Lecture 9 Activity: Trigonometric Derivatives, Chain Rule Preview

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math.dartmouth.edu/~blogsdon/activity09.pdf

1. Compute derivatives of the following functions.
   1.1 $\csc x = \frac{1}{\sin x}$
   1.2 $\sec x = \frac{1}{\cos x}$
   1.3 $\cot x = \frac{\cos x}{\sin x}$
   1.4 $e^x \sin x$
   1.5 $\frac{\sin x \cos x}{x^2+2x+1}$
2. Let $f(x) = a\cos x + b\sin x$.
   2.1 Suppose that $f(0) = 3$ and $f(\pi/2) = -2$. What are a and b?
   2.2 Suppose that $f(0) = -1$ and $f'(\pi) = 4$. What are a and b?
   2.3 Suppose that $f''(\pi) = 0$ and $f'''(2\pi) = 3$. What are a and b?
3. For each of the functions below, determine whether or not you would use the chain rule to take the derivative of the function. If it is, what is u?
   3.1 $\sin(e^x)$
   3.2 $e^x \sin x$
   3.3 $\cos(1/x)$
   3.4 $\frac{1}{x^2}$
   3.5 $5x + e^{(x^2)}$
   3.6 $\sin\cos x$

Solution 1.1

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{\sin x}\right) &= \frac{\sin x \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(\sin x)}{\sin^2 x} \\ &= \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2 x} \\ &= \frac{-\cos x}{\sin^2 x}.\end{aligned}$$

Solution 1.2

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{\cos x}\right) &= \frac{\cos x \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x}.
\end{aligned}$$

Solution 1.3

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) &= \frac{\sin x \cdot \frac{d}{dx}(\cos x) - \cos x \cdot \frac{d}{dx}(\sin x)}{\sin^2 x} \\
&= \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\cos^2 x} \\
&= -\frac{1}{\cos^2 x}.
\end{aligned}
$$

The last line is from the identity $\sin^2 x + \cos^2 x = 1$.

Solution 1.4

This will use the product rule.

$$\begin{aligned}\frac{d}{dx}\left(e^x \sin x\right) &= e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x)\sin x \\ &= e^x \cos x + e^x \sin x.\end{aligned}$$

Solution 1.5

For complicated functions like this, it's often helpful to zoom on part of it at the beginning. Let's start by taking the derivative of the numerator.

$$\begin{aligned}\frac{d}{dx}(\sin x \cos x) &= \sin x \frac{d}{dx}(\cos x) + \frac{d}{dx}(\sin x)\cos x \\ &= \sin x(-\sin x) + \cos x \cos x \\ &= -\sin^2 x + \cos^2 x \\ &= \cos^2 x - \sin^2 x.\end{aligned}$$

Now let's look at the whole thing.

$$\begin{aligned}&\frac{d}{dx}\left(\frac{\sin x \cos x}{x^2+2x+1}\right) \\ &= \frac{(x^2+2x+1)\cdot \frac{d}{dx}(\sin x \cos x) - \sin x \cos x \cdot \frac{d}{dx}(x^2+2x+1)}{(x^2+2x+1)^2} \\ &= \frac{(x^2+2x+1)\cdot(\cos^2 x - \sin^2 x) - \sin x \cos x \cdot (2x+2)}{(x^2+2x+1)^2}.\end{aligned}$$

There isn't much we can do to simplify this, so we'll just leave it like that.

Solution 2.1

For all of these, it will be useful to calculate f', f'', and f''' first.

$$f'(x) = -a \sin x + b \cos x$$

$$f''(x) = -a \cos x - b \sin x$$

$$f'''(x) = a \sin x - b \cos x$$

We know that $f(0) = 3$ and $f(\pi/2) = -2$. Since $f(0) = a \cos 0 + b \sin 0 = a \cdot 1 + b \cdot 0 = a$ and $f(\pi/2) = a \cos \frac{\pi}{2} + b \sin \frac{\pi}{2} = a \cdot 0 + b \cdot 1 = b$, we know that $a = 3$ and $b = -2$.

Solution 2.2

We know that $f(0) = -1$ and $f'(\pi) = 4$. Since $f(0) = a\cos 0 + b\sin 0 = a \cdot 1 + b \cdot 0 = a$, we know that $a = -1$. Since $f'(\pi) = -a\sin\pi + b\cos\pi = -a \cdot 0 + b \cdot (-1) = -b$, we know that $-b = 4$, so $b = -4$.

Solution 2.3

We know that $f''(\pi) = 0$ and $f'''(2\pi) = 3$. Since $f''(\pi) = -a\cos\pi - b\sin\pi = -a \cdot (-1) - b \cdot 0 = a$, we know that $a = 0$. Since $f'''(2\pi) = a\sin 2\pi - b\cos 2\pi = a \cdot 0 - b \cdot 1 = -b$, we know that $-b = 3$, so $b = -3$.

Solution 3

1. $\frac{d}{dx}\left(\sin(e^x)\right)$ requires the chain rule. The inside is $u = e^x$.
2. $\frac{d}{dx}\left(e^x \sin x\right)$ does not require the chain rule—only the product rule.
3. $\frac{d}{dx}\left(\cos(1/x)\right)$ requires the chain rule. The inside is $u = 1/x$.
4. $\frac{d}{dx}\left(\frac{1}{x^2}\right)$ does not require the chain rule—we can just use the power rule.
5. $\frac{d}{dx}\left(5x + e^{(x^2)}\right)$ requires the chain rule. The inside is x^2.
6. $\frac{d}{dx}\left(\sin \cos x\right)$ requires the chain rule. The inside is $u = \cos x$.