigma, x).$$

▮

(b) Reverse the order of the previous summation in order to convert it into a single sum involving the function χ given by

$$\chi(\sigma) = \text{the number of elements of } S \text{ left fixed by } \sigma.$$

Solution: $m = \frac{1}{|G|}\sum_{\sigma:\ \sigma\in G} \chi(\sigma)$. ▮

In Problem 299 you gave a formula for the number of orbits of a group G acting on a set X. This formula was first worked out by Cauchy in the case of the symmetric group, and then for more general groups by Frobenius. In his pioneering book on Group Theory, Burnside used this result as a lemma, and while he attributed the result to Cauchy and Frobenius in the first edition of his book, in later editions, he did not. Later on, other mathematicians who used his book named the result "Burnside's Lemma," which is the name by which it is still most commonly known. Let us agree to call this result the Cauchy-Frobenius-Burnside Theorem, or CFB Theorem for short in a compromise between historical accuracy and common usage.

➔301. In how many ways may we string four (identical) red, six (identical) blue, and seven (identical) green beads on a necklace?

Solution: We are stringing 17 beads on our necklace, so we are asking for the number of orbits of the group D_{17} on lists of four Rs, six Bs, and seven Gs. For a rotation ρ^i to fix a list, it must take an R to a R. The powers of ρ^i form a group. Thus the set of places that contain an R in a list that is fixed by ρ^i must be an orbit of that group. But the size of the orbit must be a divisor of the size of the subgroup, which must be a divisor of seventeen, so the size of the orbit is one or 17. If it is 1, then $i = 0$. Thus no elements are fixed by any nontrivial rotation, and all $\binom{17}{4,6,7}$ lists of Rs, Bs and Gs are fixed by the identity. Each flip will be a flip around a line from a bead to the spot between the two "opposite" beads. This line divides the list into the eight beads on its left, the eight beads on its right and the one bead it goes through. If a flip fixes a list, the eight beads on the left must be identical to the eight beads on the right, meaning that the bead the line goes through must be one of the seven green beads. Thus the number of lists that are fixed by a flip is the number of ways to place three green beads, two red beads, and three blue beads in 8 slots, which is $\binom{8}{3,2,3}$, or $\frac{8!}{3!2!3!}$. There are 17 flips, so there are $\frac{17\cdot 8!}{3!2!3!}$ elements fixed by flips. Therefore we have

$$\frac{1}{2\cdot 17}\left(\frac{17!}{4!6!7!}+\frac{17\cdot 8!}{3!2!2!}\right)=\frac{8\cdot 15!}{4!6!7!}+\frac{4\cdot 7!}{3!2!3!}=120,400$$

necklaces. The numerical answer, which is unimportant here, was obtained from Maple. ▮

➔302. If we have an unlimited supply of identical red beads and identical blue beads, in how many ways may we string 17 of them on a necklace?

Solution: We are asking for the number of orbits of D_{17} on the set of colorings of [17] by $\{R, B\}$. Every coloring is fixed by the identity. The only colorings fixed by a nontrivial rotation are the constant colorings that assign the same color to each bead. Each flip is around a line from a bead to the space between the two "opposite" beads. The bead the line goes through can be either color, and then the eight beads to the left of this one must be identical to the eight beads to the right of this one. There are 2^8 ways to assign beads to the positions on the left, so a flip fixes 2^9 colorings. Therefore by the CFB theorem, we have

$$\frac{1}{2\cdot 17}\left(2^{17}+17\cdot 2^9+16\cdot 2\right)=\frac{2^{16}}{17}+2^8+\frac{2^4}{17}=16\frac{4097}{17}+256=4112$$

necklaces. ▮

➔ 303. If we have five (identical) red, five (identical) blue, and five (identical) green beads, in how many ways may we string them on a necklace?

Solution: Here we need to consider the action of D_{15} on colorings of [15] by $\{R, G, B\}$ with five Rs, five Bs, and five Gs. The identity will fix $\binom{15}{5,5,5}$ colorings. A rotation through 3, 6, 9, or 12 places will fix any coloring that has the same color in places 1, 4, 7, 10, and 13, the same color in places 2, 5, 8, 11, and 14, and the same color in places 3, 6, 9, 12, and 15. There are 3! such colorings. There is no other rotation that fixes any colorings. Each flip is around an axis that goes from a bead to the space between two "opposite beads." If it fixed a coloring, the seven colors to the left of the axis would have to equal the seven colors to the right of the axis. Thus the number of beads of each color on the left and right sides would have to be equal. So except for the color of the bead the axis goes through, we would have to have an even number of beads of each color for a flip to fix a coloring. Thus no flip fixes any colorings. Therefore by the CFB theorem we have

$$\frac{1}{2\cdot 15}\left(\frac{15!}{5!5!5!}+4\cdot 3!\right)=25,226$$

necklaces. (The numerical answer, which was obtained from Maple, is not important here.) ▮

➔ 304. In how many ways may we paint the faces of a cube with six different colors, using all six?

Solution: Here we must consider the action of the rotation group of the cube on lists of six distinct colors. But no nontrivial rotation will fix a cube with all its faces colored differently. The identity rotation will fix all 6! lists, and there are 24 members of the rotation group, so we have $6!/24 = 6 \cdot 5 = 30$ ways to paint the faces of a cube with six distinct colors, using each color. ∎

305. In how many ways may we paint the faces of a cube with two colors of paint? What if both colors must be used?

Solution: We must consider the action of the rotation group of the cube on colorings of $[6]$ by $\{R, B\}$ or some other two-element set of colors. There are five kinds of elements in the rotation group of the cube. There is one identity, there are six rotations by 90 degrees or 270 degrees around an axis connecting the centers of two opposite faces, there are three rotations of 180 degrees around such an axis, there are 8 rotations (of 120 degrees and 240 degrees, respectively) around an axis connecting two diagonally opposite vertices, and there are 6 rotations of 180 degrees around an axis connecting the centers of two opposite edges. The identity fixes 2^6 colorings. There are eight colorings fixed by a 90 degree or 270 degree rotation. There are 16 colorings fixed by a 180 degree rotation along an axis through two faces. There are 8 colorings fixed by a 180 degree rotation along an axis joining the centers of two opposite sides. There are 4 colorings fixed by a 120 degree or 240 degree rotation. Thus by the CFB theorem, we have

$$\frac{1}{24}(64 + 6 \cdot 8 + 3 \cdot 16 + 8 \cdot 4 + 6 \cdot 8) = 10$$

ways to paint the faces of a cube with two colors of paint. Two of these colorings use only one color, so there are eight colorings that use both colors. ∎

➔306. In how many ways may we color the edges of a (regular) $(2n+1)$-gon free to move around in the *plane* (so it cannot be flipped) if we use red n times and blue $n+1$ times? If this is a number you have seen before, identify it.

Solution: The set of all powers of a rotation is a subgroup of the rotation group of the $(2n+1)$-gon. If a given rotation fixes a coloring, all powers of that rotation fix the coloring. The set of edges to which a given edge is taken by that rotation is an orbit of the group of powers of the rotation. The size of this orbit is a divisor of the size of the subgroup, which is a divisor of $2n+1$. If the edge is colored red, the size of the orbit is also a divisor of n, and if the edge is colored blue, the size of the orbit is also a divisor of $n+1$. But neither n nor $n+1$ has common divisors with $2n+1$, except for one. Therefore the only rotation that fixes a coloring is the identity rotation, and it fixes all $\binom{2n+1}{n}$ colorings. Thus the number of orbits is

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n},$$

a Catalan number. ▮

➔ $*$307. In how many ways may we color the edges of a (regular) $(2n+1)$-gon free to move in *three-dimensional space* so that n edges are colored red and $n+1$ edges are colored blue? Your answer may depend on whether n is even or odd.

Solution: The set of all powers of a rotation is a subgroup of the dihedral group of the $(2n+1)$-gon. If a given rotation fixes a coloring, all powers of that rotation fix the coloring. The set of edges to which a given edge is taken by that rotation is an orbit of the group of powers of the rotation. The size of this orbit is a divisor of the size of the subgroup, which is a divisor of $2n+1$. If the edge is colored red, the size of the orbit is also a divisor of n, and if the edge is colored blue, the size of the orbit is also a divisor of $n+1$. But neither n nor $n+1$ has common divisors with $2n+1$, except for one. Therefore the only rotation that fixes a coloring is the identity rotation, and it fixes all $\binom{2n+1}{n}$ colorings.

A flip, on the other hand, can fix some colorings. In particular, if n is even, we color one edge blue, leaving an even number n of additional edges to be colored blue and an even number n of edges to be colored red. It the flip over the axis perpendicular to the side we picked fixes the coloring, the n edges to the right of the chosen edge must be colored identically with the corresponding n edges to the left of the chosen edge. There are $\binom{n}{n/2}$ ways to color the n edges to the right of the chosen edge (choose which edges get red), and so this is the number of colorings fixed by this flip. We have $2n+1$ flips, and since we have one flip of the type described for each edge, all flips have the form just given. Thus each flip fixes $\binom{n}{n/2}$ colorings.

Thus the number of orbits is

$$\frac{1}{2(2n+1)}\left(\binom{2n+1}{n} + (2n+1)\binom{n}{n/2}\right).$$

If n is odd, we color one edge red, leaving an even number $n-1$ of edges to be colored red and an even number $n+1$ of edges to be colored blue. With an argument similar to the previous one we see that there are

$$\frac{1}{2(2n+1)}\left(\binom{2n+1}{n}+(2n+1)\binom{n}{(n-1)/2}\right)$$

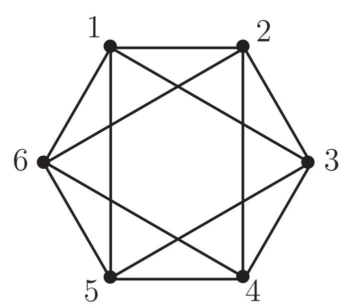
orbits. ▮

➔ ∗308. (Not unusually hard for someone who has worked on chromatic polynomials.) How many different proper colorings with four colors are there of the vertices of a graph which is a cycle on five vertices? (If we get one coloring by rotating or flipping another one, they aren't really different.)

Solution: We are asking for the number of orbits of D_5 on lists of five colors chosen from the given four with no two adjacent colors equal; we consider the first and last position adjacent as well. The identity fixes all such colorings. For those who are familiar with the chromatic polynomial, as in Problem 243, the number of such colorings is the chromatic polynomial of the cycle on five vertices evaluated at 4. The number of ways to properly color a cycle on five vertices is the number of ways to color a path on five vertices minus the number of ways to color a path on five vertices so that its first and last vertices are identical, which is the number of ways to color a cycle on four vertices. The number of ways to color a cycle on four vertices is the number of ways to color a path on four vertices minus the number of ways to color a path on four vertices so that the first and last vertices are the same color, which is the same as the number of ways to color a cycle on three vertices. The number of ways to properly color a path on five vertices with four colors is $4\cdot 3^4$. The number of ways to properly color a path on four vertices with four colors is $4\cdot 3^3$. The number of ways to properly color a cycle on three vertices with four colors is $4\cdot 3\cdot 2$. Thus the number of proper colorings of a five vertex cycle with four colors is $4\cdot 3^4 - 4\cdot 3^3 + 4\cdot 3\cdot 2 = 240$. This is the number of colorings fixed by the identity. No proper coloring is fixed by a nontrivial rotation, because if a rotation of the five cycle fixes a coloring, all the vertices must have the same color. No proper coloring is fixed by a flip, because for a flip to fix a coloring, the coloring must give the same color to two adjacent vertices. Thus the number of really different proper four-colorings of a cycle on 5 vertices is $\frac{240}{10} = 24$. ▮

➔ ∗309. How many different proper colorings with four colors are there of the graph in Figure 6.7? Two graphs are the same if we can redraw one of the graphs, not changing the vertex set or edge set, so that it is identical to the other one. This is equivalent to permuting the vertices in some way so that when we apply the permutation to the endpoints of the edges to get a new edge set, the new edge set is equal to the old one. Such a permutation is called an *automorphism* of the graph. Thus two colorings are different if there is no automorphism of the graph that carries one to the other one.

Figure 6.7: A graph on six vertices.



Solution: We want the number of orbits of the set of proper colorings under the action of the group of automorphisms of the graph. An automorphism σ maps vertex 1 to any of six vertices. Vertex 2 can be mapped to any of the four vertices adjacent to the image of vertex 1. Vertex 3 is adjacent to vertices 1 and 2, so it must be mapped to a vertex adjacent to the images of both vertex 1 and vertex 2; by checking cases you can see that there are always exactly two vertices adjacent to the images of vertex 1 and vertex 2, and mapping vertex 3 to either of these vertices preserves all the edges among vertices 1, 2, and 3. However, each of the other three vertices is adjacent to exactly two vertices of the set $\{1, 2, 3\}$, and thus it must be mapped to the unique vertex adjacent to the corresponding two of $\sigma(1)$, $\sigma(2)$ and $\sigma(3)$. (It is always the case that each vertex is adjacent to exactly two of these, as you can see by considering the cases with $\sigma(1) = 1$.) Thus there are $6 \cdot 4 \cdot 2 = 48$ elements in the group. Now to apply the CFB theorem we would need to know how many proper colorings are fixed by each group element, so we would need to know what the group elements are. We have observed that a permutation that preserves the edges is determined by where the triangle $\{1, 2, 3\}$ goes. We can see eight triangles in the graph, triangles of the form $\{i, i+1, i+2\}$, where we identify 7 with 1 and 8 with 2, and the triangles $\{1, 3, 5\}$ and $\{2, 4, 6\}$. We can map the set $\{1, 2, 3\}$ to any of these eight sets by six one-to-one maps, so each group element is determined uniquely by one of these mappings. However, focusing on these triangles makes our job here simpler in another way. In a proper coloring, vertices 1, 2, and 3 must be colored differently. We have four choices for the color of vertex 1, three different ones for vertex 2 and two still different ones for vertex 3, so there are 24 ways to color this triangle. Clearly the only difference among these ways is the actual names of the colors. That is, we can assume that vertex 1 is colored red, vertex 2 is colored blue and vertex 3 is colored green, then determine the proper colorings starting with these three colors, and up to changing the names of the colors, we will have determined all the proper colorings. Then we can ask which group elements fix a coloring rather than which colorings are fixed by a group element. This turns out to be easier. Let us write $RBGRBG$ for the coloring that colors vertices 1 and 4 red, two and five blue, and three and six green. An examination of the figure shows that this is a proper coloring. In fact, it is the only proper coloring that starts RBG and uses only three colors. Suppose we were to use a fourth color, Y. Then among vertices 4, 5, and 6, it could be used in just one place, because those three vertices are mutually adjacent. Each of the other two vertices is adjacent to two of the original three vertices colored RBG, and so there is only one color available to use on it. In summary, the colorings that start RBG are

(a) $RBGRBG$
(b) $RBGYBG$
(c) $RBGRYG$
(d) $RBGRBY$.

Thus for any of the 24 choices of colorings of the first three vertices, there are four ways to complete it to a proper coloring of the whole graph, so there are 96 proper colorings of the graph. Among the ones that start RGB, let us analyze which group elements fix them. Note that the 2-cycles (1 4), (2 5) and (3 6) are all permutations that fix coloring one. (Note that we are identifying $(i\ j)$ with the permutation of [6] that has the cycle $(i\ j)$ and four-cycles of size one.)

Further, interchanging vertices 1 and 4 does not change the endpoints of any edges, nor does interchanging 2 and 5 nor 3 and 6. So all these two-cycles are automorphisms of the graph. A composition of automorphisms must be an automorphism (this follows directly from the definition of automorphism) and so the eight permutations ι, (1 4), (2 5), (3 6), (1 4)(2 5), (1 4)(3 6), (2 5)(3 6), and (1 4)(2 5)(3 6) all are automorphisms of the graph, and they are all in the subgroup of the automorphism group that fixes Coloring (a). Any permutation not in the list will take some vertex to a vertex of another color, and so the eight permutations we listed are the subgroup fixing Coloring (a). The subgroup fixing Coloring (b) is ι, (1 3), (2 4), and (1 3)(2 4). The subgroups fixing Coloring (c) and Coloring (d) also have size 4. Thus there are $8 + 12 = 20$ pairs of a coloring with R, B, and G, in that order, on vertices 1, 2, and 3 and an automorphism fixing that coloring. Since there are $4 \cdot 3 \cdot 2 = 24$ ways to color vertices 1, 2, and 3 properly, and each gives rise to 20 pairs of a proper coloring and an automorphism fixing that coloring, there are $20 \cdot 24 = 480$ such pairs. Since the automorphism group has size 48, this means that there are 10 proper colorings of this graph, up to automorphisms. Note that we did not really use the CFB theorem, though we did use the fact that its formula is proved by dividing the number of ordered pairs of a set member and a group element fixing that member by the size of the group. ∎

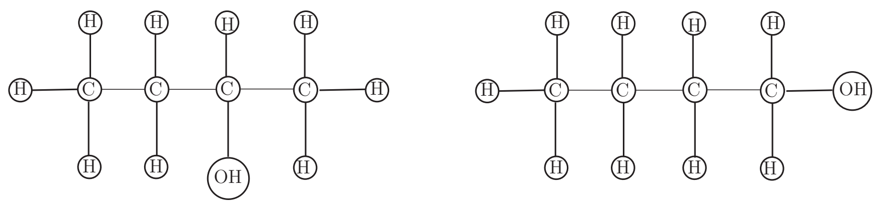
6.3 Pólya-Redfield Enumeration Theory

George Pólya and Robert Redfield independently developed a theory of generating functions that describe the action of a group G on colorings of a set S by a set T when we know the action of G on S. Pólya's work on the subject is very accessible in its exposition, and so the subject has become popularly known as Pólya theory, though Pólya-Redfield theory would be a better name. In this section we develop the elements of this theory.

The idea of coloring a set S has many applications. For example, the set S might be the positions in a hydrocarbon molecule which are occupied by hydrogen, and the group could be the group of spatial symmetries of the molecule (that is, the group of permutations of the atoms of the molecule that move the molecule around so that in its final position the molecule cannot be distinguished from the original molecule). The colors could then be radicals (including hydrogen itself) that we could substitute for each hydrogen position in the molecule. Then the number of orbits of colorings is the number of chemically different compounds we could create by using these substitutions.[6]

In Figure 6.8 we show two different ways to substitute the OH radical for a hydrogen atom in the chemical diagram we gave for butane in Chapter 2. We have colored one vertex of degree 1 with the radical OH and the rest with the atom H. There are only two distinct ways to do this, as the OH must either connect to an "end" C or a "middle" C. This shows that there are two different forms, called *isomers* of the compound shown. This compound is known as butyl alcohol.

Figure 6.8: The two different isomers of butyl alcohol.



So think intuitively about some "figure" that has places to be colored. (Think of the faces of a cube, the beads on a necklace, circles at the vertices of an n-gon, etc.) How can we picture the coloring? If we number the places to be colored, say 1 to n, then we have a standard way to represent our coloring. For example, if our colors are blue, green and red, then $BBGRRGBG$ describes a typical coloring of 8 such places. Unless the places are somehow "naturally" numbered, this idea of a coloring imposes structure that is not really there. Even if the structure is there, visualizing our colorings in this way doesn't "pull together" any common features of different colorings; we are simply visualizing all possible colorings. We have a group (think of it as symmetries of the figure you are imagining) that acts on the places. That group then acts in a natural way on the colorings of the places and we are

[6]There is a fascinating subtle issue of what makes two molecules different. For example, suppose we have a molecule in the form of a cube, with one atom at each vertex. If we interchange the top and bottom faces of the cube, each atom is still connected to exactly the same atoms as before. However, we cannot achieve this permutation of the vertices by a member of the rotation group of the cube. It could well be that the two versions of the molecule interact with other molecules in different ways, in which case we would consider them chemically different. On the other hand, if the two versions interact with other molecules in the same way, we would have no reason to consider them chemically different. This kind of symmetry is an example of what is called *chirality* in chemistry.

interested in orbits of the colorings. Thus we want a picture that pulls together the common features of the colorings in an orbit. One way to pull together similarities of colorings would be to let the letters we are using as pictures of colors commute as we did with our pictures in Chapter 4; then our picture $BBGRRGBG$ becomes $B^3G^3R^2$, so our picture now records simply how many times we use each color. Think about how we defined the action of a group on the colorings of a set on which the group acts. You will see that acting with a group element won't change how many times each color is used; it simply moves colors to different places. Thus the picture we now have of a given coloring is an equally appropriate picture for each coloring in an orbit. One natural question for us to ask is "How many orbits have a given picture?"

•310. Suppose we draw identical circles at the vertices of a regular hexagon. Suppose we color these circles with two colors, red and blue.

(a) In how many ways may we color the set $\{1, 2, 3, 4, 5, 6\}$ using the colors red and blue?

Solution: There are 64 functions from a six-element set to a two-element set, so there are 64 colorings. ▮

(b) These colorings are partitioned into orbits by the action of the rotation group on the hexagon. Using our standard notation, write down all these orbits and observe how many orbits have each picture, assuming the picture of a coloring is the product of commuting variables representing the colors.

Solution: $\{RRRRRR\}$,
$\{RRRRRB, BRRRRR, RBRRRR, RRBRRR, RRRBRR, RRRRBR\}$,
$\{RRRRBB, BRRRRB, BBRRRR, RBBRRR, RRBBRR, RRRBBR\}$,
$\{RRRBRB, BRRRBR, RBRRRB, BRBRRR, RBRBRR, RRBRBR\}$,
$\{RRBRRB, BRRBRR, RBRRBR\}$,
$\{RRRBBB, BRRRBB, BBRRRB, BBBRRR, RBBBRR, RRBBBR\}$,
$\{RRBRBB, BRRBRB, BBRRBR, RBBRRB, BRBBRR, RBRBBR\}$,
$\{RRBBRB, BRRBBR, RBRRBB, BRBRRB, BBRBRR, RBBRBR\}$
$\{RBRBRB, BRBRBR\}$,
$\{RRBBBB, BRRBBB, BBRRBB, BBBRRB, BBBBRR, RBBBBR\}$,
$\{RBRBBB, BRBRBB, BBRBRB, BBBRBR, RBBBRB, BRBBBR\}$,
$\{RBBRBB, BRBBRB, BBRBBR\}$,
$\{RBBBBB, BRBBBB, BBRBBB, BBBRBB, BBBBRB, BBBBBR\}$,
$\{BBBBBB\}$

We have one orbit with picture R^6, one with picture R^5B, three with picture R^4B^2, four with picture R^3B^3, three with picture R^2B^4, one with picture RB^5, and one with picture B^6. ▮

(c) Using the picture function of the previous part, write down the picture enumerator for the orbits of colorings of the vertices of a hexagon under the action of the rotation group.

Solution: $R^6+R^5B+3R^4B^2+4R^3B^3+3R^2B^4+RB^5+B^6$. ▮

In Problem 310c we saw a picture enumerator for pictures of orbits of the action of a group on colorings. As above, we ask how many orbits of the colorings have any given picture. We can think of a multivariable generating function in which the letters we use to picture individual colors are the variables, and the coefficient of a picture is the number of orbits with that picture. Such a generating function provides an answer to our natural question, and so it is this sort of generating function we will seek. Since the CFB theorem was our primary tool for saying how many orbits we have, it makes sense to think about whether the CFB theorem has an analog in terms of pictures of orbits.

6.3.1 The Orbit-Fixed Point Theorem

•311. Suppose now we have a group G acting on a set and we have a picture function on that set with the additional feature that for each orbit of the group, all its elements have the same picture. In this circumstance we define the picture of an orbit or multiorbit to be the picture of any one of its members. The **orbit enumerator** $\mathrm{Orb}(G,S)$ is the sum of the pictures of the orbits. (Note that this is the same as the sum of the pictures of the *multiorbits*.) The **fixed point enumerator** $\mathrm{Fix}(G,S)$ is the sum of the pictures of each of the fixed points of each of the elements of G. We are going to construct a generating function analog of the CFB theorem. The main idea of the proof of the CFB theorem was to try to compute in two different ways the number of elements (i.e. the sum of all the multiplicities of the elements) in the union of all the multiorbits of a group acting on a set. Suppose instead we try to compute the sum of all the pictures of all the elements in the union of the multiorbits of a group acting on a set. By thinking about how this sum relates to $\mathrm{Orb}(G,S)$ and $\mathrm{Fix}(G,S)$, find an analog of the CFB theorem that relates these two enumerators. State and prove this theorem.

Solution: Let E, for enumerator, be the sum of all the pictures of all the elements in the union of the multiorbits of G acting on a set S.

Recall that for any multiorbit M the picture $P(M)$ is the picture $P(x)$ of any element x of M, and the number of elements of a multiorbit M is always the size of G. This lets us write

$$\begin{aligned} E &= \sum_{M:M \text{ is a multiorbit of } G} \sum_{x:x\in M} P(x) \\ &= \sum_{M:M \text{ is a multiorbit of } G} |G|P(M) \\ &= |G| \sum_{M:M \text{ is a multiorbit of } G} P(M) \\ &= |G|\mathrm{Orb}(G,S). \end{aligned}$$

Recall also that the multiplicity of an element x in its multiorbit, and thus in the union of the multiorbits, is $|\mathrm{Fix}(x)|$. This lets us write $E = \sum_{x:x\in S} |\mathrm{Fix}(x)|P(x)$. Now we have two possible approaches. First, the sum we just gave for E is the sum, over all ordered pairs (σ, x) such that σ fixes x, of $P(x)$. But this is exactly $\mathrm{Fix}(G,S)$. Second, we can get the same formula for E by using $\chi(\sigma, x)$ as in Problem 300. We may write

$$\begin{aligned} E &= \sum_{x:x\in S} |\mathrm{Fix}(x)|P(x) \\ &= \sum_{x:x\in S} \sum_{\sigma:\sigma\in G} \chi(\sigma,x)P(x) \\ &= \sum_{\sigma:\sigma\in G} \sum_{x:\sigma x = x} P(x) \\ &= \mathrm{Fix}(G,S). \end{aligned}$$

Setting our two values of E equal and solving for $\mathrm{Orb}(G,S)$ gives us

$$\mathrm{Orb}(G,S) = \frac{1}{|G|}\mathrm{Fix}(G,S).$$

∎

We will call the theorem of Problem 311 the **Orbit-Fixed Point Theorem**. In order to apply the Orbit-Fixed Point Theorem, we need some basic facts about picture enumerators.

•312. Suppose that P_1 and P_2 are picture functions on sets S_1 and S_2 in the sense of Section 4.1.2. Define P on $S_1 \times S_2$ by $P(x_1, x_2) =$

$P_1(x_1)P_2(x_2)$. How are E_{P_1}, E_{P_1}, and E_P related? (You may have already done this problem in another context!)

Solution:

$$\begin{aligned} E_P(S_1 \times S_2) &= \sum_{x_1 \in S_1,\ x_2 \in S_2} P(x_1)P(x_2) = \sum_{x_1 \in S_1} \sum_{x_2 \in S_2} P(x_1)P(x_2) \\ &= \sum_{x_1 \in S_1} P_1(x_1) \sum_{x_2 \in S_2} P_2(x_2) = E_{P_1}(S_1)E_{P_2}(S_2). \end{aligned}$$

∎

•313. Suppose P_i is a picture function on a set S_i for $i = 1, \ldots, k$. We define the picture of a k-tuple $(x_1, x_2, \ldots, x_k)$ to be the product of the pictures of its elements, i.e.

$$\widehat{P}((x_1, x_2, \ldots x_k)) = \prod_{i=1}^{k} P_i(x_i).$$

How does the picture enumerator $E_{\widehat{P}}$ of the set $S_1 \times S_2 \times \cdots \times S_k$ of all k-tuples with $x_i \in S_i$ relate to the picture enumerators of the sets S_i? In the special case that $S_i = S$ for all i and $P_i = P$ for all i, what is $E_{\widehat{P}}(S^k)$?

Solution: Based on the previous problem, we expect that

$$E_{\widehat{P}}(S_1 \times S_2 \times \cdots \times S_k) = \prod_{i=1}^{k} E_{P_i}(S_i).$$

To prove it, we represent a k-tuple as the ordered pair

$$(x_1,\ (x_2, x_3, \ldots, x_k))$$

and apply induction and Problem 312. For the special case, $E_{\widehat{P}}(S^k) = (E_P(S))^k$. ∎

•314. Use the Orbit-Fixed Point Theorem to determine the Orbit Enumerator for the colorings, with two colors (red and blue), of six circles placed at the vertices of a hexagon which is free to move in the plane. Compare the coefficients of the resulting polynomial with the various orbits you found in Problem 310.

Solution: Let us take the pictures of red and blue to be R and B. Since the hexagon is free to move in the plane, our group G is the group R_6 of rotations of a regular hexagon. There are four kinds of elements of G: the identity, the rotation ρ through 60 degrees and the rotation ρ^5 through 300 degrees (the corresponding permutations are six-cycles), the rotations ρ^2 and ρ^4 through 120 and 240 degrees respectively (the cycle decompositions of the corresponding permutations have two three-cycles), and the rotation ρ^3 through 180 degrees (its cycle decomposition consists of three two-cycles). All 2^6 colorings of the vertices of the hexagon are fixed by the identity. Since a given vertex can be red or blue, the picture enumerator for a given vertex is $R+B$. By Problem 313 the enumerator for the colorings fixed by the identity is then $(R+B)^6$. Only the two constant colorings (that color every circle the same color) are fixed by ρ or ρ^5. The picture enumerator of each of the two constant colorings is R^6+B^6. If the vertices are numbered one through six clockwise, then the cycle decomposition of ρ^2 is $(1\ 3\ 5)(2\ 4\ 6)$. Thus for a coloring (S,f) to be fixed by ρ^2, $f(1)=f(3)=f(5)$ and $f(2)=f(4)=f(6)$. In other words, the color of vertex 1 is repeated on vertex 3 and 5, and the color of vertex 2 is repeated on vertices 4 and 6. Therefore the picture enumerator for colorings fixed by ρ^2 and ρ^4 is $R^3R^3+R^3B^3+B^3R^3+B^3B^3=(R^3+B^3)^2$. We can also think of this as the picture enumerator for colorings defined on the cycles of the permutation. The picture of assigning red to a three-cycle is R^3 and the picture of assigning blue to a three-cycle is B^3. Since there are two possible pictures of a colored three-cycle, R^3 and B^3, the picture enumerator for one three-cycle is R^3+B^3. By Problem 313, the picture enumerator for colorings defined on the two different cycles is then $(R^3+B^3)^2$. If a coloring (S,f) is fixed by ρ^3, which has the cycle decomposition $(1\ 4)(2\ 5)(3\ 6)$, then $f(1)=f(4)$, $f(2)=f(5)$, and $f(3)=f(6)$, so f is determined by the three values $f(1)$, $f(2)$, and $f(3)$. The picture enumerator for these colorings is, by Problem 313 applied to colorings of the cycles, $(R^2+B^2)^3$. Therefore the fixed point enumerator for the action of G on the colorings is

$$\mathrm{Fix}(G,S)=(R+B)^6+(R^2+B^2)^3+2(R^3+B^3)+2(R^6+B^6).$$

Then the orbit enumerator for the action of G on the colorings is

$$\mathrm{Orb}(G,S)=\frac{1}{6}\Big((R+B)^6+(R^2+B^2)^3+2(R^3+B^3)^2+2(R^6+B^6)\Big).$$

Expanding this gives us

$$R^6+R^5B+3R^4B^2+4R^3B^3+3R^2B^4+RB5+B^6.$$

Thus we have one all-red orbit, one all-blue orbit, one orbit with five reds and a blue, one with five blues and a red, three orbits with four reds and two blues as well as three with two reds and four blues, and four orbits with three reds and three blues. ▮

315. Find the generating function (in variables R, B) for colorings of the faces of a cube with two colors (red and blue). What does the generating function tell you about the number of ways to color the cube (up to spatial movement) with various combinations of the two colors?

Solution: We want to think of the rotation group of the cube acting on the faces of the cube, in order to see what kinds of colorings are left fixed. For this purpose we note that each element of the rotation group gives us a permutation of the faces of the cube, and if two faces are in the same cycle of this permutation, they must have the same color, but if they are in different cycles, they may get different colors. The 90 and 270 degree rotations around an axis through two faces have a four-cycle and two one-cycles. The 180 degree rotation around an axis through two faces has two two-cycles and two one-cycles. The 120 and 240 degree rotations around an axis connecting diagonally opposite vertices have two three-cycles. The 180 degree rotations around an axis connecting two opposite edges have three two-cycles, and the identity has six one-cycles. A coloring is fixed by a permutation if and only if it is constant on (i.e. assigns the same color to all elements of) each cycle of the permutation. The picture enumerator for (constant) colorings of an i-cycle is $R^i + B^i$. Thus, using Problem 313, if σ is a 90 or 270 degree rotation, its fixed point enumerator is $(R^4 + B^4)(R + B)^2$, if it is a 180 degree rotation around an axis connecting two opposite faces, its fixed point enumerator is $(R^2 + B^2)^2(R + B)^2$, if it is a 180 degree rotation around an axis connecting two opposite edges, its fixed point enumerator is $(R^2 + B^2)^3$, if it is a 120 or 240 degree rotation its fixed point enumerator is $(R^3 + B^3)^2$, and if it is the identity, then its fixed point enumerator is $(R + B)^6$. Therefore the generating function is

$$\frac{1}{24}((R+B)^6 + 8(R^3+B^3)^2 + 3(R^2+B^2)^2(R+B)^2 +$$
$$6(R^2+B^2)^3 + 6(R^4+B^4)(R+B)^2),$$

which expands to

$$R^6 + R^5B + 2R^4B^2 + 2R^3B^3 + 2R^2B^4 + RB^5 + B^6.$$

There is one way to color the cube all red or all blue, one way to color it with exactly five red or exactly five blue faces, there are two ways to color it with exactly four red or four blue faces and two ways to color it with exactly three red (and three blue) faces. ∎

6.3.2 The Pólya-Redfield Theorem

Pólya's (and Redfield's) famed enumeration theorem deals with situations such as those in Problems 314 and 315 in which we want a generating function for the set of all colorings a set S using a set T of colors, where the picture of a coloring is the product of the multiset of colors it uses. We are again thinking of the colors as variables. The point of the next series of problems is to analyze the solutions to Problems 314 and 315 in order to see what Pólya and Redfield saw (though they didn't see it in this notation or using this terminology).

•316. In Problem 314 we have four kinds of group elements: the identity (which fixes every coloring), the rotations through 60 or 300 degrees, the rotations through 120 and 240 degrees, and the rotation through 180 degrees. The fixed point enumerator for the rotation group acting on the colorings of the hexagon is by definition the sum of the fixed point enumerators of colorings fixed by the identity, of colorings fixed by 60 or 300 degree rotations, of colorings fixed by 120 or 240 degree rotations, and of colorings fixed by the 180 degree rotation. To the extent that you haven't already done it in an earlier problem, write down each of these enumerators (one for each kind of permutation) individually and factor each one (over the integers) as completely as you can.

Solution: In the solution to Problem 314 we wrote: "The picture enumerator of each of the two constant colorings is R^6+B^6. If the vertices are numbered one through six clockwise, then the cycle decomposition of ρ^2 is $(1\ 3\ 5)(2\ 4\ 6)$. Thus for a coloring (S,f) to be fixed by ρ^2, $f(1)=f(3)=f(5)$ and $f(2)=f(4)=f(6)$. In other words, the color of vertex 1 is repeated on vertex 3 and 5, and the color of vertex 2 is repeated on vertices 4 and 6. Therefore the picture enumerator for colorings fixed by ρ^2 and ρ^4 is $R^3R^3+R^3B^3+B^3R^3+B^3B^3=(R^3+B^3)^2$. We can also think of this as the picture enumerator for colorings defined on the cycles of the permutation. The picture of assigning red to a three-cycle is R^3 and the picture of assigning blue to a three-cycle is B^3. Since there are two possible pictures of a colored three-cycle,

R^3 and B^3, the picture enumerator for one three-cycle is $R^3 + B^3$. By Problem 313, the picture enumerator for colorings defined on the two different cycles is then $(R^3 + B^3)^2$. If a coloring (S, f) is fixed by ρ^3, which has the cycle decomposition $(1\ 4)(2\ 5)(3\ 6)$, then $f(1) = f(4)$, $f(2) = f(5)$, and $f(3) = f(6)$, so f is determined by the three values $f(1)$, $f(2)$, and $f(3)$. The picture enumerator for these colorings is, by Problem 313 applied to colorings of the cycles, $(R^2 + B^2)^3$." In factored form, these enumerators are $R^6 + B^6$, $(R + B)^6$, $(R^3 + B^3)^2$, and $(R^2 + B^2)^3$. ▮

317. In Problem 315 we have five different kinds of group elements. For each kind of element, to the extent that you haven't already done it in an earlier problem, write down the fixed point enumerator for the elements of that kind. Factor the enumerators as completely as you can.

Solution: In the solution to Problem 315, we wrote "Thus, using Problem 313, if σ is a 90 or 270 degree rotation, its fixed point enumerator is $(R^4 + B^4)(R + B)^2$, if it is a 180 degree rotation around an axis connecting two opposite faces, its fixed point enumerator is $(R^2 + B^2)^2(R + B)^2$, if it is a 180 degree rotation around an axis connecting two opposite edges, its fixed point enumerator is $(R^2 + B^2)^3$, if it is a 120 or 240 degree rotation its fixed point enumerator is $(R^3 + B^3)^2$, and if it is the identity, then its fixed point enumerator is $(R + B)^6$." We just wrote the enumerators out in factored form. ▮

•318. In Problem 316, each "kind" of group element has a "kind" of cycle structure. For example, a rotation through 180 degrees has three cycles of size two. What kind of cycle decomposition does a rotation through 60 or 300 degrees have? What kind of cycle decomposition does a rotation through 120 or 240 degrees have? Discuss the relationship between the cycle structures and the factored enumerators of fixed points of the permutations in Problem 316.

Solution: A rotation through 60 or 300 degrees is a five-cycle; a rotation through 120 or 240 degrees has two three-cycles. The cycle decomposition determines the factored enumerator; a cycle of size i gives a factor of $(R^i + B^i)$. That is because a coloring fixed by a group element has to be constant on the cycles of that group element. If a cycle has size i, it contributes a summand of P^i to the picture enumerator for colorings of that cycle for each picture P of a possible color. Problem 313 tells us to multiply these individual picture enumerators together. ▮

Recall that we said that a group of permutations acts on a set S if, for each member σ of G there is a permutation $\overline{\sigma}$ of S such that

$$\overline{\sigma \circ \varphi} = \overline{\sigma} \circ \overline{\varphi}$$

for all members σ and φ of G. Since $\overline{\sigma}$ is a permutation of S, $\overline{\sigma}$ has a cycle decomposition as a permutation of S (as well as whatever its cycle decomposition is in the original permutation group G).

319. In Problem 317, each "kind" of group element has a "kind" of cycle decomposition in the action of the rotation group of the cube on the faces of the cube. For example, a rotation of the cube through 180 degrees around a vertical axis through the centers of the top and bottom faces has two cycles of size two and two cycles of size one. To the extent that you haven't already done it in an earlier problem, answer the following questions. How many such rotations does the group have? What are the other "kinds" of group elements, and what are their cycle structures? Discuss the relationship between the cycle decomposition and the factored enumerator in Problem 317.

Solution: We effectively answered this question in our solution to Problem 315. In particular, there are three rotations of 180 degrees through the centers of opposite faces. The other kinds of group elements are as follows.

- the 90 and 270 degree rotations around an axis through two faces, of which we have six. Their cycle decomposition consists of a four-cycle and two one-cycles.
- the 120 and 240 degree rotations around an axis connecting two diagonally opposite vertices. Their cycle decomposition consists of two three-cycles. We have eight of these.
- the 180 degree rotations around an axis connecting two opposite edges; their cycle decomposition consists of three two-cycles. We have six of these.
- the identity, whose cycle decomposition is six one-cycles.

As we said in the solution to Problem 318, "The cycle decomposition determines the factored enumerator; a cycle of size i gives a factor of $(R^i + B^i)$. That is because a coloring fixed by a group element has to be constant on the cycles of that group element. If a cycle has size i, it contributes a summand of P^i to the picture enumerator for colorings of that cycle for each picture P of a possible color. Problem 313 tells us to multiply these individual picture enumerators together." ▮

•320. The usual way of describing the Pólya-Redfield enumeration theorem involves the "cycle indicator" or "cycle index" of a group acting on a set. Suppose we have a group G acting on a finite set S. Since each group element σ gives us a permutation $\overline{\sigma}$ of S, as such it has a decomposition into disjoint cycles as a permutation of S. Suppose σ has c_1 cycles of size 1, c_2 cycles of size 2, ..., c_n cycles of size n. Then the *cycle monomial* of σ is

$$z(\sigma) = z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}.$$

The **cycle indicator** or **cycle index** of G acting on S is

$$Z(G,S) = \frac{1}{|G|} \sum_{\sigma:\sigma\in G} z(\sigma).$$

•(a) What is the cycle index for the group D_6 acting on the vertices of a hexagon?

Solution: For D_6, we get

$$\frac{1}{12}\left(z_1^6 + 2z_6 + 2z_3^2 + z_2^3 + 3z_2^3 + 3z_2^2 z_1^2\right) = \frac{1}{12}\left(z_1^6 + 2z_6 + 2z_3^2 + 4z_2^3 + 3z_1^2 z_2^2\right)$$

▮

(b) What is the cycle index for the group of rotations of the cube acting on the faces of the cube?

Solution: For the rotation group of the cube, we get

$$\frac{1}{24}\left(3z_1^2 z_2^2 + 6z_1^2 z_4 + 8z_3^2 + 6z_3^2 + z_1^6\right) = \frac{1}{24}\left(3z_1^2 z_2^2 + 6z_1^2 z_4 + 14z_3^2 + z_1^6\right)$$

▮

➔•321. How can you compute the Orbit Enumerator of G acting on colorings of S by a finite set T of colors from the cycle index of G acting on S?

(Use t, thought of as a variable, as the picture of an element t of T.) State and prove the relevant theorem! This is Pólya's and Redfield's famous enumeration theorem.

Solution: The Pólya-Redfield Theorem states the following. To compute the orbit enumerator of G acting on functions from S to a finite set T, we substitute $\sum_{t:t\in T} t^i$ for z_i in the cycle index for G acting on S. Here is the proof. By the Orbit-Fixed Point Theorem, we need to sum the fixed point enumerators of the permutations in G. However, if σ fixes a coloring (S, f), then f is constant on the cycles of σ. If an i-cycle is colored by the member t of T, then the picture of the restriction of our coloring to that cycle is t^i. The picture enumerator for all colorings defined and constant on that cycle is then $\sum_{t:t\in T} t^i$. By Problem 313 the picture enumerator for all colorings constant on the cycles of σ, then, is the result of substituting $\sum_{t:t\in T} t^i$ for z_i in $z(\sigma)$. This proves the Pólya-Redfield Theorem. ▮

➔ 322. Suppose we make a necklace by stringing 12 pieces of brightly colored plastic tubing onto a string and fastening the ends of the string together. We have ample supplies of blue, green, red, and yellow tubing available. Give a generating function in which the coefficient of $B^iG^jR^kY^h$ is the number of necklaces we can make with i blues, j greens, k reds, and h yellows. How many terms would this generating function have if you expanded it in terms of powers of B, G, R, and Y? Does it make sense to do this expansion? How many of these necklaces have 3 blues, 3 greens, 2 reds, and 4 yellows?

Solution: We are asking for a generating function for the orbits of a colored 12-gon under the action of D_{12}. To apply the Pólya-Redfield theorem we need the cycle index for D_{12}. If ρ is a 30 degree rotation, then ρ, ρ^5, ρ^7 and ρ^{11} are 12-cycles. The elements ρ^2 and ρ^{10} have two six-cycles. The elements ρ^3 and ρ^9 have three four-cycles. The elements ρ^4 and ρ^8 have four three-cycles. The element ρ^6 has six two-cycles. The element $\iota = \rho^0$ has 12 one-cycles. There are six flips around axes through opposite vertices; each has five two-cycles and two one-cycles. There are six flips around axes perpendicular to two opposite sides; each has six two-cycles. Summarizing this in the cycle index, we write

$$\begin{aligned} Z(G,S) &= \frac{1}{24}\left(z_1^{12} + z_2^6 + 2z_3^4 + 2z_4^3 + 2z_6^2 + 4z_{12} + 6z_2^5z_1^2 + 6z_2^6\right) \\ &= \frac{1}{24}\left(z_1^{12} + 7z_2^6 + 2z_3^4 + 2z_4^3 + 2z_6^2 + 4z_{12} + 6z_2^5z_1^2\right). \end{aligned}$$

When we substitute $B^i + G^i + R^i + Y^i$ for z_i and expand, we would get 12^4 terms, one for each possible term $B^iG^jR^kY^h$. Thus it does not make sense to expand the polynomial. The unexpanded form is $\frac{1}{24}\Big((B+G+R+Y)^{12}+7(B^2+G^2+R^2+Y^2)^6+2(B^3+G^3+R^3+Y^3)^4+$ $2(B^4+G^4+R^4+Y^4)^3+2(B^6+G^6+R^6+Y^6)^2+$ $4(B^{12}+G^{12}+R^{12}+Y^{12})+6((B^2+G^2+R^2+Y^2)^5)(B+G+R+Y)^2\Big)$.

We can compute the coefficient of $B^3G^3R^2Y^4$ by computing the contribution of each term of the sum to the coefficient. We get

$$\begin{aligned}\frac{1}{24}\left(\binom{12}{3,3,2,4}+6\binom{5}{1,1,1,2}\binom{2}{1,1}\right) &= \frac{12!}{24\cdot 3!3!2!4!}+\frac{6\cdot 5!2!}{24\cdot 2!}\\ &= \frac{11!}{2\cdot 3!3!2!4!}+5\cdot 3\cdot 2\\ &= 11\cdot 5\cdot 3\cdot 2\cdot 7\cdot 5+30\\ &= 11,550+30=11,580.\end{aligned}$$

▮

➔ •323. What should we substitute for the variables representing colors in the orbit enumerator of G acting on the set of colorings of S by a set T of colors in order to compute the total number of orbits of G acting on the set of colorings? What should we substitute into the variables in the cycle index of a group G acting on a set S in order to compute the total number of orbits of G acting on the colorings of S by a set T? Find the number of ways to color the faces of a cube with four colors.

Solution: Substitute the number one for each color (variable) in the picture enumerator. Substitute $|T|$ for each variable in the cycle index. The cycle index for the rotation group of the cube acting on the faces is

$$\frac{1}{24}\left(3z_1^2z_2^2+6z_1^2z_4+14z_3^2+z_1^6\right).$$

Substituting 4 for each variable gives us $\frac{1}{24}\left(3\cdot 4^4+6\cdot 4^3+14\cdot 4^2+4^6\right)$ $=228$. ▮

➔ 324. We have red, green, and blue sticks all of the same length, with a dozen sticks of each color. We are going to make the skeleton of a cube by taking eight identical lumps of modeling clay and pushing three sticks into each lump so that the lumps become the vertices of the cube. (Clearly we won't need all the sticks!) In how many different ways could we make our cube? How many cubes have four edges of each color? How many have two red, four green, and six blue edges?

Solution: For this problem we are interested in the action of the rotation group of the cube on the edges. Now we think of the group elements as permutations of the edges and analyze their cycle structure.

- The identity has 12 one-cycles.
- A 90 or 270 degree rotation around an axis perpendicular to two opposite faces has three four-cycles.
- A 180 degree rotation around an axis perpendicular to two opposite faces has six two-cycles.
- A 180 degree rotation around an axis perpendicular to two opposite edges has five two-cycles and two one-cycles.
- A 120 degree rotation around an axis connecting two diagonally opposite vertices has four three-cycles.

Thus the cycle index is

$$\frac{1}{24}\left(z_1^{12} + 6z_4^3 + 3z_2^6 + 6z_2^5 z_1^2 + 8z_3^4\right).$$

We substitute the number three for each of the variables to get

$$\frac{1}{24}\left(3^{12} + 6\cdot 3^3 + 3\cdot 3^6 + 6\cdot 3^7 + 8\cdot 3^4\right) = 22815$$

ways to make the cube. To compute the number of ways to make the cube with four sticks of each color, we need to apply the Pólya-Redfield theorem. Substituting $R^i + B^i + G^i$ for z_i in the cycle index gives us

$$\tfrac{1}{24}\Big((R+B+G)^{12} + (R^4+B^4+G^4)^3 + 3(R^2+B^2+G^2)^6 +$$
$$6(R^2+B^2+B^2)^5(R+B+G)^2 + 8(R^3+B^3+G^3)^4\Big).$$

The coefficient of $R^4B^4G^4$ is

$$\begin{aligned}\frac{1}{24}\left(\binom{12}{4,4,4} + 6\binom{3}{1,1,1} + 3\binom{6}{2,2,2} + 6\binom{3}{1}\binom{5}{2,2,1}\binom{2}{2,0,0}\right)\\ = \frac{1}{24}\left(\frac{12!}{4!4!4!} + 6\cdot 3! + 3\frac{6!}{2!2!2!} + 6\cdot 3\frac{5!}{2!2!1!}\right) = 1479.\end{aligned}$$

The coefficient of $R^2B^4G^6$ is

$$\frac{1}{24}\left(\binom{12}{2,4,6}+3\binom{6}{1,2,3}+6\left(\binom{5}{0,2,3}+\binom{5}{1,1,3}+\binom{5}{1,2,2}\right)\right)$$
$$=\frac{1}{24}\left(\frac{12!}{2!4!6!}+3\frac{6!}{1!2!3!}+6\left(\frac{5!}{2!3!}+\frac{5!}{1!1!3!}+\frac{5!}{1!2!2!}\right)\right)=600.$$

▮

➔325. How many cubes can we make in Problem 324 if the lumps of modeling clay can be any of four colors?

Solution: We can first consider all ways of coloring the vertices of the cube with the four colors; once those vertices are in place, the number of ways to put the sticks in is the result of Problem 324, so by the product principle the number of ways to choose the colors of the vertices and edges is the product of the number of ways to choose each. Thus we just need to consider the action of the rotation group of the cube on the vertices:

- The identity is the product of eight one-cycles.
- A 90 or 270 degree rotation around an axis perpendicular to two opposite faces has two four-cycles.
- A 180 degree rotation around an axis perpendicular to two opposite faces has four two-cycles.
- A 180 degree rotation around an axis joining two opposite edges has four two-cycles.
- A 120 or 240 degree rotation around an axis through two diagonally opposite vertices has two three-cycles and two one-cycles.

Thus the cycle index for this action is

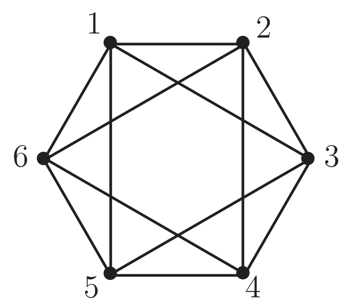
$$\frac{1}{24}\left(z_1^8+6z_4^2+9z_2^4+8z_3^2z_1^2\right).$$

Substituting 4 for each variable gives us

$$\frac{1}{24}\left(4^8+6\cdot 4^2+9\cdot 4^4+8\cdot 4^2\cdot 4^2\right)=2916.$$

Thus we have $22815\cdot 2916=66,528,540$ possible colored cubes. ▮

Figure 6.9: A possible computer network.



➔ $*326$. In Figure 6.9 we see a graph with six vertices. Suppose we have three different kinds of computers that can be placed at the six vertices of the graph to form a network. In how many different ways may the computers be placed? (Two graphs are not different if we can redraw one of the graphs so that it is identical to the other one.) This is equivalent to permuting the vertices in some way so that when we apply the permutation to the endpoints of the edges to get a new edge set, the new edge set is equal to the old one. Such a permutation is called an *automorphism* of the graph. Then two computer placements are the same if there is an automorphism of the graph that carries one to the other.

Solution: The computer placements are colorings of the vertices of the graph by the set of three kinds of computers, say $\{A, B, C\}$. Thus we are asking for the number of orbits of the automorphism group of the graph on colorings of the vertices with the colors $\{A, B, C\}$. To find this number we need to compute the cycle index of the automorphism group. An automorphism will send vertices 1, 2, and 3 to three vertices that are mutually connected; i.e. a triangle. Further, each of vertices 4, 5, and 6 is adjacent to exactly two of vertices of 1, 2, and 3. In fact, for any of the eight triangles in the graph, each vertex not in the triangle is adjacent to exactly two vertices of the triangle (and a different two for each vertex). Therefore we can send the vertices 1, 2, and 3, to any of the eight triangles in any of six orders, and this completely determines an automorphism. Thus there are $6 \cdot 8 = 48$ elements in the group of automorphisms. The dihedral group D_6 is a subgroup of the group of automorphisms. The permutations with four one-cycles and the two-cycle (1 4), (2 5), or (3 6) are also in the group of automorphisms. (For example, 1 is adjacent to everything but 4, and 4 is adjacent to everything but 1, so interchanging them leaves us with exactly the same edges.) We use the notation $(i\ i)$ not only to stand for a two-cycle, but also to stand for the permutation with four one-cycles and the two-cycle $(i,\ j)$. We will write out the cycle decomposition of each element of D_6, and then do the same for the set $(1\ 4)D_6 = \{(1\ 4)\sigma | \sigma \in D_6\}$, and from those we will be able to get the cycle index of the automorphism group acting on the vertices. The cycle decompositions of the elements of D_6 are

(1 2 3 4 5 6)	(1 3 5)(2 4 6)	(1 4)(2 5)(3 6)	(1 5 3)(2 6 4)
(1 6 5 4 3 2)	(1)(2)(3)(4)(5)(6)	(1)(4)(2 6)(3 5)	(2)(5)(1 3)(4 6)
(3)(6)(1 5)(2 4)	(1 2)(3 6)(4 5)	(1 6)(2 5)(3 4)	(1 4)(2 3)(5 6)

and the cycle decompositions of the elements of $(1\ 4)D_6$ are

(1 2 3)(4 5 6)	(1 3 5 4 6 2)	(1)(4)(2 5)(3 6)	(1 5 3 4 2 6)
(1 6 5)(2 4 3)	(1 4)(2)(3)(5)(6)	(1 4)(2 6)(3 5)	(2)(5)(1 3 4 6)
(3)(6)(1 5 4 2)	(1 2 4 5)(3 6)	(1 6 4 3)(2 5)	(1)(4)(2 3)(5 6).

Notice that neither (2 5) nor (3 6) is in either of our sets. From this, it is a straightforward series of steps to show that our group is the union

$$D_6 \cup (1\ 4)D_6 \cup (2\ 5)D_6 \cup (3\ 6)D_6.$$

But by symmetry, the cycle decomposition of $(1\ 4)D_6$, $(2\ 5)D_6$ and $(3\ 6)D_6$ will be the same, so the cycle index for our group is

$$\frac{1}{48}\left(Z_1^6 + 8Z_3^2 + 8z_6^1 + 7z_2^3 + 9z_1^2z_2^2 + 3z_1^4z_2 + 6z_2z_4 + 6z_1^2z_4\right).$$

Since we have three kinds of computers, we substitute 3 for each variable to get

$$\frac{1}{48}\left(3^6 + 8\cdot 3^2 + 8\cdot 3^1 + 7\cdot 3^3 + 9\cdot 3^4 + 3\cdot 3^5 + 6\cdot 3^2 + 6\cdot 3^3\right) = 56.$$

∎

➔ 327. Two simple graphs on the set $[n] = \{1, 2, \ldots, n\}$ with edge sets E and E' (which we think of as sets of two-element sets for this problem) are said to be *isomorphic* if there is a permutation σ of $[n]$ which, in its action of two-element sets, carries E to E'. We say two graphs are different if they are not isomorphic. Thus the number of different graphs is the number of orbits of the set of all sets of two-element subsets of $[n]$ under the action of the group S_n. We can represent an edge set by its characteristic function (as in problem 33). That is, we define

$$\chi_E(\{u,v\}) = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Thus we can think of the set of graphs as a set of colorings with colors 0 and 1 of the set of all two-element subsets of $[n]$. The number of different graphs with vertex set $[n]$ is thus the number of orbits of this set of colorings under the action of the symmetric group S_n on the set of two-element subsets of $[n]$. Use this to find the number of different graphs on five vertices.

Solution: For this problem we need the cycle index for the action of the symmetric group on five letters acting on two-element subsets of those five letters. Each way of partitioning the number five describes the cycle decomposition of an element of S_5 acting on [5]. The cycle decomposition of a permutation σ on the two-element subsets of [5] will be determined by its cycle decomposition on [5]. The partitions of five and the cycle structures they give on two-element subsets are:

- $(1,1,1,1,1) = 1^5$ is the cycle structure of the identity; acting on two-element subsets the identity has $\binom{5}{2} = 10$ one-cycles. Of course there is only one identity permutation.
- A permutation with cycle structure of the form $2^1 1^3$ will fix the two element set in the two-cycle and each pair of the three elements outside for a total of four one-cycles; each pair of an element in the two-cycle and an element not in the two-cycle will be in a two-cycle. There are 6 such pairs and thus three two-cycles, so such a permutation has 3 two-cycles and four one-cycles. There are $\binom{5}{2} = 10$ such permutations.
- A permutation with cycle structure of the form $2^2 1^1$ will have a one-cycle of two-sets for each two-cycle, any other two-set will be in a two-cycle. There are $4 + 2 + 2 = 8$ such two-sets and thus four two-cycles, so such a permutation has four two-cycles and two one-cycles. There are $\binom{5}{2}\binom{3}{2}/2 = 15$ such permutations.
- A permutation with cycle structure of the form $3^1 1^2$ will have one cycle of size 1 on two-sets from the two one-element cycles of the

original action; each other two-element subset will be in a three-cycle. There are $\binom{3}{2} + 3 \cdot 2 = 9$ such pairs, and so there are three three-cycles of two-element subsets. Thus such a permutation has one one-cycle and three three-cycles in its action of two-sets. There are $2\binom{5}{3} = 20$ such permutations.

- A permutation with cycle structure of the form $3^1 2^1$ will have one one-cycle on two-sets from the two-cycle of the original action, it will have $\binom{3}{2}$ two-sets in three-cycles, and will have six two-sets in six-cycles, and so will have just one six-cycle of two-sets. Thus such a permutation has one one-cycle, one three-cycle and one six-cycle when it acts on two-sets. There are $2\binom{5}{3} = 20$ such permutations.
- A permutation with cycle structure of the form $4^1 1^1$ has a four-cycle of two-subsets and one two-cycle of two-subsets from its own four-cycle and another four-cycle of pairs of members of the four-cycle and one-cycle. Thus it has two four-cycles and two one-cycles. There are $6\binom{5}{4} = 30$ such permutations.
- A permutation with cycle structure of the form 5^1 will have two five-cycles in its action on two-sets. There are $4! = 24$ such permutations.

Thus for the group S_5 acting on pairs from [5], the cycle index is

$$\frac{1}{120}\left(z_1^{10} + 10z_2^3 z_1^4 + 15z_2^4 z_2 + 20z_3^3 z_1 + 20z_6 z_3 z_1 + 30z_4^2 z_2 + 24z_5^2\right).$$

Substituting 2 for each variable gives us that there are

$$\frac{1}{120}\left(2^{10} + 10 \cdot 2^7 + 15 \cdot 2^5 + 20 \cdot 2^4 + 20 \cdot 2^3 + 30 \cdot 2^3 + 24 \cdot 2^2\right) = 30$$

graphs on five vertices. ▮

6.4 Supplementary Problems

1. Show that a function from S to T has an inverse (defined on T) if and only if it is a bijection.

2. How many elements are in the dihedral group D_3? The symmetric group S_3? What can you conclude about D_3 and S_3?

   Solution: Six, six. D_3 and S_3 are the same group. ▮

3. A tetrahedron is a three-dimensional geometric figure with four vertices, six edges, and four triangular faces. Suppose we start with a tetrahedron in space and consider the set of all permutations of the vertices of the tetrahedron that correspond to moving the tetrahedron in space and returning it to its original location, perhaps with the vertices in different places.

   (a) Explain why these permutations form a group.

   (b) What is the size of this group?

   (c) Write down in two row notation a permutation that is *not* in this group.

4. Find a three-element subgroup of the group S_3. Can you find a different three-element subgroup of S_3?

   Solution: $\{\iota, (1\ 2\ 3), (1\ 3\ 2)$. This is the only three-element subgroup because the other elements of S_3 are two cycles, so one of them forms a two-element subgroup with ι, and two of them together with ι are not a subgroup. Since a subgroup acts on S_3 by composition, any subgroup must have 1, 2, 3, or 6 elements, because S_3 is a union of orbits of that group and the orbits all have the same size. ▮

5. Prove true or demonstrate false with a counterexample: "In a permutation group, $(\sigma\varphi)^n = \sigma^n\varphi^n$."

6. If a group G acts on a set S, and if $\sigma(x) = y$, is there anything interesting we can say about the subgroups $\text{Fix}(x)$ and $\text{Fix}(y)$?

   Solution: They have the same size; in fact
   $\sigma\text{Fix}(x) = \{\sigma \circ \tau | \tau \in \text{Fix}(x)\} = \text{Fix}(y)$ ▮

7. (a) If a group G acts on a set S, does $\overline{\sigma}(f) = f \circ \sigma$ define a group action on the functions from S to a set T? Why or why not?

   Solution: No, because $\overline{\sigma\tau}(f) = f \circ \sigma \circ \tau$, but $\overline{\sigma} \circ \overline{\tau}(f) = f \circ \tau \circ \sigma$. Thus if our group is not commutative, this is not a group action. ▮

   (b) If a group G acts on a set S, does $\overline{\sigma}(f) = f \circ \sigma^{-1}$ define a group action on the functions from S to a set T? Why or why not?

   Solution: Yes, because the action gives a permutation of the functions and $\overline{\sigma\tau}(f) = f \circ (\sigma \circ \tau)^{-1} = f \circ \tau^{-1} \circ \sigma^{-1}$, while $\overline{\sigma} \circ \overline{\tau}(f) = f \circ \tau^{-1} \circ \sigma^{-1}$. Thus this is a group action. ▮

(c) Is either of the possible group actions essentially the same as the action we described on colorings of a set, or is that an entirely different action?

Solution: The action proposed in part (b) is the same as our action on colorings. To see why, if

$$\begin{aligned} & \{(1, g(1), (2, g(2), \ldots, (n, g(n))\} \\ = & \{(\sigma(1), f(1)), (\sigma(2, f(2)), \ldots, (\sigma(n) f(n))\}, \end{aligned}$$

and $i = \sigma(j)$, then $j = \sigma^{-1}(i)$ and so

$$(i, g(i)) = (\sigma(j), f(j)) = (\sigma(\sigma^{-1}(i)), f(\sigma^{-1}(i)) = (i, f(\sigma^{-1}(i)).$$

Therefore, $g = f \circ \sigma^{-1}$. ∎

8. Find the number of ways to color the faces of a tetrahedron with two colors.

9. Find the number of ways to color the faces of a tetrahedron with four colors so that each color is used.

10. Find the cycle index of the group of spatial symmetries of the tetrahedron acting on the vertices. Find the cycle index for the same group acting on the faces.

11. Find the generating function for the number of ways to color the faces of the tetrahedron with red, blue, green and yellow.

➔ 12. Find the generating function for the number of ways to color the faces of a cube with four colors so that all four colors are used.

➔ 13. How many different graphs are there on six vertices with seven edges?

➔ 14. Show that if H is a subgroup of the group G, then H acts on G by $\overline{\sigma}(\tau) = \sigma \circ \tau$ for all σ in H and τ in G. What is the size of an orbit of this action? How does the size of a subgroup of a group relate to the size of the group?

Solution: Composition of the elements of a permutation group on the left by σ permutes the elements of the group, so $\overline{\sigma}$ is a permutation of G. The size of an orbit is the size of the subgroup, because if $\sigma \circ \tau_1 = \sigma \circ \tau_2$, then $\tau_1 = \tau_2$. Since G is the union of the orbits of H and these orbits all have the same size, by the quotient principle the size of H divides the size of G. ∎

Appendix A

Relations

A.1 Relations as Sets of Ordered Pairs

A.1.1 The relation of a function

328. Consider the functions from $S = \{-2, -1, 0, 1, 2\}$ to $T = \{1, 2, 3, 4, 5\}$ defined by $f(x) = x + 3$, and $g(x) = x^5 - 5x^3 + 5x + 3$. Write down the set of ordered pairs $(x, f(x))$ for $x \in S$ and the set of ordered pairs $(x, g(x))$ for $x \in S$. Are the two functions the same or different?

     Solution: We get $\{(-2, 1), (-1, 2), (0, 3), (1, 4), (2, 5)\}$ in both cases, so the functions are the same. ▮

Problem 328 points out how two functions which appear to be different are actually the same on some domain of interest to us. Most of the time when we are thinking about functions it is fine to think of a function casually as a relationship between two sets. In Problem 328 the set of ordered pairs you wrote down for each function is called the *relation* of the function. When we want to distinguish between the casual and the careful in talking about relationships, our casual term will be "relationship" and our careful term will be "relation." So *relation* is a technical word in mathematics, and as such it has a technical definition. A *relation* from a set S to a set T is a set of ordered pairs whose first elements are in S and whose second elements are in T. Another way to say this is that a *relation* from S to T is a subset of $S \times T$.

A typical way to define a *function* f from a set S, called the *domain* of the function, to a set T, called the *range*, is that f is a relationship from S to T that relates one and only one member of T to each element of X. We use $f(x)$ to stand for the element of T that is related to the element x of

S. If we wanted to make our definition more precise, we could substitute the word "relation" for the word "relationship" and we would have a more precise definition. For our purposes, you can choose whichever definition you prefer. However, in any case, there is a relation associated with each function. As we said above, the relation of a function $f : S \to T$ (which is the standard shorthand for "f is a function from S to T" and is usually read as f *maps* S to T) is the set of all ordered pairs $(x, f(x))$ such that x is in S.

329. Here are some questions that will help you get used to the formal idea of a relation and the related formal idea of a function. S will stand for a finite set of size s and T will stand for a finite set of size t.

   (a) What is the size of the largest relation from S to T?

   Solution: st because that is the size of the relation that has all the ordered pairs (x, y) with $x \in S$ and $y \in T$. ▮

   (b) What is the size of the smallest relation from S to T?

   Solution: 0, because the empty set of ordered pairs is a relation. ▮

   (c) The relation of a function $f : S \to T$ is the set of all ordered pairs $(x, f(x))$ with $x \in S$. What is the size of the relation of a function from S to T? That is, how many ordered pairs are in the relation of a function from S to T?

   Solution: The size of the relation of $f : S \to T$ is s. ▮

   (d) We say f is a *one-to-one*[1] function or *injection* from S to T if each member of S is related to a *different* element of T. How many different elements must appear as second elements of the ordered pairs in the relation of a one-to-one function (injection) from S to T?

   Solution: s different elements must appear, one for each element of S. ▮

   (e) A function $f : S \to T$ is called an *onto function* or *surjection* if each element of T is $f(x)$ for some $x \in S$. What is the minimum size that S can have if there is a surjection from S to T?

   Solution: In order to have a surjection from S to T, the size of S must be at least t. ▮

[1] The phrase one-to-one is sometimes easier to understand when one compares it to the phrase many-to-one. John Fraliegh, an author of popular textbooks in abstract and linear algebra, suggests that two-to-two might be a better name that one-to-one.

330. When f is a function from S to T, the sets S and T play a big role in determining whether a function is one-to-one or onto (as defined in Problem 329). For the remainder of this problem, let S and T stand for the set of nonnegative real numbers.

   (a) If $f : S \to T$ is given by $f(x) = x^2$, is f one-to-one? Is f onto?

   Solution: With S as domain and T as range, f is both one-to-one and onto. ▮

   (b) Now assume for the rest of the problem that S' is the set of all real numbers and $g : S' \to T$ is given by $g(x) = x^2$. Is g one-to-one? Is g onto?

   Solution: The function g is not one-to-one, but it is onto. ▮

   (c) Assume for the rest of the problem that T' is the set of all real numbers and $h : S \to T'$ is given by $h(x) = x^2$. Is h one-to-one? Is h onto?

   Solution: The function h is one-to-one but not onto. ▮

   (d) And if the function $j : S' \to T'$ is given by $j(x) = x^2$, is j one-to-one? Is j onto?

   Solution: The function j is neither one-to-one nor onto. ▮

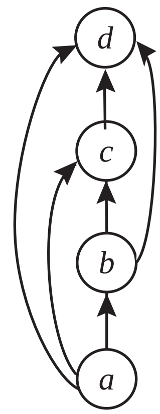
331. If $f : S \to T$ is a function, we say that f*maps* x to y as another way to say that $f(x) = y$. Suppose $S = T = \{1, 2, 3\}$. Give a function from S to T that is not onto. Notice that two different members of S have mapped to the same element of T. Thus when we say that f associates one and only one element of T to each element of S, it is quite possible that the one and only one element $f(1)$ that f maps 1 to is exactly the same as the one and only one element $f(2)$ that f maps 2 to.

   Solution: The function given by $f(x) = 1$ for all x in S is not onto. ▮

A.1.2 Directed graphs

We visualize numerical functions like $f(x) = x^2$ with their graphs in Cartesian coordinate systems. We will call these kinds of graphs *coordinate graphs* to distinguish them from other kinds of graphs used to visualize relations that are non-numerical. In Figure A.1 we illustrate another kind of graph, a "directed graph" or "digraph" of the "comes before in alphabetical order" relation on the letters a, b, c, and d. To draw a *directed graph* of a relation

Figure A.1: The alphabet digraph.

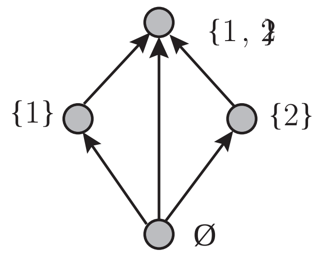


on a finite[2] set S, we draw a circle (or dot, if we prefer), which we call a *vertex*, for each element of the set, we usually label the vertex with the set element it corresponds to, and we draw an arrow from the vertex for a to that for b if a is related to b, that is, if the ordered pair (a, b) is in our relation. We call such an arrow an *edge* or a *directed edge*. We draw the arrow from a to b, for example, because a comes before b in alphabetical order. We try to choose the locations where we draw our vertices so that the arrows capture what we are trying to illustrate as well as possible. Sometimes this entails redrawing our directed graph several times until we think the arrows capture the relationship well.

332. Draw the digraph of the "is a proper subset of" relation on the set of subsets of a two element set. How many arrows would you have had to draw if this problem asked you to draw the digraph for the subsets of a three-element set?

    Solution:

[2]We could imagine a digraph on an infinite set, but we could never draw all the vertices and edges, so people sometimes speak of digraphs on infinite sets. One just has to be more careful with the definition to make sure it makes sense for infinite sets.

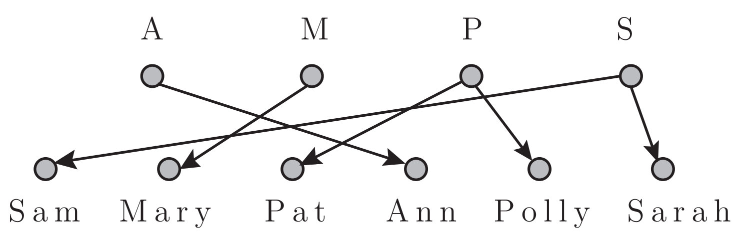


We would need to draw 19 arrows, some of them curved, for the subsets of a three-element set. ▮

We also draw digraphs for relations from a finite set S to a finite set T; we simply draw vertices for the elements of S (usually in a row) and vertices for the elements of T (usually in a parallel row) and draw an arrow from x in S to y in T if x is related to y. Notice that instead of referring to the vertex representing x, we simply referred to x. This is a common shorthand.

333. Draw the digraph of the relation from the set {A, M, P, S} to the set {Sam, Mary, Pat, Ann, Polly, Sarah} given by "is the first letter of."

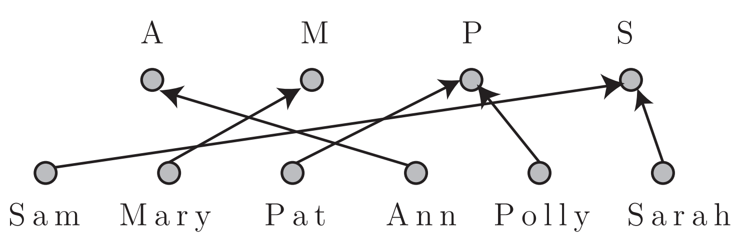
Solution:



▮

334. Draw the digraph of the relation from the set {Sam, Mary, Pat, Ann, Polly, Sarah} to the set {A, M, P, S} given by "has as its first letter."

Solution:



▮

335. Draw the digraph of the relation on the set {Sam, Mary, Pat, Ann, Polly, Sarah} given by "has the same first letter as."

Solution:

▮

A.1.3 Digraphs of Functions

336. When we draw the digraph of a function f, we draw an arrow *from* the vertex representing x *to* the vertex representing $f(x)$. One of the relations you considered in Problems 333 and 334 is the relation of a function.

(a) Which relation is the relation of a function?

Solution: It is the relation of Problem 334. ▮

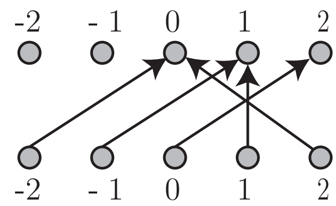
(b) How does the digraph help you visualize that one relation is a function and the other is not?

Solution: In Problem 334, each vertex representing a name has just one arrow leaving it. This means exactly one letter is related to each name, so this relation is the relation of a function from the set of names to the set of letters. In Problem 333, the vertex P has two arrows leaving it. Therefore two elements are associated with the letter P, so this relation is not the relation of a function from the set of letters to the set of names. ▮

337. Digraphs of functions help us to visualize whether or not they are onto or one-to-one. For example, let both S and T be the set $\{-2, -1, 0, 1, 2\}$ and let S' and T' be the set $\{0, 1, 2\}$. Let $f(x) = 2 - |x|$.

(a) Draw the digraph of the function f assuming its domain is S and its range is T. Use the digraph to explain why or why not this function maps S onto T.

Solution:



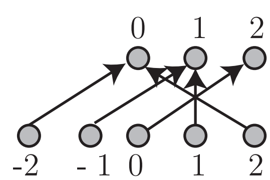
The function is not onto because there are vertices in the range that do not have an arrow going to them. ▮

(b) Use the digraph of the previous part to explain whether or not the function is one-to one.

Solution: The function is not one-to-one because there are two arrows entering vertex 0 and two arrows entering vertex 1. ▮

(c) Draw the digraph of the function f assuming its domain is S and its range is T'. Use the digraph to explain whether or not the function is onto.

Solution:



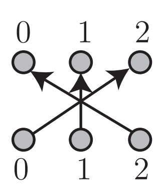
The function is onto because each vertex in the range that has at least one arrow going to it. ▮

(d) Use the digraph of the previous part to explain whether or not the function is one-to-one.

Solution: The function is not one-to-one because there are two arrows entering vertex 0 and two arrows entering vertex 1. ▮

(e) Draw the digraph of the function f assuming its domain is S' and its range is T'. Use the digraph to explain whether the function is onto.

Solution:



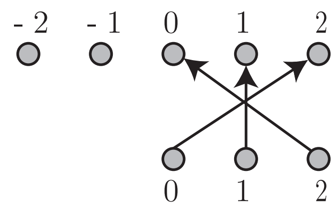
The function is onto because each vertex in the range has an arrow entering it. ▮

(f) Use the digraph of the previous part to explain whether the function is one-to-one.

Solution: The function is one-to-one because each element of the range has at most one arrow entering it. ▮

(g) Suppose that the function f has domain S' and range T. Draw the digraph of f and use it to explain whether f is onto.

Solution:



The function is not onto because there are vertices in the range that do not have an arrow entering them. ▮

(h) Use the digraph of the previous part to explain whether f is one-to-one.

Solution: f is one-to-one because each vertex of T has at most one arrow entering it. ▮

A one-to-one function from a set X onto a set Y is frequently called a *bijection*, especially in combinatorics. Your work in Problem 337 should show you that a digraph is the digraph of a bijection from X to Y

- if the vertices of the digraph represent the elements of X and Y,
- if each vertex representing an element of X has one ond only one arrow leaving it, and
- each vertex representing an element of Y has one and only one arrow entering it.

338. If we reverse all the arrows in the digraph of a bijection f, we get the digraph of another function g. Is g a bijection? What is $f(g(x))$? What is $g(f(x))$?

Solution: g is a bijection, because each vertex representing an element of Y has one arrow leaving it and each vertex representing an element of X has one arrow entering it. For both of the next questions, the answer is x. ∎

If f is a function from S to T, if g is a function from T to S, and if $f(g(x)) = x$ for each x in T and $g(f(x)) = x$ for each x in S, then we say that g is an inverse of f (and f is an inverse of g).

More generally, if f is a function from a set R to a set S, and g is a function from S to T, then we define a new function $f \circ g$, called the *composition* of f and g, by $f \circ g(x) = f(g(x))$. Composition of functions is a particularly important operation in subjects such as calculus, where we represent a function like $h(x) = \sqrt{x^2+1}$ as the composition of the square root function and the square and add one function in order to use the chain rule to take the derivative of h.

The function ι (the Greek letter iota is pronounced eye-oh-ta) from a set S to itself, given by the rule $\iota(x) = x$ for every x in S, is called the *identity function* on S. If f is a function from S to T and g is a function from T to S such that $g(f(x)) = x$ for every x in S, we can express this by saying that $g \circ f = \iota$, where ι is the identity function of S. Saying that $f(g(x)) = x$ is the same as saying that $f \circ g = \iota$, where now ι stands for the identity function on T. We use the same letter for the identity function on two different sets when we can use context to tell us on which set the identity function is being defined.

339. If f is a function from S to T and g is a function from T to S such that $g(f(x)) = x$, how can we tell from context that $g \circ f$ is the identity function on S and not the identity function on T?

     Solution: Since f has S as its domain, $g \circ f$ must have S as its domain as well, so that it is possible to compute $f(x))$ in order to compute $g(f(x))$. Thus $g \circ f$ is the identity function on S. ∎

340. Explain why a function that has an inverse must be a bijection.

     Solution: Suppose the function f from S to T has an inverse g. If $f(x) = f(y)$, then $g(f(x)) = g(f(y))$ and so $x = y$. Therefore f is one-to-one. If y is in T, then $f(g(y)) = y$, and so $g(y)$ is an element x of S such that $f(x) = y$. Therefore f is a bijection. ∎

341. Is it true that the inverse of a bijection is a bijection?

     Solution: If f is the inverse of g, then g is the inverse of f (because the definition is symmetric in f and g). Thus the inverse of a bijection has an inverse and therefore by Problem 340 it is a bijection as well. ∎

342. If g and h are inverses of f, then what can we say about g and h?

Solution: They must be equal. To see why, notice first that each element y of T must be $f(x)$ for some x in S, because f is a bijection. But then since $h(f(x)) = x = g(f(x))$, we have $h(y) = g(y)$ for every y in T. ▮

343. Explain why a bijection must have an inverse.

Solution: From the point of view of digraphs, if f is a bijection, we can get the digraph of an inverse by reversing all the arrows in the digraph of f. If you prefer to use the definitions directly, suppose f is a bijection. For each y in T, let $g(y)$ be the x such that $f(x) = y$. (There must be such an x because f is a bijection.) Then $g(f(x)) = x$ by definition. To compute $f(g(y))$, we have to take $f(x)$, where x is the unique element of S such that $f(x) = y$. Thus $f(g(y)) = y$. Therefore g is an inverse to f. ▮

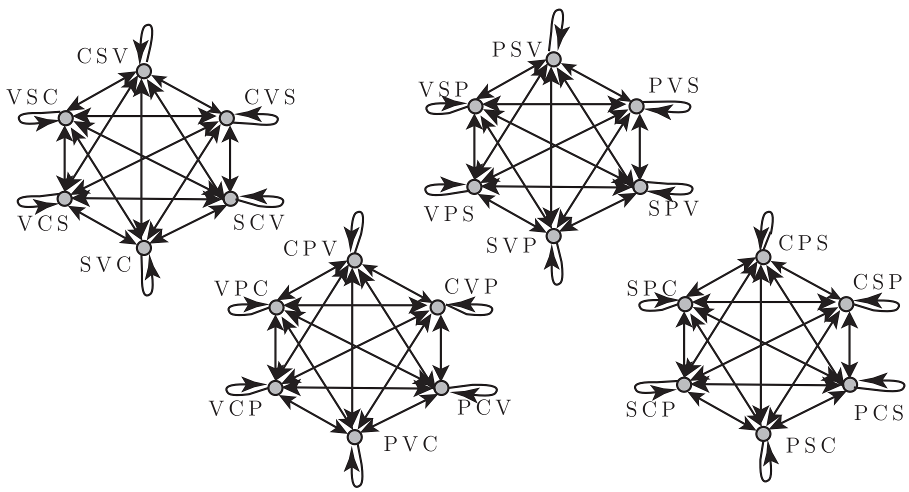
Since a function with an inverse has exactly one inverse g, we call g *the* inverse of f. From now on, when f has an inverse, we shall denote its inverse by f^{-1}. Thus $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Equivalently $f \circ f^{-1} = \iota$ and $f^{-1} \circ f = \iota$.

A.2 Equivalence Relations

So far we've used relations primarily to talk about functions. There is another kind of relation, called an equivalence relation, that comes up in the counting problems with which we began. In Problem 8 with three distinct flavors, it was probably tempting to say there are 12 flavors for the first pint, 11 for the second, and 10 for the third, so there are $12 \cdot 11 \cdot 10$ ways to choose the pints of ice cream. However, once the pints have been chosen, bought, and put into a bag, there is no way to tell which is first, which is second and which is third. What we just counted is lists of three distinct flavors—one-to-one functions from the set $\{1, 2, 3\}$ in to the set of ice cream flavors. Two of those lists become equivalent once the ice cream purchase is made if they list the same ice cream. In other words, two of those lists become equivalent (are related) if they list same subset of the set of ice cream flavors. To visualize this relation with a digraph, we would need one vertex for each of the $12 \cdot 11 \cdot 10$ lists. Even with five flavors of ice cream, we would need one vertex for each of $5 \cdot 4 \cdot 3 = 60$ lists. So for now we will work with the easier to draw question of choosing three pints of ice cream of different flavors from four flavors of ice cream.

344. Suppose we have four flavors of ice cream, V(anilla), C(hocolate), S(trawberry) and P(each). Draw the directed graph whose vertices consist of all lists of three distinct flavors of the ice cream, and whose edges connect two lists if they list the same three flavors. This graph makes it pretty clear in how many "really different" ways we may choose 3 flavors out of four. How many is it?

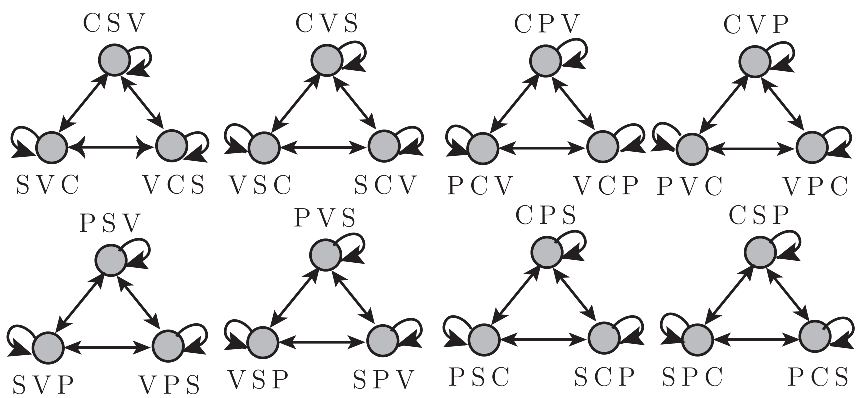
Solution:



We used double-headed arrows in place of one arrow going in each direction to reduce the clutter in the picture. Note that there is an arrow from each vertex to itself. We may choose 3 flavors in four ways. ▮

➔ 345. Now suppose again we are choosing three distinct flavors of ice cream out of four, but instead of putting scoops in a cone or choosing pints, we are going to have the three scoops arranged symmetrically in a circular dish. Similarly to choosing three pints, we can describe a selection of ice cream in terms of which one goes in the dish first, which one goes in second (say to the right of the first), and which one goes in third (say to the right of the second scoop, which makes it to the left of the first scoop). But again, two of these lists will sometimes be equivalent. Once they are in the dish, we can't tell which one went in first. However, there is a subtle difference between putting each flavor in its own small dish and putting all three flavors in a circle in a larger dish. Think about what makes the lists of flavors equivalent, and draw the directed graph whose vertices consist of all lists of three of the flavors of ice cream and whose edges connect two lists between which we cannot distinguish as dishes of ice cream. How many dishes of ice cream can we distinguish from one another?

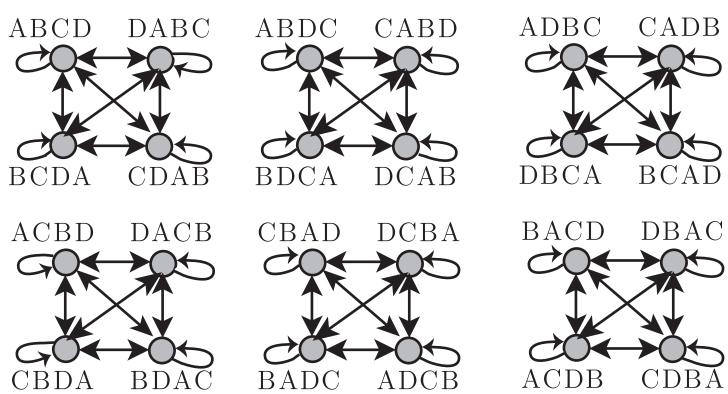
Solution:



We can distinguish eight different dishes of ice cream. ▮

346. Draw the digraph for Problem 38 in the special case where we have four people sitting around the table.

Solution:



▮

In Problems 344, 345, and 346 (as well as Problems 34c, 38, and 39d) we can begin with a set of lists, and say when two lists are equivalent as representations of the objects we are trying to count. In particular, in Problems 344, 345, and 346 you drew the directed graph for this relation of equivalence. Your digraph had an arrow from each vertex (list) to itself (or else you want to go back and give it these arrows). This is what we mean when we say a relation is *reflexive*. Whenever you had an arrow from one vertex to a second, you had an arrow from the second back to the first. This is what we mean when we say a relation is *symmetric*.

When people sit around a round table, each list is equivalent to itself: if List1 and List2 are identical, then everyone has the same person to the right in both lists (including the first person in the list being to the right of the last person). To see the symmetric property of the equivalence of seating arrangements, if List1 and List2 are different, but everyone has the same person to the right when they sit according to List2 as when they sit according to List1, then everybody better have the same person to the right when they sit according to List1 as when they sit according to List2.

In Problems 344, 345 and 346 there is another property of those relations you may have noticed from the directed graph. Whenever you had an arrow from L_1 to L_2 and an arrow from L_2 to L_3, then there was an arrow from L_1 to L_3. This is what we mean when we say a relation is *transitive*. You also undoubtedly noticed how the directed graph divides up into clumps of mutually connected vertices. This is what equivalence relations are all about. Let's be a bit more precise in our description of what it means for a relation to be reflexive, symmetric or transitive.

- If R is a relation on a set X, we say R is *reflexive* if $(x, x) \in R$ for every $x \in X$.
- If R is a relation on a set X, we say R is *symmetric* if (x, y) is in R whenever (y, x) is in R.
- If R is a relation on a set X, we say R is *transitive* if whenever (x, y) is in R and (y, z) is in R, then (x, z) is in R as well.

Each of the relations of equivalence you worked with in the Problem 344, 345 and 346 had these three properties. Can you visualize the same three properties in the relations of equivalence that you would use in Problems 34c, 38, and 39d? We call a relation an **equivalence relation** if it is reflexive, symmetric and transitive.

After some more examples, we will see how to show that equivalence relations have the kind of clumping property you saw in the directed graphs. In our first example, using the notation $(a, b) \in R$ to say that a is related to B is going to get in the way. It is really more common to write aRb to mean that a is related to b. For example, if our relation is the less than relation on $\{1, 2, 3\}$, you are much more likely to use $x < y$ than you are $(x, y) \in <$, aren't you? The reflexive law then says xRx for every x in X, the symmetric law says that if xRy, then yRx, and the transitive law says that if xRy and yRz, then xRz.

347. For the necklace problem, Problem 43, our lists are lists of beads. What makes two lists equivalent for the purpose of describing a necklace? Verify explicitly that this relationship of equivalence is reflexive, symmetric, and transitive.

    Solution: Two lists are equivalent if I can get one from the other by some combination of cyclic permutation (putting the last thing in the list at the front and moving everything else one place right) and reversals. The combination could include no operation at all. Since it is possible to have no operation, the relation is reflexive. (Even without the opportunity to do no operation, if we do two reversals to a list we get the original list back so it is equivalent to itself.) Suppose we have n beads. Then if I can get from list A to list B with a cyclic permutation, then $n-1$ more cyclic permutations give us the original list. Also if I get from a list A to a list B by a reversal, then another reversal takes B to A. Thus any sequence of cyclic permutations and reversals can be undone. Therefore if list A is equivalent to list B, then list B is equivalent to list A. Following one combination of operations with another one still gives a combination of operations, so our relation is transitive. ▮

348. Which of the reflexive, symmetric and transitive properties does the $<$ relation on the integers have?

    Solution: It is transitive, but not reflexive or symmetric. ▮

349. A relation R on the set of ordered pairs of positive integers that you learned about in grade school in another notation is the relation that says (m, n) is related to (h, k) if $mk = hn$. Show that this relation is an equivalence relation. In what context did you learn about this relation in grade school?

    Solution: $mn = mn$ so the relation is reflexive. If $mk = hn$, then $hn = mk$, so if (m, n) is related to (h, k), then (h, k) is related to (m, n). If (m, n) is related to (h, k) and (h, k) is related to (p, q), then $mk = hn$ and $hq = pk$, which gives us $mkhq = hnpk$, and cancelling h

and k gives us $mq = np = pn$, so (m, n) is related to (p, q). Therefore, the relation is transitive. This is the relation of equality of the fractions $\frac{m}{n}$ and $\frac{h}{k}$. ∎

350. Another relation that you may have learned about in school, perhaps in the guise of "clock arithmetic," is the relation of equivalence modulo n. For integers (positive, negative, or zero) a and b, we write

$$a \equiv b \pmod{n}$$

to mean that $a - b$ is an integer multiple of n, and in this case, we say that a is *congruent to* b *modulo* n. Show that the relation of congruence modulo n is an equivalence relation.

Solution: $a - a = 0 = 0 \cdot n$, so $a \equiv a \pmod{n}$. Thus the relation is reflexive. If $a - b = kn$ for some integer k, then $b - a = -kn$, and $-k$ is an integer, so if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$. If $a - b = kn$ and $b - c = jn$, then $a - b + b - c = kn + jn$, so $a - c = (k + j)n$ and since $k + j$ is an integer this means that $a \equiv c \pmod{n}$. Therefore the relation of congruence mod n is an equivalence relation. ∎

351. Define a relation on the set of all lists of n distinct integers chosen from $\{1, 2, \ldots, n\}$, by saying two lists are related if they have the same elements (though perhaps in a different order) in the first k places, and the same elements (though perhaps in a different order) in the last $n - k$ places. Show this relation is an equivalence relation.

Solution: The relation is reflexive, for a list L has the same elements as the list L in the first k places and the last $n - k$ places. If L_1 and L_2 have the same elements in the first k places and have the same elements in the last k places, then L_2 and L_1 have the same elements in the first k places and have the same elements in the last $n-k$ places, so our relation is symmetric. If L_1 and L_2 have the same elements in the first k places and L_2 and L_3 have the same elements in the first k places, then L_1 and L_3 have the same elements in the first k places. Similarly with the last $n-k$ places. Therefore our relation is transitive, and so it is an equivalence relation. ∎

352. Suppose that R is an equivalence relation on a set X and for each $x \in X$, let $C_x = \{y | y \in X \text{ and } yRx\}$. If C_x and C_z have an element y in common, what can you conclude about C_x and C_z (besides the fact that they have an element in common!)? Be explicit about what property(ies) of equivalence relations justify your answer. Why is every

element of X in some set C_x? Be explicit about what property(ies) of equivalence relations you are using to answer this question. Notice that we might simultaneously denote a set by C_x and C_y. Explain why the union of the sets C_x is X. Explain why two distinct sets C_x and C_z are disjoint. What do these sets have to do with the "clumping" you saw in the digraph of Problem 344 and 345?

Solution: If C_x and C_y have the element z in common, then by symmetry and transitivity, all elements in C_x are related to z and by symmetry and transitivity, all elements in C_y are related to z. Then by symmetry and transitivity again, all elements of C_y are related to x, so $C_y \subseteq C_x$. By the same kind of reasoning, $C_x \subseteq C_y$. Therefore, $C_x = C_y$. Every element x is in the set C_x by reflexivity. Thus the union of the sets C_x is X. The sets C_x and C_y are disjoint if they are different, because if they have a common element z then they are equal. By definition, the sets C_x form a partition of X. The clumps that we saw in those problems are the blocks of the partition. ∎

In Problem 352 the sets C_x are called *equivalence classes* of the equivalence relation R. You have just proved that if R is an equivalence relation of the set X, then each element of X is in exactly one equivalence class of R. Recall that a *partition* of a set X is a set of disjoint sets whose union is X. For example, $\{1, 3\}$, $\{2, 4, 6\}$, $\{5\}$ is a partition of the set $\{1, 2, 3, 4, 5, 6\}$. Thus another way to describe what you proved in Problem 352 is the following:

Theorem 10 *If R is an equivalence relation on X, then the set of equivalence classes of R is a partition of X.*

Since a partition of S is a set of subsets of S, it is common to call the subsets into which we partition S the *blocks* of the partition so that we don't find ourselves in the uncomfortable position of referring to a set and not being sure whether it is the set being partitioned or one of the blocks of the partition.

353. In each of Problems 38, 39d, 43, 344, and 345, what does an equivalence class correspond to? (Five answers are expected here.)

Solution: In Problem 38 the equivalence classes correspond to seating arrangements. In Problem 39d the equivalence classes correspond to the k-element subsets of our n-element set S. In Problem 43, the equivalence classes correspond to necklaces. In Problem 344 the equivalence classes correspond to choices of three flavors of ice cream out of a possible four flavors. In Problem 345 the equivalence classes correspond to the ways we can choose scoops of ice cream of three different flavors out of four and put them into a dish in a symmetric fashion. ∎

354. Given the partition $\{1,3\}$, $\{2,4,6\}$, $\{5\}$ of the set $\{1,2,3,4,5,6\}$, define two elements of $\{1,2,3,4,5,6\}$ to be related if they are in the same part of the partition. That is, define 1 to be related to 3 (and 1 and 3 each related to itself), define 2 and 4, 2 and 6, and 4 and 6 to be related (and each of 2, 4, and 6 to be related to itself), and define 5 to be related to itself. Show that this relation is an equivalence relation.

    Solution: We have said for each element of our set that it is related to itself, so the relation is reflexive. If x and y are in a given one of those sets, then y and x are in that same given set. If x and y are in the same set, and if y and z are in the same set, then x and z must be in the same set because there is one and only one set that y is in. Thus the relation is an equivalence relation. ∎

355. Suppose $P = \{S_1, S_2, S_3, \ldots, S_k\}$ is a partition of S. Define two elements of S to be related if they are in the same set S_i, and otherwise not to be related. Show that this relation is an equivalence relation. Show that the equivalence classes of the equivalence relation are the sets S_i.

    Solution: Each element is in a set S_i with itself, so the relation is reflexive. If x and y are in a given one of those sets S_i, then y and x are in that same set S_i. Therefore the relation is symmetric. If x and y are in the same set S_i, and if y and z are in the same set S_j, then S_i must equal S_j because y is in one and only one block of the partition. Therefore, x and z must be in the same set S_i. Thus the relation is an equivalence relation. If $x \in S_i$, then by definition S_i consists of all elements related to x, so it is the equivalence class containing x. ∎

In Problem 355 you just proved that each partition of a set gives rise to an equivalence relation whose classes are just the parts of the partition. Thus in Problem 352 and Problem 355 you proved the following Theorem.

Theorem 11 *A relation R is an equivalence relation on a set S if and only if S may be partitioned into sets S_1, S_2, $\ldots$, S_n in such a way that x and y are related by R if and only if they are in the same block S_i of the partition.*

In Problems 344, 345, 38 and 43 what we were doing in each case was counting equivalence classes of an equivalence relation. There was a special structure to the problems that made this somewhat easier to do. For example, in 344, we had $4 \cdot 3 \cdot 2 = 24$ lists of three distinct flavors chosen from V, C, S, and P. Each list was equivalent to $3 \cdot 2 \cdot 1 = 3! = 6$ lists, including itself, from the point of view of serving 3 small dishes of ice cream. The order in which we selected the three flavors was unimportant. Thus the set of all $4 \cdot 3 \cdot 2$ lists was a union of some number n of equivalence classes, each of size 6. By the product principle, if we have a union of n disjoint sets, each of size 6, the union has $6n$ elements. But we already knew that the union was the set of all 24 lists of three distinct letters chosen from our four letters. Thus we have $6n = 24$, so that we have $n = 4$ equivalence classes.

In Problem 345 there is a subtle change. In the language we adopted for seating people around a round table, if we choose the flavors V, C, and S, and arrange them in the dish with C to the right of V and S to the right of C, then the scoops are in different relative positions than if we arrange them instead with S to the right of V and C to the right of S. Thus the order in which the scoops go into the dish is somewhat important—somewhat, because putting in V first, then C to its right and S to its right is the same as putting in S first, then V to its right and C to its right. In this case, each list of three flavors is equivalent to only three lists, including itself, and so if there are n equivalence classes, we have $3n = 24$, so there are $24/3 = 8$ equivalence classes.

356. If we have an equivalence relation that divides a set with k elements up into equivalence classes each of size m, what is the number n of equivalence classes? Explain why.

Solution: The number of equivalence classes is k/m, because by the product principle, $mn = k$. ▮

357. In Problem 351, what is the number of equivalence classes? Explain in words the relationship between this problem and the Problem 39d.

Solution: There are $n!$ lists, and each is in an equivalence class of size $k!(n-k)!$, so the number of equivalence classes is $\frac{n!}{k!(n-k)!}$ by Problem 356. This is a way of computing the number of k-element subsets that shows why the final answer we got in Problem 39d is symmetric in k and $n - k$. ▮

358. Describe explicitly what makes two lists of beads equivalent in Problem 43 and how Problem 356 can be used to compute the number of different necklaces.

Solution: Two lists are equivalent if I can get one from the other by some combination of cyclic shifts and reversals. A cyclic shift on the list $a_1, a_2, \ldots, a_{n-1}, a_n$ gives either the list $a_n, a_1, a_2, \ldots, a_{n-1}$ or the list $a_2, \ldots, a_{n-1}, a_n, a_1$. There are n possible results of repeated cyclic shifts, and each of them may be reversed to give a new list if $n \geq 3$. Further, these are the only lists we can get from shifts and reversals. (a_1 must go to one of n places, and that leaves two choices for where a_2 goes. Then the rest of the list is determined.) Thus we can get exactly $2n$ lists from combinations of cyclic shifts and reversals. We define two lists to be equivalent if they give the same necklace; we've seen that this is an equivalence relation and that it has $2n$ elements per equivalence class. Since there are $n!$ lists, this gives us $(n-1)!/2$ equivalence classes, or necklaces. ▮

359. What are the equivalence classes (write them out as sets of lists) in Problem 45, and why can't we use Problem 356 to compute the number of equivalence classes?

Solution: The equivalence classes are

$$\{RRBB, BRRB, BBRR, RBBR\} \text{ and } \{RBRB, BRBR\}.$$

We can't use Problem 356 to compute the number of equivalence classes because the equivalence classes don't have the same size. ▮

In Problem 356 you proved our next theorem. In Chapter 1 (Problem 42) we discovered and stated this theorem in the context of partitions and called it the *Quotient Principle*.

Theorem 12 *If an equivalence relation on a set of size k has equivalence classes each of size m, then the number of equivalence classes is k/m.*

Appendix B

Mathematical Induction

B.1 The Principle of Mathematical Induction

B.1.1 The ideas behind mathematical induction

There is a variant of one of the bijections we used to prove the Pascal Equation that comes up in counting the subsets of a set. In the next problem it will help us compute the total number of subsets of a set, regardless of their size. Our main goal in this problem, however, is to introduce some ideas that will lead us to one of the most powerful proof techniques in combinatorics (and many other branches of mathematics), the principle of mathematical induction.

360. (a) Write down a list of the subsets of $\{1, 2\}$. Don't forget the empty set! Group the sets containing containing 2 separately from the others.

Solution: $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$. ▮

(b) Write down a list of the subsets of $\{1, 2, 3\}$. Group the sets containing 3 separately from the others.

Solution: $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$, $\{3\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$. ▮

(c) Look for a natural way to match up the subsets containing 2 in Part (a) with those not containing 2. Look for a way to match up the subsets containing 3 in Part (b) containing 3 with those not containing 3.

Solution: Adjoin 2 to each subset not containing 2 and you get each set containing 2. Adjoin 3 to each subset not containing 3, and you get each subset containing 3. ▮

(d) On the basis of the previous part, you should be able to find a bijection between the collection of subsets of $\{1, 2, \ldots, n\}$ containing n and those not containing n. (If you are having difficulty figuring out the bijection, try rethinking Parts (a) and (b), perhaps by doing a similar exercise with the set $\{1, 2, 3, 4\}$.) Describe the bijection (unless you are very familiar with the notation of sets, it is probably easier to describe to describe the function in words rather than symbols) and explain why it is a bijection. Explain why the number of subsets of $\{1, 2, \ldots, n\}$ containing n equals the number of subsets of $\{1, 2, \ldots, n-1\}$.

Solution: If we adjoin n to the subsets not containing n we get the subsets containing n. This is a bijection because if we start with two different sets, adjoining n to them can't make them the same, and every subset S containing n must arise in this way from the set $S - \{n\}$ not containing n. ∎

(e) Parts (a) and (b) suggest strongly that the number of subsets of a n-element set is 2^n. In particular, the empty set has 2^0 subsets, a one-element set has 2^1 subsets, itself and the empty set, and in Parts a and b we saw that two-element and three-element sets have 2^2 and 2^3 subsets respectively. So there are certainly some values of n for which an n-element set has 2^n subsets. One way to prove that an n-element set has 2^n subsets for all values of n is to argue by contradiction. For this purpose, suppose there is a nonnegative integer n such that an n-element set doesn't have exactly 2^n subsets. In that case there may be more than one such n. Choose k to be the smallest such n. Notice that $k - 1$ is still a positive integer, because k can't be 0, 1, 2, or 3. Since k was the smallest value of n we could choose to make the statement "An n-element set has 2^n subsets" false, what do you know about the number of subsets of a $(k-1)$-element set? What do you know about the number of subsets of the k-element set $\{1, 2, \ldots, k\}$ that don't contain k? What do you know about the number of subsets of $\{1, 2, \ldots, k\}$ that do contain k? What does the sum principle tell you about the number of subsets of $\{1, 2, \ldots, k\}$? Notice that this contradicts the way in which we chose k, and the only assumption that went into our choice of k was that "there is a nonnegative integer n such that an n-element set doesn't have exactly 2^n subsets." Since this assumption has led us to a contradiction, it must be false. What can you now conclude about the statement "for every nonnegative integer n, an n-element set has exactly 2^n subsets?"

Solution: We know that the number of subsets of a $(k-1)$-element set is 2^{k-1}. The number of subsets of $\{1, 2, \ldots, k\}$ that do not contain k is the number of subsets of the $k-1$-element set $\{1, 2, \ldots, k-1\}$, so we know this number is 2^{k-1}. We know that the number of subsets that do contain k equals the number that don't, so the number that do contain k is also 2^{k-1}. The sum principle tells us that the number os subsets of $\{1, 2, \ldots, k\}$ is $2^{k-1} + 2^{k-1} = 2^k$. We can conclude that the statement "for every nonnegative integer n, an n-element set has exactly 2^n subsets" is true. ∎

361. The expression
$$1 + 3 + 5 + \cdots + 2n - 1$$
is the sum of the first n odd integers (notice that the nth odd integer is $2n-1$). Experiment a bit with the sum for the first few positive integers and guess its value in terms of n. Now apply the technique of Problem 360 to prove that you are right.

Solution: We guess that $1 + 3 + 5 + \cdots + 2n - 1 = n^2$. Clearly this is true when n is 1, 2, or 3. Suppose there is an n for which this formula is not true, and let k be the smallest such n. Then $1+3+5+\cdots+2(k-1)-1 = (k-1)^2$. Simplifying, $1+3+5+\cdots+2k-3 = (k-1)^2$. Now suppose we add $2k-1$ to both sides of this equation. Then we get
$$1+3+5+\cdots+2k-3+2k-1 = (k-1)^2+2k-1 = k^2-2k+1+2k-1 = k^2.$$
But this is a contradiction, because we assumed that k was the smallest value of n for which the sum on the left is not n^2. Therefore the assumption that there is an n for which $1 + 3 + 5 + \cdots + 2n - 1 \neq n^2$ must be false, so the equation $1 + 3 + 5 + \cdots + 2n - 1 = n^2$ must be true for all positive integers n. ∎

In Problems 360 and 361 our proofs had several distinct elements. We had a statement involving an integer n. We knew the statement was true for the first few nonnegative integers in Problem 360 and for the first few positive integers in Problem 361. We wanted to prove that the statement was true for all nonnegative integers in Problem 360 and for all positive integers in Problem 361. In both cases we used the method of proof by contradiction; for that purpose we assumed that there was a value of n for which our formula wasn't true. We then chose k to be the smallest value of n for which our formula wasn't true.[1] This meant that when n was $k-1$, our formula was true, (or else that $k-1$ wasn't a nonnegative integer in Problem 360 or that $k-1$ wasn't a positive integer in Problem 361). What we did next was the crux of the proof. We showed that the truth of our statement for $n = k-1$ implied the truth of our statement for $n = k$. This gave us a contradiction to the assumption that there was an n that made the statement false. In fact, we will see that we can bypass entirely the use of proof by contradiction. We used it to help you discover the central ideas of the technique of proof by mathematical induction.

The central core of mathematical induction is the proof that the truth of a statement about the integer n for $n = k-1$ implies the truth of the statement for $n = k$. For example, once we know that a set of size 0 has 2^0 subsets, if we have proved our implication, we can then conclude that a set of size 1 has 2^1 subsets, from which we can conclude that a set of size 2 has 2^2 subsets, from which we can conclude that a set of size 3 has 2^3 subsets, and so on up to a set of size n having 2^n subsets for any nonnegative integer n we choose. In other words, although it was the idea of proof by contradiction that led us to think about such an implication, we can now do without the contradiction at all. What we need to prove a statement about n by this method is a place to start, that is a value b of n for which we know the statement to be true, and then a proof that the truth of our statement for $n = k-1$ implies the truth of the statement for $n = k$ whenever $k > b$.

B.1.2 Mathematical induction

The **principle of mathematical induction** states that

> In order to prove a statement about an integer n, if we can
>
> 1. Prove the statement when $n = b$, for some fixed integer b
> 2. Show that the truth of the statement for $n = k-1$ implies the truth of the statement for $n = k$ whenever $k > b$,
>
> then we can conclude the statement is true for all integers $n \geq b$.

As an example, let us return to Problem 360. The statement we wish to prove is the statement that "A set of size n has 2^n subsets."

[1]The fact that every set of positive integers has a smallest element is called the Well-Ordering Principle. In an axiomatic development of numbers, one takes the Well-Ordering Principle or some equivalent principle as an axiom.

> Our statement is true when $n = 0$, because a set of size 0 is the empty set and the empty set has $1 = 2^0$ subsets. (This step of our proof is called a *base step.*)
>
> Now suppose that $k > 0$ and every set with $k - 1$ elements has 2^{k-1} subsets. Suppose $S = \{a_1, a_2, \ldots a_k\}$ is a set with k elements. We partition the subsets of S into two blocks. Block B_1 consists of the subsets that do not contain a_n and block B_2 consists of the subsets that do contain a_n. Each set in B_1 is a subset of $\{a_1, a_2, \ldots a_{k-1}\}$, and each subset of $\{a_1, a_2, \ldots a_{k-1}\}$ is in B_1. Thus B_1 is the set of all subsets of $\{a_1, a_2, \ldots a_{k-1}\}$. Therefore by our assumption in the first sentence of this paragraph, the size of B_1 is 2^{k-1}. Consider the function from B_2 to B_1 which takes a subset of S including a_n and removes a_n from it. This function is defined on B_2, because every set in B_2 contains a_n. This function is onto, because if T is a set in B_1, then $T \cup \{a_k\}$ is a set in B_2 which the function sends to T. This function is one-to-one because if V and W are two different sets in B_2, then removing a_k from them gives two different sets in B_1. Thus we have a bijection between B_1 and B_2, so B_1 and B_2 have the same size. Therefore by the sum principle the size of $B_1 \cup B_2$ is $2^{k-1} + 2^{k-1} = 2^k$. Therefore S has 2^k subsets. This shows that if a set of size $k - 1$ has 2^{k-1} subsets, then a set of size k has 2^k subsets. Therefore by the principle of mathematical induction, a set of size n has 2^n subsets for every nonnegative integer n.

The first sentence of the last paragraph is called the *inductive hypothesis.* In an inductive proof we always make an inductive hypothesis as part of proving that the truth of our statement when $n = k-1$ implies the truth of our statement when $n = k$. The last paragraph itself is called the *inductive step* of our proof. In an inductive step we derive the statement for $n = k$ from the statement for $n = k - 1$, thus proving that the truth of our statement when $n = k - 1$ implies the truth of our statement when $n = k$. The last sentence in the last paragraph is called the *inductive conclusion.* All inductive proofs should have a base step, an inductive hypothesis, an inductive step, and an inductive conclusion.

There are a couple details worth noticing. First, in this problem, our base step was the case $n = 0$, or in other words, we had $b = 0$. However, in other proofs, b could be any integer, positive, negative, or 0. Second, our proof that the truth of our statement for $n = k - 1$ implies the truth of our statement for $n = k$ required that k be at least 1, so that there would be an element a_k we could take away in order to describe our bijection. However, condition (2) of the principle of mathematical induction only requires that we be able to prove the implication for $k > 0$, so we were allowed to assume $k > 0$.

362. Use mathematical induction to prove your formula from Problem 361.

 Solution: Our formula says that $1+3+5+\cdots+2n-1 = n^2$. When $n = 1$, this formula says $1 = 1$, so our formula holds when $n = 1$. Now in order to prove that the truth of the formula when $n = k-1$ implies its truth when $n = k$, assume that $k > 1$ and $1+3+5+\cdots+2(k-1)-1 = (k-1)^2$. Then by addition of $2k-1$ to both sides of the equation, $1+3+5+\cdots+2k-1 = (k-1)^2+2k-1 = k^2-2k+1+2k-1 = k^2$. Therefore the truth of the formula for $n = k-1$ implies its truth for $n = k$. Thus, by the principle of mathematical induction, the formula holds for all positive n. ▮

B.1.3 Proving algebraic statements by induction

363. Use mathematical induction to prove the well-known formula that for all positive integers n,

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

 Solution: When $n = 0$, $0 = 0(0+1)/2$, so our formula holds. Now suppose that $k > 0$ and that our formula holds when $n = k-1$, so that $1+2+\cdots+k-1 = (k-1)k/2$. Add k to both sides of this equation to get

$$\begin{aligned} 1+2+\cdots+(k-1)+k &= (k-1)k/2+k \\ &= k^2/2-k/2+k \\ &= k^2/2+k/2 \\ &= k(k+1)/2. \end{aligned}$$

 Thus the truth of our formula for $n = k-1$ implies its truth for $n = k$. Therefore by the principle of mathematical induction, our formula holds for all nonnegative integers n. ▮

364. Experiment with various values of n in the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)} = \sum_{i=1}^{n}\frac{1}{i\cdot(i+1)}.$$

Guess a formula for this sum and prove your guess is correct by induction.

Solution: We guess the formula

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

When $n = 1$ this formula says that $\frac{1}{1\cdot 2} = \frac{1}{1\cdot 2}$, so our formula holds when $n = 1$. Now assume that $k > 1$ and that our formula holds when $n = k - 1$. Then

$$\sum_{i=1}^{k-1} \frac{1}{i(i+1)} = \frac{k-1}{k}.$$

Adding $\frac{1}{k(k+1)}$ to both sides of this equation gives us

$$\begin{aligned}
\sum_{i=1}^{k-1} \frac{1}{i(i+1)} + \frac{1}{k(k+1)} &= \frac{k-1}{k} + \frac{1}{k(k+1)} \\
\sum_{i=1}^{k} \frac{1}{i(i+1)} &= \frac{(k-1)(k+1)}{k(k+1)} + \frac{1}{k(k+1)} \\
&= \frac{k^2-1+1}{k(k+1)} \\
&= \frac{k}{k+1}.
\end{aligned}$$

Thus whenever our formula is true with $n = k - 1$, it is true with $n = k$ as well. Therefore by the principle of mathematical induction, our formula is true for all positive integers. ▮

365. For large values of n, which is larger, n^2 or 2^n? Use mathematical induction to prove that you are correct.

Solution: We note that $0^2 = 0$, while $2^0 = 1$, that $1^2 = 1$, while $2^1 = 2$, that $2^2 = 4$, while $2^2 = 4$, that $3^2 = 9$ while $2^3 = 8$, that $4^2 = 16$ while $2^4 = 16$, and $5^2 = 25$ while $2^5 = 32$. We suspect that $2^n > n^2$ for $n \geq 5$, so we try to prove this by induction. We have already shown that $2^5 > 5^2$. Now suppose that $k > 5$ and $2^{k-1} > (k-1)^2$. Then $2^k = 2 \cdot 2^{k-1} > 2(k-1)^2 = 2k^2 - 4k + 1$. Now since $k > 5$, $k^2 > 5k$, so that $k^2 - 4k + 1 = k^2 + k^2 - 4k + 1 > k^2 + 5k - 4k + 1 = k^2 + k + 1 > k^2$. Thus for $k > 5$, the statement $2^{k-1} > (k-1)^2$ implies the statement $2^k > k^2$. Therefore, by the principle of mathematical induction, $2^n > n^2$ for all $n \geq 5$. ▮

366. What is wrong with the following attempt at an inductive proof that all integers in any consecutive set of n integers are equal for every positive integer n? For an arbitrary integer i, all integers from i to i are equal, so our statement is true when $n = 1$. Now suppose $k > 1$ and all integers in any consecutive set of $k - 1$ integers are equal. Let S be a set of k consecutive integers. By the inductive hypothesis, the first $k - 1$ elements of S are equal and the last $k - 1$ elements of S are equal. Therefore all the elements in the set S are equal. Thus by the principle of mathematical induction, for every positive n, every n consecutive integers are equal.

 Solution: One possible value of k that is greater than 1 is 2. When we have a set S of two elements and we argue that the first $k - 1$ elements are equal and the last $k - 1$ elements are equal, we cannot conclude from those equalities that all elements of S are equal, because there is no overlap among the first $k - 1 = 1$ elements of S and the last $k - 1 = 1$ elements of S. Thus our inductive step does not cover the possibility that $k = 2$. Therefore our inductive step does not show that the truth of our statement for $n = k - 1$ implies the truth of our statement for $n = k$ for **all** integers $n > 1$. Therefore the principle of mathematical induction does not apply. ∎

B.2 Strong Induction

One way of looking at the principle of mathematical induction is that it tells us that if we know the “first” case of a theorem and we can derive each other case of the theorem from a smaller case, then the theorem is true in all cases. However, the particular way in which we stated the theorem is rather restrictive in that it requires us to derive each case from the immediately preceding case. This restriction is not necessary, and removing it leads us to a more general statement of the principle of mathematical induction which people often call the **strong principle of mathematical induction**. It states:

> In order to prove a statement about an integer n if we can
>
> 1. prove our statement when $n = b$ and
> 2. prove that the statements we get with $n = b$, $n = b + 1$, $\ldots n = k - 1$ imply the statement with $n = k$,
>
> then our statement is true for all integers $n \geq b$.

367. What postage do you think we can make with five and six cent stamps? Do you think that there is a number N such that if $n \geq N$, then we can make n cents worth of postage?

Solution: We can make 10, 11, and 12 cents in postage, but not 13 cents. We can also make 15, 16, 17, and 18, but not 19 cents. However, when we try starting with 20 cents, we can make 20, 21, 22, 23, 24, 25, 26, 27,...cents, and so it seems for all $n \geq 20$, we can make n cents in stamps. Once we know we can make 20 cents through 24 cents, by adding 5 cents to each of these we can get 25 through 29 cents, and so we expect to be able to keep going. However, making 29 cents does not depend on our ability to make 28 cents; rather we know we can make 29 cents because we know we can make 24 cents and $24+5 = 29$ or we know we can make 23 cents and $23 + 6 = 29$. Thus it certainly seems as if for all $n \geq 20$ we can make n cents in postage. ∎

You probably see that we can make n cents worth of postage as long as n is at least 20. However, you didn't try to make 26 cents in postage by working with 25 cents; rather you saw that you could get 20 cents and then add six cents to that to get 26 cents. Thus if we want to prove by induction that we are right that if $n \geq 20$, then we can make n cents worth of postage, we are going to have to use the strong version of the principle of mathematical induction.

We know that we can make 20 cents with four five-cent stamps. Now we let k be a number greater than 20, and assume that it is possible to make any amount between 20 and $k - 1$ cents in postage with five and six cent stamps. Now if k is less than 25, it is 21, 22, 23, or 24. We can make 21 with three fives and one six. We can make 22 with two fives and two sixes, 23 with one five and three sixes, and 24 with four sixes. Otherwise $k - 5$ is between 20 and $k - 1$ (inclusive) and so by our inductive hypothesis, we know that $k - 5$ cents can be made with five and six cent stamps, so with one more five cent stamp, so can k cents. Thus by the (strong) principle of mathematical induction, we can make n cents in stamps with five and six cent stamps for each $n \geq 20$.

Some people might say that we really had five base cases, $n = 20$, 21, 22, 23, and 24, in the proof above and once we had proved those five consecutive base cases, then we could reduce any other case to one of these base cases by successively subtracting 5. That is an appropriate way to look at the proof. In response, a logician might say that it is also the case that, for example, by proving we could make 22 cents, we also proved that if we can make 20 cents and 21 cents in stamps, then we could also make 22 cents. We just didn't bother to use the assumption that we could make 20 cents and 21 cents! On the other hand a computer scientist might say that if we want to write a program that figures out how to make n cents in postage, we use one method for the cases $n = 20$ to $n = 24$, and then a general method for all the other cases. So to write a program it is important for us to think in terms of having multiple base cases. How do you know what your base cases are? They are the ones that you solve without using the inductive hypothesis. So long as one point of view or the other satisfies you, you are ready to use this kind of argument in proofs.

368. A number greater than one is called prime if it has no factors other than itself and one. Show that each positive number is either a power of a prime (remember what p^0 and p^1 are) or a product of powers of prime numbers.

    Solution: We note that $1 = 2^0$, so 1 is a power of a prime. Now suppose that all positive numbers less than n are primes, powers of primes, or products of powers of primes. If n has no proper factors, it is a prime. If it does have proper factors, say $n = mk$, both factors are less than n and greater than 1. Therefore each factor is a prime, a power of a prime, or a product of powers of primes. When we multiply m and k together, the result will still be a power of a prime or a product of powers of primes. Thus the statement that all positive numbers less than n are primes, powers of primes, or products of powers of primes implies the statement that n is a prime, a power of a prime, or a product of powers of primes. Therefore by the strong principle of mathematical induction, all positive numbers are either primes, powers of primes, or products of powers of primes. ∎

369. Show that the number of prime factors of a positive number $n \geq 2$ is less than or equal to $\log_2 n$. (If a prime occurs to the kth power in a factorization of n, you can consider that power as k prime factors.) (There is a way to do this by induction and a way to do it without induction. It would be ideal to find both ways.)

    Solution: First, we will prove this by induction. The number of prime factors of 2 is 1, which is less than or equal to $\log_2 2 = 1$. Now assume that the number of prime factors of any number k greater than 1 and less than n is no more than $\log_2 k$. If n is prime, then its number of prime factors is less than or equal to $\log_2 n$. Otherwise n is a product of two factors, $n = mk$. Then by our inductive hypothesis, the number of prime factors of m is less than or equal to $\log_2 m$ and the number of prime factors of k is less than or equal to $\log_2 k$. But the number of prime factors of the product is the sum of the number of prime factors of each factor, so the number of prime factors of n is no more than $\log_2 m + \log_2 k = \log_2 mk = \log_2 n$. Thus the statement that "the number of prime factors of any number k between 2 and $n-1$ inclusive is no more than $\log_2 k$" implies the statement that "the number of prime factors of n is no more than $\log_2 n$." Therefore, by the principle of mathematical induction, the number of prime factors of n is less than or equal to $\log_2 n$ for all integers $n \geq 2$.

For a noninductive proof, note that all factors of n are at least 2. If n is a power of two, then the number of times 2 is a factor of n is exactly $\log_2 n$. But if n is not a power of 2, we still have that $2^{\log_2 n} = n$, so the product of $\lceil \log_2 n \rceil$ numbers including some greater than 2 must be greater than n. Therefore, the number of prime factors of n is no more than $\log_2 n$. Thus for all $n \geq 2$, the number of prime factors of n must be less than or equal to $\log_2 n$. ▮

370. One of the most powerful statements in elementary number theory is Euclid's Division Theorem[2]. This states that if m and n are positive integers, then there are unique nonnegative intergers q and r with $0 \leq r < n$, such that $m = nq + r$. The number q is called the quotient and the number r is called the remainder. In computer science it is common to denote r by $m \bmod n$. In elementary school you learned how to use long division to find q and r. However, it is unlikely that anyone ever proved for you that for any pair of positive intgers, m and n, there is such a pair of nonnegative numbers q and r. You now have the tools needed to prove this. Do so.

Solution: We prove our result by induction on m. If $m = 1$, then either $n = 1$ and we can choose $q = 1$ and $r = 0$, or $n > 1$ and we can choose $q = 0$ and $r = 1$. Furthermore, if $m = 1$ and $n = 1$, then r must equal zero (to be less than 1), and so q must equal 1. In $m = 1$

[2]In a curious twist of language, mathematicians have long called The Division Algorithm or Euclid's Division Algorithm. However as computer science has grown in importance, the word *algorithm* has gotten a more precise definition: an algorithm is now a method to do something. There is a method (in fact there are more than one) to get the q and r that Euclid's Division Theorem gives us, and computer scientists would call these methods algorithms. Your author has chosen to break with mathematical tradition and restrict his use of the word algorithm to the more precise interpretation as a computer scientist probably would. We aren't giving a method here, so this is why the name used here is "Euclid's Division Theorem."

and $n > 1$, then we must choose $q = 0$ (because otherwise $nq+r$ would have to be bigger than m) and therefor must choose $r = m$. Therefore Euclid's theorem is true when $m = 1$. Now assume that $m > 1$ and that for any positive integer $m' < m$, there are unique nonnegative integers q' and r' such that $m' = q'n + r'$, with $0 \leq r' < n$. If $m < n$, then we can let $q = 0$ and $r = m$, and we have our q and r. If $m = n$, we may choose $q = 1$ and $r = 0$ and we have our q and r again. Therefore we may assume that $m > n$. In this case, $m-n$ is a positive integer, and so since $m - n < n$, we have that, for unique nonnegative integers q' and r',

$$m - n = q'n + r',$$

with $0 \leq r < n$. But then adding n to both sides of the equation gives us $m = (q' + 1)n + r'$, with $0 \leq r' < n$. Therefore if we take $q = q'+1$ and $r = r'$ we have that $m = qn+r$, with $0 \leq r < n$. On the other hand, q' and r' are unique by our inductive hypothesis. Thus if $m = qn + r$, subtracting n from both sides of the equations gives us $m-n = (q-1)n+r$. This tells us that q *must* equal $q'+1$ and r *must* equal r'. Therefore by the strong principle of mathematical induction, Euclid's Theorem holds for all positive integers m. ▮

Appendix C

Exponential Generating Functions

C.1 Indicator Functions

When we introduced the idea of a generating function, we said that the formal sum

$$\sum_{i=0}^{\infty} a_i x^i$$

may be thought of as a convenient way to keep track of the sequence a_i. We then did quite a few examples that showed how combinatorial properties of arrangements counted by the coefficients in a generating function could be mirrored by algebraic properties of the generating functions themselves. The monomials x^i are called *indicator polynomials*. (They indicate the position of the coefficient a_i.) One example of a generating function is given by

$$(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i.$$

Thus we say that $(1+x)^n$ is the generating function for the binomial coefficients $\binom{n}{i}$. The notation tells us that we are assuming that only i varies in the sum on the right, but that the equation holds for each fixed integer n. This is implicit when we say that $(1+x)^n$ is the generating function for $\binom{n}{i}$, because we haven't written i anywhere in $(1+x)^n$, so it is free to vary.

Another example of a generating function is given by

$$x^{\underline{n}} = \sum_{i=0}^{\infty} s(n,i) x^i.$$

Thus we say that $x^{\underline{n}}$ is the generating function for the Stirling numbers of the first kind, $s(n,i)$. There is a similar equation for Stirling numbers of the second kind, namely

$$x^n = \sum_{i=0}^{\infty} S(n,i)x^{\underline{i}}.$$

However, with our previous definition of generating functions, this equation would not give a generating function for the Stirling numbers of the second kind, because $S(n,i)$ is not the coefficient of x^i. If we were willing to consider the falling factorial powers $x^{\underline{i}}$ as indicator polynomials, then we could say that x^n is the generating function for the numbers $S(n,i)$ relative to these indicator polynomials. This suggests that perhaps different sorts of indicator polynomials go naturally with different sequences of numbers.

The binomial theorem gives us yet another example.

∘371. Write $(1+x)^n$ as a sum of multiples of $\frac{x^i}{i!}$ rather than as a sum of multiples of x^i.

Solution:

$$(1+x)^n = \sum_{i=0}^{\infty}\binom{n}{i}x^i = \sum_{i=0}^{\infty}\frac{n!}{i!(n-i)!}x^i = \sum_{i=0}^{\infty}\frac{n!}{(n-i)!}\frac{x^i}{i!} = \sum_{i=0}^{\infty} n^{\underline{i}}\frac{x^i}{i!}.$$

∎

This example suggests that we could say that $(1+x)^n$ is the generating function for the falling factorial powers $n^{\underline{i}}$ relative to the indicator polynomials $\frac{x^i}{i!}$. In general, a sequence of polynomials is called a family of **indicator polynomials** if there is one polynomial of each nonnegative integer degree in the sequence. Those familiar with linear algebra will recognize that this says that a family of indicator polynomials forms a basis for the vector space of polynomials. This means that each polynomial can be expressed as a sum of numerical multiples of indicator polynomials in one and only one way. One could use the language of linear algebra to define indicator polynomials in an even more general way, but a definition in such generality would not be useful to us at this point.

C.2 Exponential Generating Functions

We say that the expression $\sum_{i=0}^{\infty} a_i\frac{x^i}{i!}$ is the **exponential generating function** for the sequence a_i. It is standard to use **EGF** as a shorthand for exponential generating function. In this context we call the generating function

$\sum_{i=0}^{n} a_i x^i$ that we originally studied the **ordinary generating function** for the sequence a_i. You can see why we use the term exponential generating function by thinking about the exponential generating function (EGF) for the all ones sequence,

$$\sum_{i=0}^{\infty} 1\frac{x^i}{i!} = \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x,$$

which we also denote by $\exp(x)$. Recall from calculus that the usual definition of e^x or $\exp(x)$ involves limits at least implicitly. We work our way around that by defining e^x to be the power series $\sum_{i=0}^{\infty} \frac{x^i}{i!}$.

∘372. Find the EGF (exponential generating function) for the sequence $a_n = 2^n$. What does this say about the EGF for the number of subsets of an n-element set?

Solution: $\sum_{i=0}^{\infty} 2^i \frac{x^i}{i!} = e^{2x}$. It says that the EGF for subsets of an n-element set is e^{2x}. ▮

∘373. Find the EGF (exponential generating function) for the number of ways to paint the n streetlight poles that run along the north side of Main Street in Anytown, USA using five colors.

Solution: The number of ways to paint n streetlight poles is 5^n, so the EGF is $\sum_{n=0}^{\infty} 5^n \frac{x^n}{n!} = e^{5x}$. ▮

374. For what sequence is $\frac{e^x - e^{-x}}{2} = \cosh x$ the EGF (exponential generating function)?

Solution: For the sequence $\frac{1-(-1)^n}{2}$ which, starting with $n = 0$ is the alternating sequence 0,1,0,1... of zeros and ones. ▮

·375. For what sequence is $\ln(\frac{1}{1-x})$ the EGF? (The notation $\ln(y)$ stands for the natural logarithm of y. People often write $\log(y)$ instead.) Hint: Think of the definition of the logarithm as an integral, and don't worry at this stage whether or not the usual laws of calculus apply, just use them as if they do! We will then define $\ln(1 - x)$ to be the power series you get.[1]

[1]It is possible to define the derivatives and integrals of power series by the formulas

$$\frac{d}{dx} \sum_{i=0}^{\infty} b_i x^i = \sum_{i=1}^{\infty} i b_i x^{i-1}$$

Solution:

$$\begin{aligned}
\ln\left(\frac{1}{1-x}\right) = -\ln(1-x) &= \int_0^x \frac{1}{1-t}dt \\
&= \int_0^x (1+t+t^2+\cdots)dt \\
&= \sum_{i=1}^{\infty} \frac{x^i}{i} \\
&= \sum_{i=1}^{\infty} (i-1)!\frac{x^i}{i!}
\end{aligned}$$

so the sequence is $a_n = (n-1)!$. ▮

·376. What is the EGF for the number of permutations of an n-element set?

Solution: $\frac{1}{1-x}$. ▮

➔ ·377. What is the EGF for the number of ways to arrange n people around a round table? Try to find a recognizable function represented by the EGF. Notice that we may think of this as the EGF for the number of permutations on n elements that are cycles.

Solution: Since there are $(n-1)!$ seatings of n people, our EGF is $\sum_{i=1}^{\infty} \frac{(n-1)!}{n!}x^n$. By Problem 375 the EGF is $\ln \frac{1}{1-x}$. ▮

➔ ·378. What is the EGF $\sum_{n=0}^{\infty} p_{2n}\frac{x^{2n}}{(2n)!}$ for the number of ways p_{2n} to pair up $2n$ people to play a total of n tennis matches (as in Problems 12a and 44)? Try to find a recognizable function represented by the EGF.

Solution: Recall that $p_{2n} = (2n-1)(2n-3)\cdots 1 = \frac{(2n)!}{2^n n!}$. Thus

$$\sum_{n=0}^{\infty} p_{2n}\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}.$$

▮

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and

$$\int_0^x \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} \frac{b_i}{i+1}x^{i+1}$$

rather than by using the limit definitions from calculus. It is then possible to prove that the sum rule, product rule, etc. apply. (There is a little technicality involving the meaning of composition for power series that turns into a technicality involving the chain rule, but it needn't concern us at this time.)

∘379. What is the EGF for the sequence $0, 1, 2, 3, \ldots$? You may think of this as the EGF for the number of ways to select one element from an n element set. What is the EGF for the number of ways to select two elements from an n-element set?

Solution: $\sum_{n=0}^{\infty} \frac{nx^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = xe^x$.
$\sum_{n=0}^{\infty} \frac{n(n-1)x^n}{2(n!)} = \sum_{i=2}^{\infty} \frac{x^n}{2(n-2)!} = x^2e^x/2$. ▮

·380. What is the EGF for the sequence $1, 1, \cdots, 1, \cdots$? Notice that we may think of this as the EGF for the number of identity permutations on an n-element set, which is the same as the number of permutations of n elements whose cycle decomposition consists entirely of 1-cycles, or as the EGF for the number of ways to select an n-element set (or, if you prefer, an empty set) from an n-element set. As you may have guessed, there are many other combinatorial interpretations we could give to this EGF.

Solution: e^x. ▮

∘381. What is the EGF for the number of ways to select n distinct elements from a one-element set? What is the EGF for the number of ways to select a positive number n of distinct elements from a one-element set? Hint: When you get the answer you will either say "of course," or "this is a silly problem."

Solution: $1 + x$, x. ▮

·382. What is the EGF for the number of partitions of a k-element set into exactly one block? (Hint: is there a partition of the empty set into exactly one block?)

Solution: There is one way to partition a set into one block, unless the set is empty, in which case it has no partition into one block. Thus our EGF is $e^x - 1$. ▮

·383. What is the EGF for the number of ways to arrange k books on one shelf (assuming they all fit)? What is the EGF for the number of ways to arrange k books on a fixed number n of shelves, assuming that all the books can fit on any one shelf? (Remember Problem 122e.)

Solution: $\sum_{k=0}^{\infty} k!\frac{x^k}{k!} = \frac{1}{1-x}$, $\sum_{k=0}^{\infty} \binom{n+k-1}{k} k!\frac{x^k}{k!} = (1-x)^{-n}$. ▮

C.3 Applications to Recurrences.

We saw that ordinary generating functions often play a role in solving recurrence relations. We found them most useful in the constant coefficient case. Exponential generating functions are useful in solving recurrence relations where the coefficients involve simple functions of n, because the $n!$ in the denominator can cancel out factors of n in the numerator.

∘384. Consider the recurrence $a_n = na_{n-1} + n(n-1)$. Multiply both sides by $\frac{x^n}{n!}$, and sum from $n = 2$ to ∞. (Why do we sum from $n = 2$ to infinity instead of from $n = 1$ or $n = 0$?) Letting $y = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$, show that the left-hand side of the equation is $y - a_0 - a_1 x$. Express the right hand side in terms of y, x, and e^x. Solve the resulting equation for y and use the result to get an equation for a_n. (A finite summation is acceptable in your answer for a_n.)

Solution:

$$\begin{aligned}\sum_{n=2}^{\infty} a_n \frac{x^n}{n!} &= \sum_{n=2}^{\infty} a_{n-1} \frac{x^n}{(n-1)!} + \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} \\ &= x\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - a_0\right) + x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= x\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right) - a_0 x + x^2 e^x\end{aligned}$$

We sum from $n = 2$ because otherwise we would have a factorial of a negative number in the denominator. Thus $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - a_0 - a_1 x = x\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right) - a_0 x + x^2 e^x$, or

$$(1-x)\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 x - a_0 x + x^2 e^x.$$

This gives us

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \frac{1}{1-x}(a_0 + a_1 x - a_0 x + x^2 e^x).$$

Computing the coefficient of a_n gives us $a_n = a_1 + \sum_{i=0}^{n-2} \frac{1}{i!}$. ▮

➔ ·385. The telephone company in a city has n subscribers. Assume a telephone call involves exactly two subscribers (that is, there are no calls to outside the network and no conference calls), and that the configuration of the telephone network is determined by which pairs of subscribers are talking. Notice that we may think of a configuration of the telephone network as a permutation whose cycle decomposition consists entirely of one-cycles and two-cycles, that is, we may think of a configuration as an involution in the symmetric group S_n.

(a) Give a recurrence for the number c_n of configurations of the network. (Hint: Person n is either on the phone or not.)

Solution: $c_n = (n-1)c_{n-2} + c_{n-1}$. (The first term counts the number of network configurations in which person n is in a phone call with someone else, and the second term counts the number of network configurations in which person n is not in a phone call.) ∎

(b) What are c_0 and c_1?

Solution: $c_0 = 1$ and $c_1 = 1$, because there is only one configuration of a network with 0 or one phones. ∎

(c) What are c_2 through c_6?

Solution: Using our recurrence and the values of c_0 and c_1, we get $c_2 = 2$, $c_3 = 2\cdot1+2 = 4$, $c_4 = 3\cdot2+4 = 10$, $c_5 = 4\cdot4+10 = 26$, and $c_6 = 5\cdot10+26 = 76$. ∎

➔ ·386. Recall that a *derangement* of $[n]$ is a permutation of $[n]$ that has no fixed points, or equivalently is a way to pass out n hats to their n different owners so that nobody gets the correct hat. Use d_n to stand for the number of derangements of $[n]$. We can think of a derangement of $[n]$ as a list of 1 through n so that i is not in the ith place for any n. Thus in a derangement, some number k different from n is in position n. Consider two cases: either n is in position k or it is not. Notice that in the second case, if we erase position n and replace n by k, we get a derangement of $[n-1]$. Based on these two cases, find a recurrence for d_n. What is d_1? What is d_2? What is d_0? What are d_3 through d_6?

Solution: $d_n = (n-1)d_{n-1} + (n-1)d_{n-2}$. $d_1 = 0$ and $d_2 = 1$. Thus d_0 must be 1 for our recurrence to be valid. (For those familiar with functions as sets of ordered pairs, the empty function is not only a permutation, but it does not map i to i for any integer i, so it is a derangement as well! Thus the definition of a derangement also gives us that $d_0 = 1$.) $d_3 = 2$, $d_4 = 3\cdot1+3\cdot2 = 9$, $d_5 = 4\cdot2+4\cdot9 = 44$, and $d_6 = 5\cdot9+5\cdot44 = 256$. ∎

C.3.1 Using calculus with exponential generating functions

➔ ·387. Your recurrence in Problem 385 should be a second order recurrence.

(a) Assuming that the left hand side is c_n and the right hand side involves c_{n-1} and c_{n-2}, decide on an appropriate power of x divided by an appropriate factorial by which to multiply both sides of the recurrence. Using the fact that the derivative of $\frac{x^n}{n!}$ is $\frac{x^{n-1}}{(n-1)!}$, write down a differential equation for the EGF $T(x) = \sum_{i=0}^{\infty} c_i \frac{x^i}{i!}$. Note that it makes sense to substitute 0 for x in $T(x)$. What is $T(0)$? Solve your differential equation to find an equation for $T(x)$.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty} c_n \frac{x^{n-1}}{(n-1)!} &= \sum_{n=2}^{\infty}(n-1)c_{n-2}\frac{x^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} c_{n-1}\frac{x^{n-1}}{(n-1)!}\\
\sum_{n=1}^{\infty} c_n \frac{x^{n-1}}{(n-1)!} - c_1 &= x\sum_{n=2}^{\infty} c_{n-2}\frac{x^{n-2}}{(n-2)!} + \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} - c_0\\
T'(x) &= xT(x) + T(x)
\end{aligned}$$

$T(0) = c_0 = 1$. Then $\frac{T'(x)}{T(x)} = x+1$, giving $\ln T(x) = x^2/2 + x + k$, and $T(x) = e^k e^{x+x^2/2} = e^{x+x^2/2}$, since $T(0) = 1$. ▮

(b) Use your EGF to compute a formula for c_n.

Solution: $T(x) = \sum_{i=0}^{\infty}(x + x^2/2)^i/i!$. By the binomial theorem, this gives

$$T(x) = \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{i}\binom{i}{j}x^j\left(\frac{x^2}{2}\right)^{i-j}}{i!} = \sum_{i=0}^{\infty}\frac{\sum_{j=0}^{i}\binom{i}{j}x^{2i-j}2^{j-i}}{i!}.$$

Then the coefficient c_n of x^n is the sum over all i and j with $2i-j = n$ and $j \le i$ of $\binom{i}{j}\frac{n!}{i!}2^{j-i}$. But if $2i-j = n$, then $j = 2i-n$, and if $2i-n \le i$, then $i \le n$, so that $c_n = \frac{n!}{2^n}\sum_{i=0}^{n}\binom{i}{2i-n}\frac{2^i}{i!}$. Note that $\binom{i}{2i-n}$ is the same as $\binom{i}{n-i}$, which is 0 unless $i \ge n/2$, which reduces our sum to $c_n = \frac{n!}{2^n}\sum_{i=\lceil n/2\rceil}^{n}\binom{i}{n-i}\frac{2^i}{i!}$. ▮

➔ ·388. Your recurrence in Problem 386 should be a second order recurrence.

(a) Assuming that the left-hand side is d_n and the right hand side involves d_{n-1} and d_{n-2}, decide on an appropriate power of x

divided by an appropriate factorial by which to multiply both sides of the recurrence. Using the fact that the derivative of $\frac{x^n}{n!}$ is $\frac{x^{n-1}}{(n-1)!}$, write down a differential equation for the EGF $D(x) = \sum_{i=0}^{\infty} d_i \frac{x^i}{i!}$. What is $D(0)$? Solve your differential equation to find an equation for $D(x)$.

(b) Use the equation you found for $D(x)$ to find an equation for d_n. Compare this result with the one you computed by inclusion and exclusion.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty} d_n \frac{x^{n-1}}{(n-1)!} &= \sum_{n=2}^{\infty} d_{n-1} \frac{x^{n-1}}{(n-2)!} + \sum_{n=2}^{\infty} d_{n-2} \frac{x^{n-1}}{(n-2)!} \\
\sum_{n=1}^{\infty} d_n \frac{x^{n-1}}{(n-1)!} - d_1 &= x \sum_{n=2}^{\infty} d_{n-1} \frac{x^{n-2}}{(n-2)!} + xD(x) \\
D'(x) - d_1 &= xD'(x) + xD(x) \\
D'(x)(1-x) &= xD(x) \\
\frac{D'(x)}{D(x)} &= \frac{x}{1-x}
\end{aligned}$$

This gives us $\ln D(x) = -\ln(1-x) - x + c$, so that $D(x) = \frac{1}{1-x} e^{-x} e^c$. Since $d_0 = 1$, we have $d(0) = 1$, so $c = 0$ and

$$\begin{aligned}
D(x) &= \frac{e^{-x}}{1-x} \\
&= \sum_{i=0}^{\infty} (-1)^i \frac{x^i}{i!} \cdot \sum_{j=0}^{\infty} x^j = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \frac{(-1)^j}{j!} \right) x^i.
\end{aligned}$$

Thus $d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$, as we computed by inclusion and exclusion. ▮

C.4 The Product Principle for EGFs

One of our major tools for ordinary generating functions was the product principle. It is thus natural to ask if there is a product principle for exponential generating functions. In Problem 383 you likely found that the EGF for the number of ways of arranging n books on one shelf was exactly the same as the EGF for the number of permutations of $[n]$, namely $\frac{1}{1-x}$ or

$(1-x)^{-1}$. Then using our formula from Problem 122e and the ordinary generating function for multisets, you probably found that the EGF for number of ways of arranging n books on some fixed number m of bookshelves was $(1-x)^{-m}$. Thus the EGF for m shelves is a product of m copies of the EGF for one shelf.

∘389. In Problem 373 what would the exponential generating function have been if we had asked for the number of ways to paint the poles with just one color of paint? With two colors of paint? What is the relationship between the EGF for painting the n poles with one color of paint and the EGF for painting the n poles with five colors of paint? What is the relationship among the EGF for painting the n poles with two colors of paint, the EGF for painting the poles with three colors of paint, and the EGF for painting the poles with five colors of paint?

Solution: With one color of paint, there would have been one way to paint each pole so our EGF would be $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, or e^x. With two colors of paint, it would be e^{2x} by analogy with the solution to Problem 373. Thus the EGF for two colors of paint would be the square of the EGF for one color of paint. The EGF for five colors of paint is the fifth power of the EGF for one color of paint and is also the product of the EGF for two colors of paint with the EGF for three colors of paint. ∎

In Problem 385 you likely found that the EGF for the number of network configurations with n customers was $e^{x+x^2/2} = e^x \cdot e^{x^2/2}$. In Problem 380 you saw that the EGF for the number of permutations on n elements whose cycle decompositions consist of only one-cycles was e^x, and in Problem 378 you likely found that the EGF for the number of tennis pairings of $2n$ people, or equivalently, the number of permutations of $2n$ objects whose cycle decomposition consists of n two-cycles is $e^{x^2/2}$.

·390. What can you say about the relationship among the EGF for the number of permutations whose cycle structure consists of disjoint two-cycles and one-cycles, i.e., which are involutions, the exponential generating function for the number of permutations whose cycle decomposition consists of disjoint two-cycles only and the EGF for the number of permutations whose cycle decomposition consists of of disjoint one-cycles only (these are identity permutations on their domain)?

Solution: The EGF for involutions is the product of the EGF for the permutations whose cycle decomposition consists of only two-cycles and the EGF for permutations whose cycle decomposition consists of only one-cycles. ▮

In Problem 388 you likely found that the EGF for the number of permutations of $[n]$ that are derangements is $\frac{e^{-x}}{1-x}$. But every permutation is a product of a derangement and a permutation whose cycle decomposition consists of one-cycles, because the permutation that sends i to i is a one-cycle, so that when you find the cycle decomposition of a permutation, the cycles of size greater than one are the cycle decomposition of a derangement (of the set of elements moved by the permutation), and the elements not moved by the permutation are one-cycles.

·391. If we multiply the EGF for derangements times the EGF for the number of permutations whose cycle decompositions consist of one-cycles only, what EGF do we get? For what set of objects have we found the EGF?

Solution: We get the EGF $\frac{1}{1-x}$ for all permutations of $[n]$. Notice that any permutation is a product of a derangement of the elements not fixed by the permutation times a permutation whose cycle decomposition consists of one-cycles. ▮

We now have four examples in which the EGF for a sequence or a pair of objects is the product of the EGFs for the individual objects making up the sequence or pair.

·392. What is the coefficient of $\frac{x^n}{n!}$ in the product of two EGFs $\sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$ and $\sum_{j=0}^{\infty} b_j \frac{x^j}{j!}$? (A summation sign is appropriate in your answer.)

Solution:

$$\sum_{i,j\colon\, i+j=n} n! \frac{a_i}{i!} \frac{b_j}{j!},$$

which can be better written as

$$\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a_i b_{n-i} = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

▮

In the case of painting streetlight poles in Problem 389, let us examine the relationship among the EGF for painting poles with two colors, the EGF for painting poles with three colors, and the EGF for painting poles with five colors, e^{5x}. To be specific, the EGF for painting poles red and white is e^{2x} and the EGF for painting poles blue, green, and yellow is e^{3x}. To decide how to paint poles with red, white, blue, green, and yellow, we can decide which set of poles is to be painted with red and white, and which set of poles is to be painted with blue, green and yellow. Notice that the number of ways to paint a set of poles with red and white depends only on the size of that set, and the number of ways to paint a set of poles with blue, green, and yellow depends only on the size of that set.

$\cdot$393. Suppose that a_i is the number of ways to paint a set of i poles with red and white, and b_j is the number of ways to paint a set of j poles with blue, green and yellow. In how many ways may we take a set N of n poles, divide it up into two sets I and J (using i to stand for the size of I and j to stand for the size of the set J, and allowing i and j to vary) and paint the poles in I red and white and the poles in J blue, green, and yellow? (Give your answer in terms of a_i and b_j. Don't figure out formulas for a_i and b_j to use in your answer; that will make it harder to get the point of the problem!) How does this relate to Problem 392?

Solution: $\binom{n}{i}a_ib_j$. You could also write the first answer as $\binom{n}{i}a_ib_{n-i}$ or in some other form. This shows that the coefficient of $\frac{x^n}{n!}$ in the EGF for painting poles with five colors is the coefficient of $\frac{x^n}{n!}$ in the product of the EGF for painting poles with two colors and the EGF for painting poles with three colors. ∎

Problem 393 shows that the formula you got for the coefficient of $\frac{x^n}{n!}$ in the product of two EGFs is the formula we get by splitting a set N of poles into two parts and painting the poles in the first part with red and white and the poles in the second part with blue, green, and yellow. More generally, you could interpret your result in Problem 392 to say that the coefficient of $\frac{x^n}{n!}$ in the product $\sum_{i=0}^{\infty} a_i\frac{x^i}{i!}\sum_{j=0}^{\infty} b_j\frac{x^j}{j!}$ of two EGFs is the sum, over all ways of splitting a set N of size n into an ordered pair of disjoint sets I of size i and J of size j, of the product a_ib_j.

There seem to be two essential features that relate to the product of exponential generating functions. First, we are considering **structures** that consist of a set and some additional mathematical construction on or relationship among the elements of that set. For example, our set might be a set of light poles and the additional construction might be a coloring function defined on that set. Other examples of additional mathematical constructions or relationships on a set could include a permutation of that set; in particular an involution or a derangement, a partition of that set, a graph on that set, a connected graph on that set, an arrangement of the elements of that set around a circle, or an arrangement of the elements of that set on the shelves of a bookcase. In fact a set with no additional construction or arrangement on it is also an example of a structure. Its additional construction is the empty set! When a structure consists of the set S plus the additional construction, we say the structure *uses* S. What all the examples we have mentioned in our earlier discussion of exponential generating functions have in common is that the number of structures that use a given set is determined by the size of that set. We will call a family $\mathcal{F}$ of structures a *species* of structures on subsets of a set X if structures are defined on finite subsets of X and if the number of structures in the family using a finite set S is finite and is determined by the size of S (that is, if there is a bijection between subsets S and T of X, the number of structures in the family that use S equals the number of structures in the family that use T). We say a structure is an $\mathcal{F}$*-structure* if it is a member of the family $\mathcal{F}$.

·394. In Problem 383, why is the family of arrangements of sets of books on a single shelf (assuming they all fit) a species?

Solution: Because the number of ways to put a set S of books onto a shelf is the same (namely $|S|!$) for all sets S of the same size. ∎

·395. In Problem 385, why is the family of sets of people actually making phone calls (assuming nobody is calling outside the telephone network) at any given time, with the added relationship of who is calling whom, a species? Why is the the family of sets of people who are not using their phones a species (with no additional construction needed)?

Solution: Because the number of ways to break a given set of $2n$ people into two-cycles depends only on n and not the particular set of $2n$ people we choose. The number of ways to break up a set of size k into one-cycles is one, so it doesn't depend on which set of size k we are breaking up. (In fact it doesn't depend on k either, but that is irrelevant here.) Of course since these people are doing nothing, the structure of one-cycles is just another way to say that our species consists of sets with no additional structure. ∎

The second essential feature of our examples of products of EGFs is that products of EGFs seem to count structures on ordered pairs of two disjoint sets (or more generally on k-tuples of mutually disjoint sets). For example, we can determine a five coloring of a set S by partitioning it in all possible ways into two sets and coloring the first set in the pair with our first two colors and our second pair with the last three colors. Or we can partition our set in all possible ways into five parts and color part i with our ith color. We don't have to do the same thing to each part of our partition; for example, we could define a derangement on one part and an identity permutation on the other; this defines a permutation on the set we are partitioning, and we have already noted that every permutation arises in this way.

Our combinatorial interpretation of EGFs will involve assuming that the coefficient of $\frac{x^i}{i!}$ counts the number of structures on a particular set of of size i in a species of structures on subsets of a set X. Thus in order to give an interpretation of the product of two EGFs we need to be able to think of ordered pairs of structures on disjoint sets or k-tuples of structures on disjoint sets as structures themselves. Thus given a structure on a set S and another structure on a disjoint set T, we define the ordered pair of structures (which is a mathematical construction!) to be a structure on the set $S \cup T$. We call this a *pair structure* on $S \cup T$. We can get many structures on a set $S \cup T$ in this way, because $S \cup T$ can be divided into many other pairs of disjoint sets. In particular, the set of pair structures whose first structure comes from $\mathcal{F}$ and whose second element comes from $\mathcal{G}$ is denoted by $\mathcal{F} \cdot \mathcal{G}$.

396. Show that if $\mathcal{F}$ and $\mathcal{G}$ are species of structures on subsets of a set X, then the pair structures of $\mathcal{F} \cdot \mathcal{G}$ form a species of structures.

Solution: We must show that if S and S' are finite subsets of X with the same size, then the number of pair structures on S and the number of pair structures on S' are the same. To get a pair structure on S we partition S into two parts, S_1 and S_2, take an $\mathcal{F}$ structure on S_1 and a $\mathcal{G}$ structure on S_2 and form the ordered pair of these structures. The number of ways to partition S into sets of size s_1 and s_2 is the same as the number of ways to partition S' into sets S'_1 and S'_2 of size s_1 and s_2 (there are intentionally no primes on the lower case s_1 and s_2) and the number of ways to choose an $\mathcal{F}$-structure on S'_1 and a $\mathcal{G}$-structure on S'_2 is the number of ways to make the same choices on S_1 and S_2. Therefore the number of pairs of structures on the disjoint sets S_1 and S_2 whose union is S is the same as the number of pairs of structures on the disjoint sets S'_1 and S'_2 whose union is S'. Summing over all ways to partition S or S' into two sets, we find that the number of pair structures on S equals the number of pair structures on S'. Thus the pair structure of $\mathcal{F} \cdot \mathcal{G}$ forms a species of structures. ∎

Given a species $\mathcal{F}$ of structures, the number of structures using any particular set of size i is the same as the number of structures in the family using any other set of size i. We can thus define the exponential generating function (EGF) for the family as the power series $\sum_{i=1}^{\infty} a_i \frac{x^i}{i!}$, where a_i is the number of structures of $\mathcal{F}$ that use one particular set of size i. In Problems 372, 373, 376, 377, 378, 380, 381, 382, 383, 387, and 388 we were computing EGFs for species of subsets of some set.

397. If $\mathcal{F}$ and $\mathcal{G}$ are species of subsets of X, how is the EGF for $\mathcal{F}\cdot\mathcal{G}$ related to the EGFs for $\mathcal{F}$ and $\mathcal{G}$? Prove you are right.

 Solution: Let $f(x)$, $g(x)$ and $h(x)$ be, respectively, the EGFs for the species $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F}\cdot\mathcal{G}$. Suppose $f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$ and $g(x) = \sum_{j=0}^{\infty} b_j \frac{x^j}{j!}$. Then the coefficient of x^n in $f(x)g(x)$ is $\sum_{k=0}^{n} \frac{a_k b_{n-k}}{i!j(n-k)!}$, so the coefficient of $\frac{x^n}{n!}$ is

$$n!\sum_{k=0}^{n} \frac{a_k b_{n-k}}{i!(n-k)!} = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

 But $\binom{n}{k} a_k b_{n-k}$ is the number of ways to partition a set of size n into a first set of size k and a second set of size $n-k$ and to choose a structure for each of them. That is, it is the number of pair structures on an ordered pair of sets of size k and $n-k$. Thus the coefficient of $\frac{x^n}{n!}$ in $h(x)$ is the number of $\mathcal{F}\cdot\mathcal{G}$ pair structures on a subset of X of size n. This proves that $h(x) = f(x)g(x)$. ▮

398. Without giving the proof, how can you compute the EGF $f(x)$ for the number of structures using a set of size n in the species $\mathcal{F}_1\cdot\mathcal{F}_2\cdots\mathcal{F}_k$ of structures on k-tuples of subsets of of X from the EGFs $f_i(x)$ for F_i for each i from 1 to k? (Here we are using the natural extension of the idea of the pair structure to the idea of a k-tuple structure.)

 Solution: $f(x) = \prod_{i=1}^{k} f_i(x)$. ▮

The result of Problem 398 will be of enough use to us that we will state it formally along with two useful corollaries.

Theorem 13 *If $\mathcal{F}_1$, $\mathcal{F}_2$, ..., $\mathcal{F}_k$ are species set X and $\mathcal{F}_i$ has EGF $f_i(x)$, then the family of k-tuple structures $\mathcal{F}_1\cdot\mathcal{F}_2\cdots\mathcal{F}_n$ has* EGF $\prod_{i=1}^{n} f_i(x)$.

We call Theorem 13 the **General Product Principle for Exponential Generating Functions**. We give two corollaries; the proof of the second is not immediate though not particularly difficult.

Corollary 3 *If $\mathcal{F}$ is a species of structures on subsets of X and $f(x)$ is the EGF for $\mathcal{F}$, then $f(x)^k/$ is the EGF for the k-tuple structures on k-tuples of $\mathcal{F}$-structures using disjoint subsets of X.*

Our next corollary uses the idea of a k-set structure. Suppose we have a species $\mathcal{F}$ of structures on nonempty subsets of X, that is, a species of structures which assigns no structures to the empty set. Then we can define a new species $\mathcal{F}^{(k)}$ of structures, called "k-set structures," using nonempty subsets of X. Given a fixed positive integer k, a *k-set structure* on a subset Y of X consists of a k-element set of nonempty disjoint subsets of X whose union is Y and an assignment of an $\mathcal{F}$-structure to each of the disjoint subsets. This is a species on the set of subsets of X; the subset used by a k-set structure is the union of the sets of the structure. To recapitulate, the set of k-set structures on a subset Y of X is the set of all possible assignments of $\mathcal{F}$-structures to k nonempty disjoint sets whose union is Y. (You can also think of the k-set structures as a family of structures defined on blocks of partitions of subsets of X into k blocks.)

Corollary 4 *If $\mathcal{F}$ is a species of structures on nonempty subsets of X and $f(x)$ is the EGF for $\mathcal{F}$, then for each positive integer k, $\frac{f(x)^k}{k!}$ is the EGF for the family $\mathcal{F}^{(k)}$ of k-set structures on subsets of X.*

399. Prove Corollary 4.

 Solution: Since the sets of a k-set structure are nonempty and disjoint, the k-element set of sets can be arranged as a k-tuple in $k!$ ways. Thus the number of k-set structures on a given set is $\frac{1}{k!}$ times the number of k-tuple structures on that set. Therefore the EGF for k-set structures is $\frac{1}{k!}$ times the EGF for k-tuple structures. By Corollary 3 the EGF for k-set structures is thus $\frac{f(x)^k}{k!}$. ∎

·400. Use the product principle for EGFs to explain the results of Problems 390 and 391.

 Solution: Every involution has a cycle decomposition as disjoint two-cycles and one-cycles, so we can think of it as an ordered pair whose first element is a set of disjoint two-cycles and whose second element is a set of disjoint one-cycles. The family of permutations whose cycle decomposition consists entirely of two-cycles and the family of permutations whose cycle decomposition consists entirely of one-cycles are both species. By the product principle for EGFs, the EGF for involutions is the product of the EGF for permutations whose cycle decomposition consists of only disjoint two-cycles and the EGF for permutations whose cycle decomposition consists of only disjoint one-cycles (i.e. identity permutations).

We noted in the solution to Problem 391 that every permutation has a cycle decomposition consisting of the cycle decomposition of a derangement (on the elements that are not fixed by the permutation) and the cycle decomposition of an identity (on the elements that are fixed by the permutation). Thus we can think of every permutation as an ordered pair consisting of a derangement and an identity (on complementary domains), and the product principle tells us that the EGF for all permutations is the product of the EGF for derangements and the EGF for identity permutations. ▮

·401. Use the general product principle for EGFs or one of its corollaries to explain the relationship between the EGF for painting streetlight poles in only one color and the EGF for painting streetlight poles in 5 colors in Problems 373 and 389. What is the EGF for the number p_n of ways to paint n streetlight poles with some fixed number k of colors of paint?

Solution: We can think of a painting of a set of street poles as a five-tuple of sets, the sets painted each of the five colors. Then Corollary 3 tells us that the EGF for such five-tuples is the fifth power of the EGF for the number of ways to paint streetlight poles with one color. The EGF for painting streetlight poles with k colors of paint is e^{kx}. ▮

·402. Use the general product principle for EGFs or one of its corollaries to explain the relationship between the EGF for arranging books on one shelf and the EGF for arranging books on n shelves in Problem 383.

Solution: An arrangement of books on n shelves may be thought of as a n-tuple of arrangements of books on one shelf. More precisely, structures of arrangements of books on a shelf or, similarly, the arrangements of books on n shelves are species of structures on the subsets of the set of available books. Corollary 3 tells us that the EGF for arrangements on n shelves is the nth power of the EGF for arranging books on one shelf, which is the EGF for permutations. Thus the EGF for arranging books on n shelves is $(1-x)^{-n}$. ▮

➔403. (Optional) Our very first example of exponential generating functions used the binomial theorem to show that the EGF for k-element permutations of an n element set is $(1+x)^n$. Use the EGF for k-element permutations of a one-element set and the product principle to prove the same thing. Hint: Review the alternate definition of a function in Section 3.1.2.

Solution: In Section 3.1.2 we remarked that an alternate definition of a function from S to T is that it is an assignment of disjoint subsets of S to elements of T so that the union of the subsets is S. Thus a function from $[k]$ to $[n]$ may be thought of as an n-tuple of disjoint subsets of S whose union is $[k]$. In particular, an injection from $[k]$ to $[n]$ (which is a k-element permutation of $[n]$) can be thought of as a n-tuple of disjoint singleton sets and empty sets whose union is $[n]$. The number of such n-tuples is therefore the number of k-element permutations of $[n]$. If $n = 1$, the possible n-tuples are $(\emptyset)$ and $(\{1\})$, and so the EGF for such n-tuples is $1 + x$. (Note that the family of structures here has no additional construction; it simply consists of the empty set and the set $\{1\}$, which is a "trivial" example of a species of structures on the subsets of $\{1\}$.) Then by Corollary 3, the EGF for the number of n-tuples $(1 + x)^n$. Thus this is the EGF for k-element permutations of $[n]$. ▮

404. What is the EGF for the number of ways to paint n streetlight poles red, white, blue, green and yellow, assuming an even number of poles must be painted green and an even number of poles must be painted yellow? Give a formula for the number of ways to paint n poles. (Don't forget the factorial!)

Solution: By the product principle for exponential generating functions it is

$$e^{3x}\left(1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\cdots\right)=e^{3x}\frac{e^{2x}-e^{-2x}}{2}=e^{3x}\cosh(2x).$$

Since $\cosh(2x) = \sum_{i=0}^{\infty}\frac{x^{2i}}{(2i)!}$ and e^{3x} is $\sum_{i=0}^{\infty}3^i\frac{x^i}{i!}$, we have that the coefficient of x^n is $\sum_{i=0}^{\lfloor n/2\rfloor}\frac{3^{n-2i}}{(n-2i)!}\frac{2^{2i}}{(2i)!}$, and so the number of ways to paint n lightpoles is $n!$ times this, which simplifies to $\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}3^{n-2i}2^{2i}$. ▮

➔ ·405. What is the EGF for the number of functions from an n-element set onto a one-element set? (Can there be any functions from the empty set onto a one-element set?) What is the EGF for the number c_n

of functions from an n-element set onto a k element set (where k is fixed)? Use this EGF to find an explicit expression for the number of functions from a k-element set onto an n-element set and compare the result with what you got by inclusion and exclusion.

Solution: There are no onto functions from a 0-element set to a 1-element set; otherwise there is exactly one function from an n-element set onto a one-element set so the EGF for functions from an n-element set onto a one-element set is $e^x - 1$. A function from an n-element set onto a k-element set may be thought of as a k-tuple of functions (from disjoint subsets whose union is the n-element set) onto the one-element subsets of the k-element set. Therefore by Corollary 3 to the product principle for EGFs the EGF for functions from an n-element set onto a k-element set is $(e^x - 1)^k$. By the binomial theorem, this is

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}e^{ix} = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=0}^{\infty}\frac{(ix)^j}{j!} = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{j=0}^{\infty}i^j\frac{(x)^j}{j!}.$$

Thus $c_n = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}n^i$, which is consistent with the formula we got by inclusion and exclusion. ▮

➔ · 406. In Problem 142 you showed that the Bell Numbers B_n satisfy the equation $B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}$ (or a similar equation for B_n). Multiply both sides of this equation by $\frac{x^n}{n!}$ and sum from $n = 0$ to infinity. On the left hand side you have a derivative of a certain EGF we might call $B(x)$. On the right hand side, you have a product of two EGFs, one of which is $B(x)$. What is the other one? What differential equation involving $B(x)$ does this give you? Solve the differential equation for $B(x)$. This is the EGF for the Bell numbers!

Solution:

$$\sum_{n=0}^{\infty}B_{n+1}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\binom{n}{k}B_{n-k}\frac{x^n}{n!} = \sum_{i=0}^{\infty}B_i\frac{x^i}{(i)!}\sum_{j=0}^{\infty}\frac{x^j}{j!}.$$

Thus $B'(x) = B(x)e^x$, which gives us $\ln B(x) = e^x + c$, or $B(x) = e^{ex+c} = e^c e^{(e^x)}$. Since $B_0 = B(0)$ and $B_0 = 1$, we have $c = -1$ and $B(x) = \exp(e^x - 1)$. ▮

➔ 407. Prove that $n2^{n-1} = \sum_{k=1}^{n}\binom{n}{k}k$ by using EGFs.

Solution: By the product principle for EGFs, the EGF for the right hand side is

$$e^x \cdot xe^x = xe^{2x} = x\sum_{i=o}^{\infty}\frac{(2x)^i}{i!} = \sum_{j=1}^{\infty}2^{j-1}\frac{x^j}{(j-1)!} = \sum_{j=1}^{\infty}j2^{j-1}\frac{x^j}{j!}.$$

Thus the coefficient of $\frac{x^n}{n!}$ is $n2^{n-1}$, as well as $\sum_{k=1}^{n}\binom{n}{k}k$. ▮

·408. In light of Problem 382, why is the EGF for the Stirling numbers $S(n,k)$ of the second kind (with n fixed and k allowed to vary) not $(e^x-1)^n$? What is it equal to instead?

Solution: Notice that a one block partition is the same thing as a function from that block onto a one-element set. However, a partition with n blocks is not an n-tuple of blocks, but rather a set of n blocks. An n-tuple of blocks corresponds to a function from the union of the blocks onto an n-element set, and $n!$ different onto functions correspond to the same partition into n blocks. Thus the EGF for partitions of an n-element set into k parts (where n is fixed but k varies) is $\frac{1}{n!}(e^x-1)^n$. We could also use Corollary 4 directly. ▮

C.5 The Exponential Formula

Exponential generating functions turn out to be quite useful in advanced work in combinatorics. One reason why is that it is often possible to give a combinatorial interpretation to the composition of two exponential generating functions. In particular, if $f(x) = \sum_{i=0}^{n} a_i\frac{x^i}{i!}$ and $g(x) = \sum_{j=1}^{\infty} b_j\frac{x^j}{j!}$, it makes sense to form the composition $f(g(x))$ because in so doing we need add together only finitely many terms in order to find the coefficient of $\frac{x^n}{n!}$ in $f(g(x))$ (since in the EGF $g(x)$ the dummy variable j starts at 1). Since our study of combinatorial structures has not been advanced enough to give us applications of a general formula for the compositions of EGFs, we will not give here the combinatorial interpretation of composition in general. However, we have seen some examples where one particular composition can be applied. Namely, if $f(x) = e^x = \exp(x)$, then $f(g(x)) = \ exp(g(x))$ is well defined when $b_0 = 0$. We have seen three examples in which an EGF is $e^{f(x)}$ where $f(x)$ is another EGF. There is a fourth example in which the exponential function is slightly hidden.

·409. If $f(x)$ is the EGF for the number of partitions of an n-set into one block, and $g(x)$ is the EGF for the total number of partitions of an

n-element set, that is, for the Bell numbers B_n, how are the two EGFs related?

Solution: The EGF for one-block partitions is $e^x - 1$ and for the Bell numbers is $\exp(e^x-1)$, and so the EGF for the Bell numbers is the composition of the exponential function with the EGF for one-block partitions. ▮

$\cdot$410. Let $f(x)$ be the EGF for the number of permutations of an n-element set with one cycle of size one or two and no other cycles, including no other one-cycles. What is $f(x)$? What is the EGF $g(x)$ for the number of permutations of an n-element set all of whose cycles have size one or two, that is, the number of involutions in S_n, or the number of configurations of a telephone network? How are these two exponential generating functions related?

Solution: There is one permutation with one cycle of size 1, and one permutation with one cycle of size 2. Therefore the EGF for such permutations is $x+\frac{x^2}{2!} = x+x^2/2$. The EGF for involutions is $e^{x+x^2/2}$. Thus $g(x) = \exp(f(x))$. ▮

➔ $\cdot$411. Let $f(x)$ be the EGF for the number of permutations of an n-element set whose cycle decomposition consists of exactly one two-cycle and no other cycles (this includes having no one-cycles). Let $g(x)$ be the EGF for the number of permutations whose cycle decomposition consists of two-cycles only, that is, for tennis pairings. What is $f(x)$? What is $g(x)$? How are these two exponential generating functions related?

Solution: The EGF $f(x)$ for permutations of an n-element set that have exactly one two-cycle (and no other cycles) is $\frac{x^2}{2!}$. By Problem 378, the EGF for permutations whose cycle structure consists of two-cycles only is $\exp(x^2/2)$. Thus $g(x) = \exp(f(x)) = e^{x^2/2}$. ▮

$\cdot$412. Let $f(x)$ be the EGF for the number of permutations of an n-element set that have exactly one cycle. Notice that if $n > 1$ this means they have no one-cycles. (This is the same as the EGF for the number of ways to arrange n people around a round table.) Let $g(x)$ be the EGF for the total number of permutations of an n-element set. What is $f(x)$? What is $g(x)$? How are $f(x)$ and $g(x)$ related?

Solution: In Problem 377 we showed that $f(x) = \ln\left(\frac{1}{1-x}\right)$. In Problem 376 we showed that $g(x) = \frac{1}{1-x}$. Therefore $g(x) = \exp(f(x))$. ▮

There was one element that our last four problems had in common. In each case our EGF $f(x)$ involved the number of structures of a certain type (partitions, telephone networks, tennis pairings, permutations) that used only one set of an appropriate kind. (That is, we had a partition with one part, a telephone network consisting either of one person or two people connected to each other, a tennis pairing of one set of two people, or a permutation with one cycle.) Our EGF $g(x)$ was the number of structures of the same "type" (we put type in quotation marks here because we don't plan to define it formally) that could consist of any number of sets of the appropriate kind. Notice that the order of these sets was irrelevant. For example, we don't order the blocks of a partition or the cycles in a cycle decomposition of a permutation. Thus we were relating the EGF for structures which were somehow "building blocks" to the EGF for structures which were sets of building blocks. For a reason that you will see later, it is common to call the building blocks *connected* structures. Notice that our connected structures were all based on nonempty sets, so we had no connected structures whose value was the empty set. Thus in each case, if $f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$, we would have $a_0 = 0$. The relationship between the EGFs was always $g(x) = e^{f(x)}$. We now give a combinatorial explanation for this relationship.

·413. Suppose that $\mathcal{F}$ is a species of structures on subsets of a set X with no structures on the empty set. Let $f(x)$ be the EGF for $\mathcal{F}$.

(a) In the power series

$$e^{f(x)} = 1 + f(x) + \frac{f(x)^2}{2!} + \cdots + \frac{f(x)^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{f(x)^k}{k!},$$

what does Corollary 4 tell us about the coefficient of $\frac{x^n}{n!}$ in $\frac{f(x)^k}{k!}$?

Solution: It tells us that the coefficient of $\frac{x^n}{n!}$ is the number of k-set structures using a particular set with n elements. ▮

(b) What does the coefficient of $\frac{x^n}{n!}$ in $e^{f(x)}$ count?

Solution: It counts the total number of sets of disjoint structures which together use a particular set with n elements. ▮

In Problem 413 we proved the following theorem, which is called the **exponential formula**.

Theorem 14 *Suppose that $\mathcal{F}$ is a species of structures on subsets of a set X with no structures on the empty set. Let $f(x)$ be the EGF for $\mathcal{F}$. Then* *the coefficient of $\frac{x^n}{n!}$ in $e^{f(x)}$ is the number of sets of structures on disjoint sets whose union is a particular set of size n.*

Let us see how the exponential formula applies to the examples in Problems 409, 410, 411 and 412. In Problem 382 our family $\mathcal{F}$ should consist of one-block partitions of finite subsets of a set, say the set of natural numbers. Since a partition of a set is a set of blocks whose union is S, a one block partition whose block is B is the set $\{B\}$. Then any nonempty finite subset of the natural numbers is the set used by exactly one structure in $\mathcal{F}$. (There is no one block partition of the empty set, so we have no structures using the empty set.) As you showed in Problem 382 the EGF for partitions with just one block is $e^x - 1$. Thus by the exponential formula, $\exp(e^x - 1)$ is the EGF for sets of disjoint subsets of the positive integers whose union is any particular set N of size n. This set of disjoint sets partitions the set N. Thus $\exp(e^x - 1)$ is the EGF for partitions of sets of size n. (As we wrote our description, it is the EGF for partitions of n-element subsets of the positive integers, but any two n-element sets have the same number of partitions.) In other words, $\exp(e^x - 1)$ is the exponential generating function for the Bell numbers B_n.

·414. Explain how the exponential formula proves the relationship we saw in Problem 412.

Solution: We take $\mathcal{F}$ to be the species of permutations of finite sets of positive integers whose cycle decomposition consists of exactly one cycle. The number of cycles using any given n-element set of positive integers is $(n-1)!$, so we have defined a species. Then by the exponential formula, if $f(x)$ is the EGF for permutations with one cycle, $\exp(f(x)) = g(x)$ is the EGF in which the coefficient of $\frac{x^n}{n!}$ is the number of sets of cycles that partition any given set N of size n. That is, the coefficient of $\frac{x^n}{n!}$ in $g(x)$ is the number of permutations whose cycles partition any given set N. Therefore, $g(x)$ is the EGF for permutations of N. ▮

·415. Explain how the exponential formula proves the relationship we saw in Problem 411.

Solution: We let $\mathcal{F}$ be the family of permutations of finite sets of positive integers whose cycle decomposition consists of exactly one two-cycle. Since the number of two-cycle structures on a two-element set is one and the number on a set of any other size is 0, we have a species of structures on the finite subsets of the positive integers.

We saw in Problem 411 that the EGF for permutations whose cycle decomposition consists of exactly one two-cycle is $x^2/2$. By the exponential formula, the EGF for finite sets of disjoint two-cycles is $\exp(x^2/2)$. But sets of disjoint two-cycles correspond bijectively with permutations of finite subsets of the positive integers whose cycle decompositions consist of two-cycles (only), and this confirms the result of Problem 378. ▮

·416. Explain how the exponential formula proves the relationship we saw in Problem 410.

Solution: We let the family $\mathcal{F}$ be the set of permutations of subsets of the positive integers whose cycle decomposition is either one one-cycle or one two-cycle. Then $\mathcal{F}$ is a species of structures on the finite subsets of the positive integers. We saw in Problem 410 that the EGF for $\mathcal{F}$ is $x + x^2/2$. By the exponential formula, the EGF for sets of disjoint one and two-cycles is $e^{x+x^2/2}$. But there is a bijection between the sets of disjoint one and two-cycles and permutations whose cycle decomposition consists of disjoint one and two-cycles. This confirms the result of Problem 387. ▮

·417. In Problem 373 we saw that the EGF for the number of ways to use five colors of paint to paint n light poles along the north side of Main Street in Anytown was e^{5x}. We should expect an explanation of this EGF using the exponential formula. Let $\mathcal{F}$ be the family of all one-element sets of light poles with the additional construction of an ordered pair consisting of a light pole and a color. Thus a given light pole occurs in five ordered pairs. Put no structure on any other finite set. Show that this is a species of structures on the finite subsets of the positive integers. What is the exponential generating function $f(x)$ for $\mathcal{F}$? Assuming that there is no upper limit on the number of light poles, what subsets of S does the exponential formula tell us are counted by the coefficient of x^n in $e^{f(x)}$? How do the sets being counted relate to ways to paint light poles?

Solution: Since each one-element set has five structures on it and each set of any other size has no structures on it, $\mathcal{F}$ is a species. The EGF for $\mathcal{F}$ is $5x$, because there are five ordered pairs using any given one-element set and none using any other set. Note that a set of ordered pairs whose first elements partition a set N of light poles is exactly a function from the set N of light poles to the set of colors. Then by the exponential formula, the EGF for the number of ways to paint n light poles with five colors is e^{5x}. ▮

One of the most spectacular applications of the exponential formula is also the reason why, when we regard a combinatorial structure as a set of building block structures, we call the building block structures *connected*. In Chapter 2 we introduced the idea of a connected graph and in Problem 104 we saw examples of graphs which were connected and were not connected. A subset C of the vertex set of a graph is called a **connected component** of the graph if

- every vertex in C is connected to every other vertex in that set by a walk whose vertices lie in C, and
- no other vertex in the graph is connected by a walk to any vertex in C.

In Problem 241 we showed that each connected component of a graph consists of a vertex and all vertices connected to it by walks in the graph.

·418. Show that every vertex of a graph lies in one and only one connected component of a graph. (Notice that this shows that the connected components of a graph form a partition of the vertex set of the graph.)

Solution: Let C be the set of all vertices connected by a walk to a vertex x. Then

- Each pair of vertices u and v in C is connected by the walk that goes from u to x and then from x to v.
- If a vertex w is connected by a walk to a vertex v in C, then it is connected to x by the walk that goes from w to v and then from v to x. Thus no vertex w in the graph other than a member of C is connected by a walk to any vertex in C.

Therefore C is a connected component containing x. If a connected component D contained x, then every vertex in D would be connected by a walk to x and then by a walk from x to v for each other vertex v in C. Similarly, each vertex in C would be connected to each vertex in D. Thus by the definition of connected component, C and D would have to be the same set. Therefore each vertex lies in one and only one connected component. ▮

·419. Explain why no edge of the graph connects two vertices in different connected components.

Solution: If an edge connected two vertices in different connected components, that edge would give a walk from a vertex in one of the connected components to a vertex in the other connected component, and thus not in the first component, violating the second part of the definition of a connected component. ▮

·420. Explain why it is that if C is a connected component of a graph and E' is the set of all edges of the graph that connect vertices in C, then the graph with vertex set C and edge set E' is a connected graph. We call this graph a *connected component graph* of the original graph.

Solution: Given a walk between two vertices in a connected component, all edges of the walk must connect two vertices in the connected component, because if there were an edge in the walk that did not do so, it would violate the second part of the definition of a connected component. Since, for each pair of vertices in the connected component there is a walk between them, there is such a walk in our connected component graph. ▮

The last sequence of problems shows that we may think of any graph as the set of its connected component graphs. (Once we know them, we know all the vertices and all the edges of the graph.) Notice that a graph is connected if and only if it has exactly one connected component. Since the connected components form a partition of the vertex set of a graph, the exponential formula will relate the EGF for the number of connected graphs on n vertices with the EGF for the number of graphs (connected or not) on n vertices. However, because we can draw as many edges as we want between two vertices of a graph, there are infinitely many graphs on n vertices, and so the problem of counting them is uninteresting. We can make it interesting by considering **simple graphs**, namely graphs in which each edge has two distinct endpoints and no two edges connect the same two vertices. It is because *connected* simple graphs form the building blocks for viewing all simple graphs as sets of connected components that we refer to the building blocks for structures counted by the EGFs in the exponential formula as *connected* structures.

➔ ·421. Suppose that $f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$ is the exponential generating function for the number of simple connected graphs on n vertices and $g(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$ is the exponential generating function for the number of simple graphs on i vertices. From this point onward, any use of the word graph means simple graph.

(a) Is $f(x) = e^{g(x)}$, is $f(x) = e^{g(x)-1}$, is $g(x) = e^{f(x)-1}$ or is $g(x) = e^{f(x)}$?

Solution: To apply the exponential formula, we must take the exponential function of an EGF whose constant term is zero, or in other words, for a species of structures that has no structures that use the empty set. We can let $\mathcal{F}$ be the set structures consisting of finite subsets of a set and (all) connected graphs on the nonempty sets. (Technically, the graph with the empty set of vertices and the empty set of edges is connected. That is why we consider only connected graphs on the nonempty sets.) Therefore $f(x)-1$ is the EGF for $\mathcal{F}$. By the exponential formula, $g(x) = e^{f(x)-1}$ because a simple graph may be thought of as a set of simple connected graphs, namely its connected component graphs. (Note that $g(x)$ has 1 for its constant term, which corresponds to thinking of the empty graph as having an empty set of nonempty connected components.) ▮

(b) One of a_i and c_n can be computed by recognizing that a simple graph on a vertex set V is completely determined by its edge set and its edge set is a subset of the set of two-element subsets of V. Figure out which it is and compute it.

Solution: To specify a simple graph on a vertex set V, we have to specify its set of edges. The possible sets of edges thus correspond bijectively to sets of two-element subsets of V. But if V has size i the set of all two-element subsets of V has $\binom{i}{2}$ elements. Thus the number of sets of two-element subsets of V is $2^{\binom{i}{2}}$. Therefore $a_i = 2^{\binom{i}{2}}$. ▮

(c) Write $g(x)$ in terms of the natural logarithm of $f(x)$ or $f(x)$ in terms of the natural logarithm of $g(x)$.

Solution: Since $g(x) = e^{f(x)-1}$, $f(x) = 1 + \ln g(x)$. ▮

(d) Write $\log(1+y)$ as a power series in y.

Solution:

$$\log(1+y) = \int_0^y \frac{1}{1+x}dx = \int_0^y \sum_{i=0}^{\infty}(-1)^i x^i = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{y^j}{j}.$$

▮

(e) Why is the coefficient of $\frac{x^0}{0!}$ in $g(x)$ equal to one? Write $f(x)$ as a power series in $g(x) - 1$.

Solution: The coefficient of $\frac{x^0}{0!}$ is 1 because there is one graph on the empty set; the one with no edges.

$$f(x) = 1 + \ln(1 + (g(x) - 1)) = 1 + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(g(x) - 1)^j}{j}.$$

∎

(f) You can now use the previous parts of the problem to find a formula for c_n that involves summing over all partitions of the integer n. (It isn't the simplest formula in the world, and it isn't the easiest formula in the world to figure out, but it is nonetheless a formula with which one could actually make computations!) Find such a formula.

Solution:

$$f(x) = 1 + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(g(x) - 1)^j}{j} = 1 + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(\sum_{i=1}^{\infty} 2^{\binom{i}{2}} \frac{x^i}{i!})^j}{j}.$$

From the right-hand expression, we get a term involving x^n whenever we have an x^n term in the jth power of $\sum_{i=1}^{\infty} 2^{\binom{i}{2}} \frac{x^i}{i!}$. So the coefficient of $\frac{x^n}{n!}$ is the sum over all j and all sequences $i_1, i_2, \ldots, i_j$ with $i_1 + i_2 + \cdots + i_j = n$ of terms of the form

$$\frac{n!}{j} (-1)^j \prod_{k=1}^{j} \frac{2^{\binom{i_k}{2}}}{i_k!},$$

where each $i_k > 0$. Notice that reordering the numbers i_1, i_2, $\ldots i_k$ does not change the value of the expression. The sequence of i_ks is a composition of n into positive parts. If we knew how many compositions of n into j parts correspond to one partition of n into j parts, we could sum over a much smaller set of terms. If we use the type vector notation for a partition, namely that it has p_1 parts of size 1, p_2 parts of size 2, $\ldots$, p_n parts of size n, then the number of compositions corresponding to that partition, i.e., the number of compositions with the type vector $(p_1, p_2, \ldots, p_n)$ is the number of ways to take j places in a vector and label p_1 of them with 1, p_2 of them with 2, and so on until we label p_n of them with n. This number is the multinomial coefficient $\binom{j}{p_1, p_2, \ldots, p_n}$. Thus our sum over all j and all compositions of n

into j parts becomes

$$\sum_{j=1}^{n} \frac{n!}{j}(-1)^j \sum_{\substack{p_1,p_2,\ldots,p_n:\sum_{r=1}^n rp_r=n\\ \text{and}\sum_{r=1}^n p_r=j}} \binom{j}{p_1,p_2,\ldots,p_n} \prod_{r=1}^{n} \frac{2^{\binom{r}{2}p_r}}{(r!)^{p_r}}.$$

We can remove one of the summation signs and the condition that $\sum_{r=1}^n p_r = j$ by substituting $\sum_{r=1}^n p_r$ for j, and we get

$$\sum_{p_1,p_2,\ldots,p_n:\sum_{r=1}^n rp_r=n} n!(-1)^{\sum_{r=1}^n p_r} \left(\sum_{r=1}^{n} p_r - 1\right)! \prod_{r=1}^{n} \frac{2^{\binom{r}{2}p_r}}{(r!)^{p_r} p_r!}$$

for the number of connected graphs on n vertices. If we want to shorten the appearance of the formula we can keep j in our sum and explain its value afterwards, as in

$$\sum_{p_1,p_2,\ldots,p_n:\sum_{r=1}^n rp_r=n} n!(-1)^j(j-1)! \prod_{r=1}^{n} \frac{2^{\binom{r}{2}p_r}}{(r!)^{p_r} p_r!},$$

where $j = \sum_{r=1}^n p_r$. ∎

The point to the last problem is that we can use the exponential formula in reverse to say that if $g(x)$ is the EGF for the number of (nonempty) connected structures of size n in a given family of combinatorial structures and $f(x)$ is the EGF for all the structures of size n in that family, then not only is $f(x) = e^{g(x)}$, but $g(x) = \ln(f(x))$ as well. Further, if we happen to have a formula for either the coefficients of $f(x)$ or the coefficients of $g(x)$, we can get a formula for the coefficients of the other one!

C.6 Supplementary Problems

1. Use product principle for EGFs and the idea of coloring a set in two colors to prove the formula $e^x \cdot e^x = e^{2x}$.

2. Find the EGF for the number of ordered functions from a k-element set to an n-element set.

3. Find the EGF for the number of ways to string n distinct beads onto a necklace.

4. Find the exponential generating function for the number of broken permutations of a k-element set into n parts.

5. Find the EGF for the total number of broken permutations of a k-element set.

6. Find the EGF for the number of graphs on n vertices in which every vertex has degree 2.

7. Recall that a cycle of a permutation cannot be empty.

   (a) What is the EGF for the number of cycles on an even number of elements (i.e. permutations of an even number n of elements that form an n-cycle)? Your answer should not have a summation sign in it. Hint: If $y = \sum_{i=0}^{\infty} \frac{x^{2i}}{2i}$, what is the derivative of y?

   (b) What is the EGF for the number of permutations on n elements whose cycle decomposition consists of even cycles?

   (c) What is the EGF for the number of cycles on an odd number of elements?

   (d) What is the EGF for the number of permutations on n elements whose cycle decomposition consists of odd cycles?

   (e) How do the EGFs in parts (b) and (d) of this problem relate to the EGF for all permutations on n elements?

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