Lecture 8 Activity: Product and Quotient Rules

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1. Compute derivatives of the following functions.
   1.1 $x^2 e^x$.
   1.2 $\frac{x-1}{x+1}$.
   1.3 $\frac{x^2+x-1}{x^3}$.
   1.4 $\frac{xe^x}{x^2+1}$.
2. What is the tangent line to $\frac{x-1}{x+1}$ at $x = 1$?
3. Suppose $f(1) = 3$, $f'(1) = -1$, $g(1) = 5$, $g'(1) = 2$, and $h'(1) = 3$. Which of the following can be determined from this information, and why?
   3.1 $(f+g)'(1)$ (the derivative of $f(x) + g(x)$ at $x = 1$).
   3.2 $(g-h)'(1)$ (the derivative of $g(x) - h(x)$ at $x = 1$).
   3.3 $(fg)'(1)$ (the derivative of $f(x)g(x)$ at $x = 1$).
   3.4 $(fh)'(1)$ (the derivative of $f(x)h(x)$ at $x = 1$).
4. Find a function $f(x)$ such that $f'(x) = xe^x$. (**Hint:** It looks like $axe^x + be^x$ for some constants a and b.)
5. **Challenge Problem:** Use the limit definition of the derivative to prove the product rule. (**Hint:** You'll start with $f(x+h)g(x+h) - f(x)g(x)$ in the numerator of the limit. Use algebra to change this to $(f(x+h) - f(x))g(x) + f(x)(g(x+h) - g(x))$.)

Solution 1.1

Using the product rule,

$$(x^2e^x)' = (x^2)'e^x + x^2(e^x)' = 2xe^x + x^2e^x.$$

Solution 1.2

Using the quotient rule,

$$\begin{aligned}\left(\frac{x-1}{x+1}\right)' &= \frac{(x+1)(x-1)' - (x-1)(x+1)'}{(x+1)^2}\\ &= \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}\\ &= \frac{2}{(x+1)^2}.\end{aligned}$$

Solution 1.3

Using the quotient rule,

$$\begin{aligned}
\left(\frac{x^2+x-1}{x^3}\right)' &= \frac{(x^3)(x^2+x-1)' - (x^2+x-1)(x^3)'}{(x^3)^2} \\
&= \frac{(x^3)\cdot(2x+1) - (x^2+x-1)\cdot(3x^2)}{x^6} \\
&= \frac{(2x^4+x^3) - (3x^4+3x^3-3x^2)}{x^6} \\
&= \frac{-x^4-2x^3+3x^2}{x^6} \\
&= \frac{-x^2-2x+3}{x^4}.
\end{aligned}$$

Solution 1.4

This one is tricky. We have a product of two things in the numerator, and there is also something in the denominator. That means we'll have to use both the product rule and the quotient rule. Let's focus for a moment on just the numerator. We can calculate its derivative with the product rule:

$$(xe^x)' = e^x + xe^x.$$

Now let's look at the whole thing:

$$\begin{aligned}
\left(\frac{xe^x}{x^2+1}\right)' &= \frac{(x^2+1)(xe^x)' - (xe^x)(x^2+1)'}{(x^2+1)^2} \\
&= \frac{(x^2+1)(e^x + xe^x) - (xe^x)(2x)}{(x^2+1)^2} \\
&= \frac{(x^2e^x + e^x + x^3e^x + xe^x) - 2x^2e^x}{(x^2+1)^2} \\
&= \frac{x^3e^x - x^2e^x + xe^x + e^x}{(x^2+1)^2}
\end{aligned}$$

Solution 2

We know from Solution 1.2 that, if $f(x) = \frac{x-1}{x+1}$, then $f'(x) = \frac{2}{(x+1)^2}$. So, if the tangent line is $y = mx + b$, then $m = f'(1) = \frac{2}{2^2} = \frac{1}{2}$. Since $f(1) = 0$, we should have $0 = \frac{1}{2}(1) + b$, so $b = -\frac{1}{2}$. Thus, the tangent line is $y = \frac{1}{2}x - \frac{1}{2}$.

Solution 3

1. We have $(f+g)'(1) = f'(1) + g'(1) = (-1) + 2 = 1$.
2. We have $(g-h)'(1) = g'(1) - h'(1) = 2 - 3 = -1$.
3. We have $(fg)'(1) = f(1)g'(1) + f'(1)g(1) = 3 \cdot 2 + (-1) \cdot 5 = 1$.
4. We have
   $(fh)'(1) = f(1)h'(1) + f'(1)h(1) = 3 \cdot 3 + (-1) \cdot h(1) = 9 - h(1)$.
   Since we don't know $h(1)$, we can't figure out what $(fh)'(1)$ is.

Solution 4

If we guess that $f(x) = axe^x + be^x$, we can differentiate:

$$\frac{d}{dx}(axe^x + be^x) = a\frac{d}{dx}(xe^x) + b\frac{d}{dx}(e^x) = a(xe^x + e^x) + be^x = axe^x + (a+b)e^x.$$

We want $f' = xe^x$, so we set $a = 1$ and $a + b = 0$, so $b = -1$. Thus, we end up with $f(x) = xe^x - e^x$.

Solution 5

First, let's think about the hint in the problem. One way to see that this is true is to simplify the second, longer expression—if you do this, you'll just end up with the first expression. We use the function fg in the limit definition of the derivative:

$$\begin{aligned}
&\lim_{h\to 0} \frac{(fg)(x+h)-(fg)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h)-f(x))g(x)+f(x)(g(x+h)-g(x))}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h)-f(x))g(x)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h)-g(x))}{h} \\
&= g(x)\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + f(x)\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= g(x)f'(x)+f(x)g'(x).
\end{aligned}$$